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On Gauss Fibonacci polynomials, on Gauss Lucas polynomials and their applications

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ABSTRACT

We define the Gauss Fibonacci polynomials. Then we give a formula for the Gauss Fibonacci polynomials by using the Fibonacci polynomials. The Gauss Lucas polynomials are described and the relation with Lucas polynomials are explained. We show that there is a relation between the Gauss Fibonacci polynomials and the Fibonacci polynomials. The Gauss Lucas polynomials are also given by using the Gauss Fibonacci polynomials. Some theorems like Cassini's theorem are proved for the polynomials. Their Binet's formulas are obtained. We also define the matrices of the Gauss Fibonacci polynomials and the Gauss Lucas polynomials. We examine properties of the matrices.

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1. Introduction

The investigation of Gaussian numbers is a research topic of great interest. The set of these numbers is denoted by Z[i]. One of the first studies on this subject belongs to Gauss. Gaussian numbers were investigated by Gauss in 1832. Then, in 1963, Horadam [6], introduced the concept of Gaussian Fibonacci numbers. And then Jordan [8] considered the two different sequences of Gaussian numbers and extended some relationships which are known about the common Fibonacci sequences.

Horadam also studied also the complex Fibonacci polynomials [7]. Polynomials that can be defined by Fibonacci-like recursion relations are called Fibonacci Polynomials and they were studied in 1883 by Catalan and Jacobsthal. More mathematicians were involved in the study of Fibonacci polynomials such as Yuan and Zhang [14], among others. Berzsenyi [2] presented a natural manner of extension of the Fibonacci numbers into the complex plane. In addition to this, Berzsenyi obtained some interesting identities for the classical Fibonacci numbers. And, he gave a closed form for Gaussian Fibonacci numbers by the Fibonacci Q matrix. In 1981, Harman gave the extension of Fibonacci numbers into the complex plane [5]. And, the author generalized the methods are given by Horadam and Berzsenyi. Stakhov and Rozin [13] derived analytical formulas for the Fibonacci p-numbers. They showed that these formulas are similar to the Binet formulas for the classical Fibonacci numbers.

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The Fibonacci polynomials, The Lucas polynomials, The Gauss sequence and their properties have been studied by some authors; see for example [1, 3, 4, 9–12].

2. Gauss Fibonacci polynomials, Gauss Lucas polynomials and their some properties

2.1. Gauss Fibonacci polynomials

In this section, we introduce Gauss Fibonacci polynomials $GF_n(x)$ and present some of their basic properties.

Definition 1. Gauss Fibonacci polynomials $GF_n(x)$ are defined by

$$GF_{n+1}(x) = xGF_n(x) + GF_{n-1}(x), n \geq 2$$

with $GF_1(x) = 1$ and $GF_2(x) = x + i$.

The first few terms are as follows:

$$\begin{aligned} GF_3(x) &= xGF_2(x) + GF_1(x) \\ &= x(x+i) + 1 = x^2 + 1 + ix \\ GF_4(x) &= xGF_3(x) + GF_2(x) \\ &= x(x^2 + 1 + ix) + x + i = x^3 + 2x + ix^2 + i \\ GF_5(x) &= xGF_4(x) + GF_3(x) \\ &= x^4 + 3x^2 + 1 + ix^3 + 2ix \\ GF_6(x) &= xGF_5(x) + GF_4(x) \\ &= x^5 + 4x^3 + 3x + ix^4 + 3ix^2 + i \\ GF_7(x) &= xGF_6(x) + GF_5(x) \\ &= x^6 + 5x^4 + 6x^2 + 1 + ix^5 + 4ix^3 + 3ix. \end{aligned}$$

And, if we write $x = 1$ in the equation $GF_{n+1}(x) = xGF_n(x) + GF_{n-1}(x)$, then we obtain $GF_n(1) = GF_n$.

