

On θ - b -Continuous Functions

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Abstract. In this paper we introduce a new class of functions called θ - b -continuous. Some characterizations and several properties concerning θ - b -continuity are obtained. Furthermore relationships among θ - b -continuity, strong θ - b -continuity, b -continuity, weak b -continuity and faint b -continuity are investigated.

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1 Introduction

The notion of b -open sets is introduced by Andrijevic [3] and variants of continuity related to this concept are studied in [5,11,12,19,21,22]. The purpose of the present paper is to introduce and investigate some of the fundamental properties of θ - b -continuous functions. In Section 3, we obtain some characterizations and basic properties of θ - b -continuous functions. In Section 4, we investigate relationships between θ - b -continuity and some other variants of b -continuity. In Section 5, we investigate relationships between θ - b -continuity and separation axioms and study relationships between θ - b -continuity and b - θ -closed graphs. In the last section we give some covering properties of θ - b -continuous functions.

2 Preliminaries

Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset S of (X, τ) , $cl(S)$ and $int(S)$ represent the closure of S and the interior of S , respectively. A subset S of a space (X, τ) is said to be regular open [14] (resp. regular closed [14]) if $S = int(cl(S))$ (resp. $S = cl(int(S))$). A point x of X is called a θ -cluster [23] point of A if $cl(U) \cap A \neq \emptyset$ for every open set U of X containing x . The set of all θ -cluster points of A is called the θ -closure [23] of A and is denoted by $cl_\theta(A)$. A set A is said to be θ -closed if $A = cl_\theta(A)$. The complement of a θ -closed set is said to be θ -open [9]. A subset S of a space (X, τ) is said to be semi-open [7] (resp. preopen [10], α -open [13], semi-preopen [2] or β -open [1], b -open [3] or γ -open [5]) if $S \subset cl(int(S))$ (resp. $S \subset int(cl(S))$, $S \subset int(cl(int(S)))$, $S \subset cl(int(cl(S)))$, $S \subset cl(int(S)) \cup int(cl(S))$). The family of all semi-open (resp. preopen, α -open, β -open, b -open) sets of X is denoted by $SO(X)$ (resp. $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$). The complement of a semi-open (resp. preopen, α -open, β -open, b -open) set is said to be semi-closed (resp. preclosed, α -closed, β -closed, b -closed). If S is a subset of a space X , then the b -closure of S , denoted by $bcl(S)$, is the smallest b -closed set containing S . The semiclosure (resp. preclosure, α -closure, β -closure) of S is similarly defined and is denoted by $scl(S)$ (resp. $pcl(S)$, $\alpha Cl(S)$, $\beta Cl(S)$). The b -interior of S , denoted by $bint(S)$, is the largest b -open set contained in S . A subset S is said to be b -regular if it is b -open and b -closed. The family of all b -closed (resp. b -regular) sets of X is denoted by $BC(X)$ (resp. $BR(X)$) and the family of all b -open (resp. b -regular) sets of X containing a point $x \in X$ is denoted by $BO(X, x)$ (resp. $BR(X, x)$).

A point x of X is called a b - θ -cluster point [19] of S if $bcl(U) \cap S \neq \emptyset$ for every $U \in BO(X, x)$. The set of all b - θ -cluster points of S is called the b - θ -closure [19] of S and is denoted by $bcl_\theta(S)$. A subset S is said to be b - θ -closed if $S = bcl_\theta(S)$. The complement of a b - θ -closed set is said to be b - θ -open and the b - θ -interior of a set S , $bint_\theta(S)$, is defined as $bint_\theta(S) = \{x \in S : bcl(U) \subset S \text{ for some } U \in BO(X, x)\}$.

Lemma 1. (Andrijevic [3]) *For a subset A of a space X , the following properties hold:*

- (a) $bcl(A) = scl(A) \cap pcl(A)$;
- (b) $bint(A) = sint(A) \cup pint(A)$;
- (c) $x \in bcl(A)$ if and only if $A \cap U \neq \emptyset$ for every $BO(X, x)$;

(d) $A \in BC(X)$ if and only if $A = bcl(A)$.

Lemma 2. (Park, [19]) Let A and B be any subsets of a space X . Then the following properties hold:

- (a) $A \in BO(X)$ if and only if $bcl(A) \in BR(X)$;
- (b) if $A \subset B$, then $bcl_\theta(A) \subset bcl_\theta(B)$;
- (c) $bcl_\theta(bcl_\theta(A)) = bcl_\theta(A)$;
- (d) $x \in bcl_\theta(A)$ if and only if $V \cap A \neq \emptyset$ for each $V \in BR(X, x)$;
- (e) A is b - θ -open in X if and only if for each $x \in A$ there exists $V \in BR(X, x)$ such that $x \in V \subset A$;
- (f) $bcl_\theta(A)$ is b - θ -closed;
- (g) if $A \in BO(X)$, then $bcl(A) = bcl_\theta(A)$.

Lemma 3. A subset U of a space X is b - θ -open in X if and only if for each $x \in U$, there exists a $W \in BO(X)$ with $x \in W$ such that $bcl(W) \subset U$.

Proof. Suppose that U is b - θ -open in X . Then $X \setminus U$ is b - θ -closed. Let $x \in U$, then $x \notin bcl_\theta(X \setminus U)$ and so there exists $W \in BO(X)$ with $x \in W$ such that $bcl(W) \cap (X \setminus U) = \emptyset$. Thus $bcl(W) \subset U$.

Conversely, assume that U is not a b - θ -open. Then $X \setminus U$ is not b - θ -closed, and so there exists $x \in bcl_\theta(X \setminus U)$ such that $x \notin X \setminus U$. Since $x \in U$, by hypothesis, there exists $W \in BO(X)$ with $x \in W$ such that $bcl(W) \subset U$. Thus $bcl(W) \cap (X \setminus U) = \emptyset$. This is a contradiction since $x \in bcl_\theta(X \setminus U)$. $\square$

3 Characterizations

Definition 4. A function $f : X \rightarrow Y$ is said to be θ - b -continuous (briefly θ - b -c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in BO(X, x)$ such that $f(bcl(U)) \subset cl(V)$.

Theorem 5. The following properties are equivalent for a function $f : X \rightarrow Y$:

- (a) f is θ - b -c.;

- (b) $bcl_\theta(f^{-1}(int(cl(B)))) \subset f^{-1}(cl(B))$ for every subset B of Y ;
- (c) $bcl_\theta(f^{-1}(int(F))) \subset f^{-1}(F)$ for every regular closed set F of Y ;
- (d) $bcl_\theta(f^{-1}(int(cl(V)))) \subset f^{-1}(cl(V))$ for every $V \in \beta O(Y)$;
- (e) $bcl_\theta(f^{-1}(int(cl(V)))) \subset f^{-1}(cl(V))$ for every $V \in BO(Y)$;
- (f) $bcl_\theta(f^{-1}(V)) \subset f^{-1}(cl(V))$ for every $V \in PO(Y)$;
- (g) $bcl_\theta(f^{-1}(V)) \subset f^{-1}(cl(V))$ for every open set V of Y ;
- (h) $f^{-1}(V) \subset bint_\theta(f^{-1}(cl(V)))$ for every open set V of Y .

Proof. (a) $\Rightarrow$ (b): Let B be any subset of Y . Assume that $x \in X \setminus f^{-1}(cl(B))$. Then $f(x) \in Y \setminus cl(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \emptyset$; hence $cl(V) \cap int(cl(B)) = \emptyset$. Since f is θ .b.c., there exists $U \in BO(X, x)$ such that $f(bcl(U)) \subset cl(V)$. Therefore we have $bcl(U) \cap f^{-1}(int(cl(B))) = \emptyset$ and hence $x \in X \setminus bcl_\theta(f^{-1}(int(cl(B))))$. Thus we obtain $bcl_\theta(f^{-1}(int(cl(B)))) \subset f^{-1}(cl(B))$.

(b) $\Rightarrow$ (c): Let F be any regular closed set of Y . Then we have $bcl_\theta(f^{-1}(int(F))) = bcl_\theta(f^{-1}(int(cl(int(F))))) \subset f^{-1}(cl(int(F))) = f^{-1}(F)$.

(c) $\Rightarrow$ (d): Let V be any β -open set of Y . It follows that $cl(V)$ is regular closed. Since f is θ .b.c., by (c) we have $bcl_\theta(f^{-1}(int(cl(V)))) \subset f^{-1}(cl(V))$.

(d) $\Rightarrow$ (e): This is clear, since $BO(Y) \subset \beta O(Y)$.