Theorem 1. *For $n > 2$, we have*

$$GF_n(x) = F_n(x) + iF_{n-1}(x).$$

Proof. We can prove the theorem by the induction method on n. For $n = 3$, we have

$$GF_3(x) = xGF_2(x) + GF_1(x) = x^2 + 1 + ix = F_3(x) + iF_2(x).$$

Now, assume that the theorem holds for $n = k$, that is

$$GF_k(x) = F_k(x) + iF_{k-1}(x).$$

Then, for $n = k + 1$, we have

$$\begin{aligned} GF_{k+1}(x) &= xGF_k(x) + GF_{k-1}(x) \\ &= x(F_k(x) + iF_{k-1}(x)) + (F_{k-1}(x) + iF_{k-2}(x)) \\ &= xF_k(x) + xiF_{k-1}(x) + F_{k-1}(x) + iF_{k-2}(x) \\ &= xF_k(x) + F_{k-1}(x) + i(xF_{k-1}(x) + F_{k-2}(x)) \\ &= F_{k+1}(x) + iF_k(x) \end{aligned}$$

Thus, the result follows. □

We defined Binet's formulas for the Gauss Fibonacci polynomials. Let $\alpha(x)$ and $\beta(x)$ be the solutions of the quadratic equation $t^2 - xt - 1 = 0$:

$$\alpha(x) = \frac{x + \sqrt[2]{x^2+4}}{2}, \beta(x) = \frac{x - \sqrt[2]{x^2+4}}{2}.$$

So we obtain

$$GF_n(x) = \frac{\alpha^{n-1}(x)(\alpha(x)+i) - \beta^{n-1}(\beta(x)+i)}{\alpha(x)-\beta(x)}.$$

Theorem 2. *For $n \geq 3$, we have*

$$GF_n^2(x) + GF_{n+1}^2(x) = F_{2n}(x)(x+2i).$$

Proof.

$$\begin{aligned}
GF_n^2(x)GF_{n+1}^2(x) &= F_{2n}(x)(x+2i) \\
&= (F_n(x) + iF_{n-1}(x))^2 + (F_{n+1}(x) + iF_n(x))^2 \\
&= F_n^2(x) + 2iF_n(x)F_{n-1}(x) + i^2F_{n-1}^2(x) \\
&\quad + F_{n+1}^2(x) + 2iF_{n+1}(x)F_n(x) + i^2F_n^2(x) \\
&= F_{n+1}^2(x) - F_{n-1}^2(x) + 2iF_n(x)(F_{n-1}(x) + F_{n+1}(x)) \\
&= (F_{n+1}(x) + F_{n-1}(x))(F_{n+1}(x) - F_{n-1}(x)) + 2iF_n(x)L_n(x) \\
&= xF_n(x)L_n(x) + 2iF_n(x)L_n(x) \\
&= F_n(x)L_n(x)(x+2i) \\
&= F_{2n}(x)(x+2i).
\end{aligned}$$

□

Theorem 3. *For $n \geq 1$, we have*

$$GF_{n+1}^2(x) - GF_{n-1}^2(x) = (x^2+2ix)F_{2n-1}(x).$$

Proof. If we use Theorem 2, we have

$$\begin{aligned}
GF_{n+1}^2(x) &= F_{2n}(x)(x+2i) - GF_n^2(x) \\
GF_{n-1}^2(x) &= F_{2n-2}(x)(x+2i) - GF_n^2(x)
\end{aligned}$$

If to the first equation, we subtract the second one we have:

$$\begin{aligned}
GF_{n+1}^2(x) - GF_{n-1}^2(x) &= F_{2n}(x)(x+2i) - F_{2n-2}(x)(x+2i) \\
&= (x+2i)(F_{2n}(x) - F_{2n-2}(x)) \\
&= (x+2i)xF_{2n-1}(x) \\
&= (x^2+2ix)F_{2n-1}(x).
\end{aligned}$$

□

Definition 2. Gauss Fibonacci polynomial matrix $gf_n(x)$ is defined by

$$gf_n(x) = \begin{bmatrix} GF_{n+1}(x) & GF_n(x) \\ GF_n(x) & GF_{n-1}(x) \end{bmatrix}, n \geq 0.$$

For example $gf_0(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $gf_1(x) = \begin{bmatrix} x+i & 1 \\ 1 & 0 \end{bmatrix}$.

Proposition 1. *For $n \geq 2$, we have*

$$\det(gf_n(x)) = (-1)^n(2 - ix).$$

Proof. We prove the proposition by the induction method on n. For $n = 2$, we have

$$\begin{vmatrix} x^2 + 1 + ix & x + i \\ x + i & 1 \end{vmatrix} = x^2 + 1 + ix - (x + i)^2 = 2 - ix$$

Now, assume that the equation holds for $n = k$, that is

$$\det(gf_k(x)) = GF_{k-1}(x)GF_{k+1}(x) - GF_k^2(x) = (-1)^k(2 - 1ix).$$

Then, for $n = k + 1$, we have

$$\begin{aligned}
\det(gf_{k+1}(x)) &= \begin{vmatrix} GF_{k+2}(x) & GF_{k+1}(x) \\ GF_{k+1}(x) & GF_k(x) \end{vmatrix} = GF_{k+2}(x)GF_k(x) - GF_{k+1}^2(x) \\
&= (xGF_{k+1}(x) + GF_k(x))GF_k(x) - GF_{k+1}(x)GF_{k+1}(x) \\
&= (x(xGF_k(x) + GF_{k-1}(x)) + GF_k(x))GF_k(x) - (xGF_k(x) + GF_{k-1}(x))GF_{k+1}(x) \\
&= x^2GF_k^2(x) + xGF_{k-1}(x)GF_k(x) + GF_k^2(x) - xGF_k(x)GF_{k+1}(x) - GF_{k-1}(x)GF_{k+1}(x) \\
&= x^2GF_k^2(x) + xGF_{k-1}(x)GF_k(x) + GF_k^2(x) - xGF_k(x)(xGF_k(x) + GF_{k+1}(x)) \\
&\quad - GF_{k-1}(x)GF_{k+1}(x) \\
&= GF_k^2(x) - GF_{k-1}(x)GF_{k+1}(x) \\
&= (-1)(GF_{k+1}(x)GF_{k-1}(x) - GF_k^2(x)) \\
&= (-1)(-1)^k(2 - ix) \\
&= (-1)^{k+1}(2 - ix).
\end{aligned}$$

Using Proposition 1 and properties of the determinant we can deduce that. □

Proposition 2. *For $n \geq 2$, we have*

$$\det(gf_n^2(x)) + \det(gf_{n+1}^2(x)) = 2(2 - ix)^2.$$

Proof.

$$\begin{aligned}
&\det(gf_n^2(x)) + \det(gf_{n+1}^2(x)) \\
&= \det(gf_n(x))\det(gf_n(x)) + \det(gf_{n+1}(x))\det(gf_{n+1}(x)) \\
&= (-1)^n(2 - ix)(-1)^n(2 - ix) + (-1)^{n+1}(2 - ix)(-1)^{n+1}(2 - ix) \\
&= (-1)^{2n}(2 - ix)^2 + (-1)^{2n+2}(2 - ix)^2 \\
&= (-1)^{2n}(2 - ix)^2(1 + (-1)^2) \\
&= 2(2 - ix)^2.
\end{aligned}$$