(e) $\Rightarrow$ (f): This is clear, since $PO(Y) \subset BO(Y)$.

(f) $\Rightarrow$ (g): This is clear, since every open set is preopen.

(g) $\Rightarrow$ (h): Let V be any open set of Y . Since $Y \setminus cl(V)$ is open in Y we have $X \setminus bint_\theta(f^{-1}(cl(V))) = bcl_\theta(X \setminus f^{-1}(cl(V))) = bcl_\theta(f^{-1}(Y \setminus cl(V))) \subset f^{-1}(cl(Y \setminus cl(V))) = f^{-1}(Y \setminus int(cl(V))) \subset X \setminus f^{-1}(V)$. Then $f^{-1}(V) \subset bint_\theta(f^{-1}(cl(V)))$.

(h) $\Rightarrow$ (a): Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and, by (h) and Lemma 3, there exists $U \in BO(X, x)$ such that $bcl(U) \subset f^{-1}(cl(V))$. Hence $f(bcl(U)) \subset cl(V)$ and f is θ -b-continuous. $\square$

Theorem 6. *The following properties are equivalent for a function $f : X \rightarrow Y$:*

- (a) f is θ .b.c.;
- (b) $f(bcl_\theta(A)) \subset cl_\theta(f(A))$ for every subset A of X ;
- (c) $bcl_\theta(f^{-1}(B)) \subset f^{-1}(cl_\theta(B))$ for every subset B of Y ;

(d) $bcl_\theta(f^{-1}(int(cl_\theta(B)))) \subset f^{-1}(cl_\theta(B))$ for every subset B of X .

Proof. (a) $\Rightarrow$ (b): Let A be a subset of X and $x \in bcl_\theta(A)$. Let V be any open set of Y containing $f(x)$. By (a) there exists $U \in BO(X, x)$ such that $f(bcl(U)) \subset cl(V)$. Then $bcl(U) \cap A \neq \emptyset$ and $\emptyset \neq f(bcl(U)) \cap f(A) \subset cl(V) \cap f(A)$ so that $f(x) \in cl_\theta(f(A))$.

(b) $\Rightarrow$ (c): Let B be a subset of Y . We have

$$f(bcl_\theta(f^{-1}(B))) \subset cl_\theta(f(f^{-1}(B))) \subset cl_\theta(B) \text{ then } bcl_\theta(f^{-1}(B)) \subset f^{-1}(cl_\theta(B)).$$

(c) $\Rightarrow$ (d): Let B be a subset of Y . Since $cl_\theta(B)$ is closed in Y , we have $bcl_\theta(f^{-1}(int(cl_\theta(B)))) \subset f^{-1}(cl(int(cl_\theta(B)))) \subset f^{-1}(cl_\theta(B))$.

(d) $\Rightarrow$ (a): Let V be any open set of Y . Then $V \subset int(cl(V)) = int(cl_\theta(V))$ and hence $bcl_\theta(f^{-1}(V)) \subset bcl_\theta(f^{-1}(int(cl_\theta(V)))) \subset f^{-1}(cl_\theta(V)) \subset f^{-1}(cl(V))$ and it follows from Theorem 5 that f is θ .b.c. $\square$

Lemma 7. (Andrijevic [3]). *The intersection of an open set and a b -open set is a b -open set.*

Theorem 8. *Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ the graph function of f , defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is θ - b -continuous if and only if f is θ .b.c.*

Proof. Necessity. Suppose that g is θ .b.c. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since g is θ .b.c., there exists $U \in BO(X, x)$ such that $g(bcl(U)) \subset cl(X \times V)$. It follows that $cl(X \times V) = X \times cl(V)$. Therefore we obtain $f(bcl(U)) \subset cl(V)$. This shows that f is θ .b.c.