□

Proposition 3. *For $n, m \geq 2$, we have*

$$\det(gf_{m+1}(x)gf_{n+1}(x)) + \det(gf_m(x)gf_n(x)) = 2(-1)^{m+n}(2 - ix)^2.$$

Proof. From definition of determinant, we have

$$\begin{aligned}
&\det(gf_{m+1}(x)gf_{n+1}(x)) + \det(gf_m(x)gf_n(x)) \\
&= (-1)^{m+1}(2-ix)(-1)^{n+1}(2-ix) + (-1)^m(2-ix)(-1)^n(2-ix) \\
&= (-1)^{m+n+2}(2-ix)^2 + (-1)^{m+n}(2-ix)^2 \\
&= (-1)^{m+n}(2-ix)^2((-1)^2+1) \\
&= 2(-1)^{m+n}(2-ix)^2.
\end{aligned}$$

□

Proposition 4. *For $n, m \geq 2$, we have*

$$(2-ix)\det(gf_{m+n}(x)) = \det(gf_m(x))\det(gf_n(x))$$

Proof.

$$\begin{aligned}
\det(gf_m(x))\det(gf_n(x)) &= (-1)^m(2-ix)(-1)^n(2-ix) \\
&= (-1)^{m+n}(2-ix)(2-ix) \\
&= (2-ix)\det(gf_{m+n}(x)).
\end{aligned}$$

□

Proposition 5. *For $n \geq 2$, we have*

$$(gf_{n+1}^n(x)) - \det(gf_n^n(x)) = \begin{cases} 0, & n \quad even \\ 2(2-ix)^n, & n \quad odd \end{cases}.$$

Proof.

$$\begin{aligned}
\det(gf_{n+1}^n(x)) - \det(gf_n^n(x)) &= ((-1)^{n+1}(2-ix))^n - ((-1)^n(2-ix))^n \\
&= (-1)^{n^2+n}(2-ix)^n - (-1)^{n^2}(2-ix)^n \\
&= (-1)^{n^2}(2-ix)((-1)^n - 1)
\end{aligned}$$

□

Then the result follows.

2.2. Gauss Lucas polynomials

In this section, we introduce Gauss Lucas polynomials $GL_n(x)$ and present some of their basic properties.

Definition 3. Gauss Lucas polynomials $GL_n(x)$ are defined by

$$GL_{n+1}(x) = xGL_n(x) + GL_{n-1}(x), n \geq 2$$

with $GL_1(x) = x + 2i$ and $GL_2(x) = x^2 + 2 + ix$.

The first few terms are as follows:

$$\begin{aligned}
GL_n(x) &= xGL_{n-1}(x) + GL_{n-2}(x) \\
GL_{n+1}(x) &= xGL_n(x) + GL_{n-1}(x)
\end{aligned}$$

$$\begin{aligned}
GL_3(x) &= xGL_2(x) + GL_1(x)\\
&= x(x^2 + 2 + ix) + x + 2i = x^3 + 3x + ix^2 + 2i\\
GL_4(x) &= xGL_3(x) + GL_2(x)\\
&= x(x^3 + 3x + ix^2 + 2i) + x^2 + 2 + ix\\
&= x^4 + 4x^2 + 2 + ix^3 + 3ix\\
GL_5(x) &= xGL_4(x) + GL_3(x)\\
&= x^5 + 5x^3 + 5x + ix^4 + 4ix^2 + 2i\\
GL_6(x) &= xGL_5(x) + GL_4(x)\\
&= x^6 + 6x^4 + 9x^2 + 2 + ix^5 + 5ix^3 + 5ix\\
GL_7(x) &= xGL_6(x) + GL_5(x)\\
&= x^7 + 7x^5 + 14x^3 + 7x + ix^6 + 6ix^4 + 9ix^2 + 2
\end{aligned}$$

Theorem 4. *For $n > 2$, we have*

$$GL_n(x) = L_n(x) + iL_{n-1}(x).$$

Proof. We can prove the theorem by the induction method on n. For $n = 3$, we have

$$GL_3(x) = xGL_2(x) + GL_1(x) = x^3 + 3x + ix^2 + 2i = L_3(x) + iL_2(x)$$

Now, assume that the theorem holds for $n = k$, that is

$$GL_k(x) = L_k(x) + iL_{k-1}(x).$$

Then, for $n = k + 1$, we have

$$\begin{aligned}
GL_{k+1}(x) &= xGL_k(x) + GL_{k-1}(x)\\
&= x(L_k(x) + iL_{k-1}(x)) + (L_{k-1}(x) + iL_{k-2}(x))\\
&= xL_k(x) + xiL_{k-1}(x) + L_{k-1}(x) + iL_{k-2}(x)\\
&= xL_k(x) + L_{k-1}(x) + i(xL_{k-1}(x) + L_{k-2}(x))\\
&= L_{k+1}(x) + iL_k(x)
\end{aligned}$$

□

We find Binet's formulas for the Gauss Lucas polynomials. Let $\alpha(x)$ and $\beta(x)$ be the solutions of the quadratic equation $t^2 - xt - 1 = 0$;

$$\alpha(x) = \frac{x + \sqrt[2]{x^2 + 4}}{2}, \beta(x) = \frac{x - \sqrt[2]{x^2 + 4}}{2}.$$

So we obtain

$$GL_n(x) = \alpha^{n-1}(x)(\alpha(x) + i) + \beta^{n-1}(x)(\beta(x) + i).$$

Theorem 5. *For $n \geq 4$, we have*

$$GL_n(x) = GF_{n+1}(x) + GF_{n-1}(x)$$

Proof.

$$\begin{aligned}
GL_n(x) &= L_n(x) + iL_{n-1}(x) = (F_{n+1}(x) + F_{n-1}(x)) + i(F_n(x) + F_{n-2}(x))\\
&= F_{n+1}(x) + F_{n-1}(x) + iF_n(x) + iF_{n-2}(x)\\
&= F_{n+1}(x) + iF_n(x) + F_{n-1}(x) + iF_{n-2}(x)\\
&= GF_{n+1}(x) + GF_{n-1}(x).
\end{aligned}$$

□

Theorem 6. *For $n \geq 5$, we have*

$$GL_{n+1}(x) + GL_{n-1}(x) = (x^2+4)GF_n(x).$$

Proof.

$$\begin{aligned}
GL_{n+1}(x) + GL_{n-1}(x) &= L_{n+1}(x) + iL_n(x) + L_{n-1}(x) + iL_{n-2}(x) \\
&= (F_{n+2}(x) + F_n(x)) + i(F_{n+1}(x) + F_{n-1}) \\
&\quad + (F_n(x) + F_{n-2}(x)) + i(F_{n-1}(x) + F_{n-3}(x)) \\
&= F_{n+2}(x) + iF_{n+1}(x) + F_n(x) + iF_{n-1}(x) \\
&\quad + F_n(x) + iF_{n-1}(x) + F_{n-2}(x) + iF_{n-3}(x) \\
&= GF_{n+2}(x) + GF_n(x) + GF_n(x) + GF_{n-2}(x) \\
&= (xGF_{n+1}(x) + GF_n(x)) + 2GF_n(x) + GF_{n-2}(x) \\
&= x(xGF_n(x) + GF_{n-1}(x)) + 3GF_n(x) + GF_{n-2}(x) \\
&= x^2GF_n(x) + xGF_{n-1}(x) + GF_{n-2}(x) + 3GF_n(x) \\
&= x^2GF_n(x) + GF_n(x) + 3GF_n(x) \\
&= (x^2+4)GF_n(x).
\end{aligned}$$

□

Theorem 7. *For $n \geq 3$, we have*

$$GF_{m+n}(x) = F_m(x)GF_{n+1}(x) + F_{m-1}(x)GF_n(x).$$

Proof.

$$\begin{aligned}
&F_m(x)GF_{n+1}(x) + F_{m-1}(x)GF_n(x) \\
&= F_m(x)(F_{n+1}(x) + iF_n(x)) + F_{m-1}(x)(F_n(x) + iF_{n-1}(x)) \\
&= F_m(x)F_{n+1}(x) + F_{m-1}(x)F_n(x) \\
&\quad + i(F_m(x)F_n(x) + F_{m-1}(x)F_{n-1}(x)) \\
&= F_{n+m}(x) + iF_{n+m-1} \\
&= GF_{m+n}(x).
\end{aligned}$$