Sufficiency. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subset X$ and $V \subset Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subset W$. Since f is θ .b.c., there exists $U_2 \in BO(X, x)$ such that $f(bcl(U_2)) \subset cl(V)$. Let $U = U_1 \cap U_2$ then $U \in BO(X, x)$ by Lemma 7. Therefore we obtain $g(bcl(U)) \subset cl(U_1) \times f(bcl(U_2)) \subset cl(U_1) \times cl(V) \subset cl(W)$. This shows that g is θ .b.c. $\square$

Lemma 9. (El-Atik [5]). *Let A and X_0 be subsets of a space (X, τ) . If $A \in BO(X)$ and $X_0 \in \alpha O(X)$, then $A \cap X_0 \in BO(X_0)$.*

Lemma 10. (Park [19]) *If $A \subset X_0 \subset X$ and X_0 is α -open in X , then $bcl(A) \cap X_0 = bcl_{X_0}(A)$, where $bcl_{X_0}(A)$ denotes the b -closure of A in the subspace X_0 .*

Theorem 11. *If $f : X \rightarrow Y$ is θ .b.c. and X_0 is an α -open subset of X , then the restriction $f|_{X_0} : X_0 \rightarrow Y$ is θ .b.c.*

Proof. For any $x \in X_0$ and any open neighborhood V of $f(x)$, there exists $U \in BO(X, x)$ such that $f(bcl(U)) \subset cl(V)$ since f is $\theta.b.c.$ Put $U_0 = U \cap X_0$, then by Lemma 9 $U_0 \in BO(X_0, x)$ and by Lemma 10 $bcl_{X_0}(U_0) \subset bcl(U_0)$. Therefore, we obtain $(f|_{X_0})(bcl_{X_0}(U_0)) = f(bcl_{X_0}(U_0)) \subset f(bcl(U_0)) \subset f(bcl(U)) \subset cl(V)$. $\square$

Definition 12. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be θ -continuous [6] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U in X containing x such that $f(cl(U)) \subset cl(V)$.

Theorem 13. If $f : X \rightarrow Y$ is $\theta.b.c.$ and $g : Y \rightarrow Z$ is θ -continuous, then the composition $g \circ f : X \rightarrow Z$ is $\theta.b.c.$

Proof. Let W be an open set of Z . $bcl_\theta((g \circ f)^{-1}(W)) = bcl_\theta(f^{-1}(g^{-1}(W))) \subset f^{-1}(cl(g^{-1}(W))) \subset f^{-1}(g^{-1}(cl(W))) \subset (g \circ f)^{-1}(cl(W))$ which proves that $g \circ f$ is $\theta.b.c.$ $\square$

Theorem 14. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. If $g \circ f : X \rightarrow Z$ is $\theta.b.c.$ and f is surjective and satisfies the condition that $f(bcl_\theta(A))$ is b - θ -closed for every $A \subset X$, then g is $\theta.b.c.$

Proof. Let W be an open set of Z . Since $g \circ f : X \rightarrow Z$ is $\theta.b.c.$, by Theorem 5 $bcl_\theta(f^{-1}(g^{-1}(W))) \subset f^{-1}(g^{-1}(cl(W)))$ and hence $f(bcl_\theta(f^{-1}(g^{-1}(W)))) \subset g^{-1}(cl(W))$. Because $f(bcl_\theta(f^{-1}(g^{-1}(W))))$ is b - θ -closed, $bcl_\theta(f(f^{-1}(g^{-1}(W)))) \subset f(bcl_\theta(f^{-1}(g^{-1}(W))))$. Finally, since f is surjective, it follows that $bcl_\theta((g^{-1}(cl(W))) \subset g^{-1}(cl(W))$. By Theorem 5, g is $\theta.b.c.$ $\square$

4 Comparisons

Definition 15. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

- (a) b -continuous [5] if $f^{-1}(V)$ is b -open in (X, τ) for every open set V of (Y, σ) ,
- (b) strongly θ - b -continuous [19] (resp. weakly b -continuous [20], almost b -continuous [22]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in BO(X, x)$ such that $f(bcl(U)) \subset V$ (resp. $f(U) \subset cl(V)$, $f(U) \subset int(cl(V))$),
- (c) strongly θ -continuous [16] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U in X containing x such that $f(cl(U)) \subset V$,

- (d) faintly b -continuous [11] if for each $x \in X$ and each θ -open set V of Y containing $f(x)$, there exists $U \in BO(X, x)$ such that $f(U) \subset V$,
- (e) θ -semi-continuous [17] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in SO(X, x)$ such that $f(scl(U)) \subset cl(V)$,
- (f) θ -pre-continuous [18] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in PO(X, x)$ such that $f(pcl(U)) \subset cl(V)$.

Remark 16. Both θ -semi-continuity and θ -precontinuity imply θ - b -continuity but the converse implication is not true in general as the following example shows.