□

Theorem 8. *For $n \geq 3$, we have*

$$GF_n(x)GL_n(x) = (x+2i)F_{2n-1}(x).$$

Proof.

$$\begin{aligned}
GL_n(x)GF_n(x) &= (L_n(x) + iL_{n-1}(x))(F_n(x) + iF_{n-1}(x)) \\
&= [(F_{n+1}(x) + F_{n-1}(x)) + i(F_n(x) + F_{n-2}(x))]F_n(x) \\
&\quad + i[(F_{n+1}(x) + F_{n-1}(x)) + i(F_n(x) + F_{n-2}(x))]F_{n-1}(x) \\
&= F_{n+1}(x)F_n(x) + F_{n-1}(x)F_n(x) - F_n(x)F_{n-1}(x) \\
&\quad - F_{n-2}(x)F_{n-1}(x) + i(F_n^2(x) + F_n(x)F_{n-2}(x) \\
&\quad + F_{n+1}(x)F_{n-1}(x) + F_{n-1}^2(x)) \\
&= F_{2n}(x) - F_{2n-2}(x) + i(F_n^2(x) + F_{n-1}^2(x) \\
&\quad + F_n(x)F_{n-2}(x) + F_{n+1}(x)F_{n-1}(x)) \\
&= xF_{2n-1}(x) + i(F_{2n-1}(x) + F_{2n-1}(x)) \\
&= (x+2i)F_{2n-1}(x).
\end{aligned}$$

□

Definition 4. Gauss Lucas Polynomial matrix $gl_n(x)$ is defined by

$$gl_n(x) = \begin{bmatrix} GL_{n+1}(x) & GL_n(x) \\ GL_n(x) & GL_{n-1}(x) \end{bmatrix}, n \geq 0.$$

For example, $gl_0 = \begin{bmatrix} x+2i & 2-ix \\ 2-ix & 2i+ix^2-x \end{bmatrix}$ and $gl_1 = \begin{bmatrix} x^2+2+ix & x+2i \\ x+2i & 2-ix \end{bmatrix}$.

Proposition 6. *For $n \geq 2$, we have*

$$\det(gl_n(x)) = (-1)^{n+1}(x^2+4)(2-ix).$$

Proof. We prove the proposition by the induction method on n. For $n=2$, we have

$$\det(gl_2) = \begin{vmatrix} x^3+3x+ix^2+2i & x^2+2+ix \\ x^2+2+ix & x+i \end{vmatrix} = -(x^2+4)(2-ix)$$

Now, assume that the equation holds for $n=k$. That is

$$gl_k(x) = \begin{bmatrix} GL_{k+1}(x) & GL_k(x) \\ GL_k(x) & GL_{k-1}(x) \end{bmatrix} = (-1)^{k+1}(x^2+4)(2-ix).$$

Then, for $n = k+1$, we have

$$\begin{aligned} gl_{k+1}(x) &= \begin{bmatrix} GL_{k+2}(x) & GL_{k+1}(x) \\ GL_{k+1}(x) & GL_k(x) \end{bmatrix} = GL_{k+2}(x)GL_k(x) - GL_{k+1}^2(x) \\ &= GL_{k+2}(x)GL_k(x) - GL_{k+1}(x)Gl_{k+1}(x) \\ &= (GL_k(x) + xGL_{k+1}(x))GL_k(x) - GL_{k+1}(x)Gl_{k+1}(x) \\ &= (x(xGL_k(x) + GL_{k-1}(x) + GL_k(x))GL_k(x) \\ &\quad - (xGL_k(x) + GL_{k-1}(x))GL_{k+1}(x) \\ &= (x^2+1)GL_k(x)GL_k(x) + xGL_k(x)GL_{k-1}(x) \\ &\quad - xGL_k(x)GL_{k+1}(x) - GL_{k-1}(x)GL_{k+1}(x) \\ &= (x^2+1)GL_k(x)GL_k(x) + xGL_k(x)GL_{k-1}(x) \\ &\quad - xGL_k(x)(xGL_k(x) + GL_{k-1}(x)) - GL_{k-1}(x)GL_{k+1}(x) \\ &= GL_k^2(x) - GL_{k-1}(x)GL_{k+1}(x) \\ &= (-1)(GL_{k-1}(x)GL_{k+1}(x) - GL_k^2(x)) \\ &= (-1)(-1)^{k+1}(x^2+4)(2-ix) \\ &= (-1)^{k+2}(x^2+4)(2-ix). \end{aligned}$$

□

Proposition 7. *For $n \geq 2$, we have*

$$\det(gl_n(x)) = (-1)(x^2+4)\det(gf_n(x)).$$

Proof. The proof is easily deduced from Proposition 1 and Proposition 6. □

3. Conclusion

We defined new polynomials by using definitions of Gauss sequence, Fibonacci Polynomial, Lucas Polynomial. The properties of those polynomials were examined. Some theorems about these polynomials were presented.

References

[1] Atanassov, K., Atanassova, L., Sasselov, D. (1985). A new perspective to the generalization of the Fibonacci sequence. *Fibonacci Q.* 23(1):21–28.

[2] Berzsenyi, G. (1977). Gaussian Fibonacci numbers. *Fibonacci Q.* 15(3):233–236.

[3] Deveci, Ö., Karaduman, E., Campbell, C. M. (2017). The Fibonacci-circulant sequences and their applications. *Iran. J. Sci. Technol. Trans. Sci.* 41(4):1033–1038. DOI: 10.1007/s40995-017-0317-7.

[4] Halici, S., Öz, S. (2016). On some Gaussian Pell and Pell-Lucas numbers. *Ordu Univ. Sci. Technol. J.* 6(1): 8–18.

[5] Harman, C. J. (1981). Complex Fibonacci numbers. *Fibonacci Q.* 19(1):82–86.

[6] Horadam, A. F. (1961). A generalized Fibonacci sequence. *Am. Math. Mon.* 68(5):455–459. DOI: 10.1080/00029890.1961.11989696.

[7] Horadam, A. F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. *Am. Math. Mon.* 70(3): 289–291. DOI: 10.2307/2313129.

[8] Jordan, J. H. (1965). Gaussian Fibonacci and Lucas numbers. *Fibonacci Q.* 3(4):315–318.

[9] Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications*. New York, NY: Wiley-Interscience Publication.

[10] Özkan, E., Altun, İ., Göçer, A. (2017). On relationship among a new family of k-Fibonacci, k-Lucas numbers, Fibonacci and Lucas numbers. *Chiang Mai J. Sci.* 44(4):1744–1750.

[11] Özkan, E., Taştan, M., Aydoğdu, A. (2018). 2-Fibonacci polynomials in the family of Fibonacci numbers. *Notes Number Theory Discret. Math.* 24(3):47–55. DOI: 10.7546/nntdm.2018.24.3.47-55.

[12] Pethe, S., Horadam, A. F. (1986). Generalized Gaussian Fibonacci numbers. *Bull. Aust. Math. Soc.* 33(1): 37–48. DOI: 10.1017/S0004972700002847.

[13] Stakhov, A. P., Rozin, B. (2006). Theory of Binet formulas for Fibonacci and Lucas p-numbers. *Chaos Solitons Fractals* 27(5):1162–1177. DOI: 10.1016/j.chaos.2005.04.106.

[14] Yuan, Y., Zhang, W. (2002). Some identities involving the Fibonacci polynomials. *Fibonacci Q.* 40(4): 314–318.