Example 17. Let τ be the usual topology for $\mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, [0, 1] \cup ((1, 2) \cap \mathbb{Q}), (-\infty, 0) \cup ((1, 2) \cap \mathbb{R} \setminus \mathbb{Q}) \cup [2, \infty)\}$. Then the identity function $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \sigma)$ is θ - b -continuous but it is neither θ -semi-continuous nor θ -pre-continuous.

Theorem 18. For a function $f : X \rightarrow Y$, the following properties hold:

- (a) if f is almost b -continuous, then it is θ . b .c.,
- (b) if f is θ . b .c., then it is weakly b -continuous.

Proof. Statement (b) is obvious. We will show statement (a). Suppose $x \in X$ and V is any open set of Y containing $f(x)$. Since f is almost b -continuous, $f^{-1}(int(cl(V)))$ is b -open and since $cl(V)$ is regular closed $f^{-1}(cl(V))$ is b -closed by [22]. Now set $U = f^{-1}(int(cl(V)))$. Then we have $U \in BO(X, x)$ and $bcl(U) \subset f^{-1}(cl(V))$. Therefore we obtain $f(bcl(U)) \subset cl(V)$. This shows that f is θ . b .c. $\square$

Remark 19. We have the following implications for a function $f : X \rightarrow Y$:

Strongly θ - b -continuous $\Rightarrow$ b -continuous $\Rightarrow$ almost b -continuous $\Rightarrow$ θ - b -continuous $\Rightarrow$ weakly b -continuous $\Rightarrow$ faintly- b -continuous.

Remark 20. None of these implications is reversible in general as the following examples and related articles [11,19,21,22] shows.

Example 21. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is weakly b -continuous but it is not θ . b .c.

Example 22. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Define $f : (X, \tau) \rightarrow (X, \tau)$ by $f(a) = c$, $f(b) = d$, $f(c) = b$ and $f(d) = a$. Then f is $\theta.b.c.$ but it is not almost b -continuous.

Corollary 23. Let Y be a regular space. Then, for a function $f : X \rightarrow Y$, the following properties are equivalent:

- (a) f is strongly $\theta.b.c.$;
- (b) f is b -continuous;
- (c) f is almost b -continuous;
- (d) f is θ - b -continuous;
- (e) f is weakly b -continuous;
- (f) f is faintly b -continuous.

Proof. This is an immediate consequence of [22, Theorem 4]. $\square$

Definition 24. A space X is said to be b -regular [19] if for each closed set F and each point $x \in X \setminus F$, there exist disjoint b -open sets U and V such that $x \in U$ and $F \subset V$.

Theorem 25. Let X be a b -regular space. Then $f : X \rightarrow Y$ is $\theta.b.c.$ if and only if f is weakly b -continuous.

Proof. Suppose that f is weakly b -continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then there exists $U \in BO(X, x)$ such that $f(U) \subset cl(V)$. Since X is b -regular, there exists $W \in BO(X, x)$ such that $x \in W \subset bcl(W) \subset U$ by [19, Lemma 4.12]. Therefore we obtain $f(bcl(W)) \subset cl(V)$. This shows f is $\theta.b.c.$ $\square$

5 Separation axioms and graph properties

Definition 26. A space X is said to be $b-T_2$ [5] if for each pair of distinct points x and y in X , there exist b -open sets U and V of X containing x and y , respectively, such that $U \cap V = \emptyset$.

Definition 27. A function $f : X \rightarrow Y$ is said to have a b - θ -closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a b -open set U of X and an open set V of Y such that $(x, y) \in U \times V$ and $(bcl(U) \times cl(V)) \cap G(f) = \emptyset$.

Lemma 28. *The graph $G(f)$ of a function $f : X \rightarrow Y$ is b - θ -closed if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a b -open set U of X containing x and an open set V in Y containing y such that $f(bcl(U)) \cap cl(V) = \emptyset$.*

Theorem 29. *If Y is a Urysohn space and $f : X \rightarrow Y$ is θ .b.c., then $G(f)$ is b - θ -closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and there exist open sets V and W of Y containing $f(x)$ and y , respectively, such that $cl(V) \cap cl(W) = \emptyset$. Since f is θ .b.c., there exists a b -open set U of X containing x such that $f(bcl(U)) \subset cl(V)$. Therefore, we obtain $f(bcl(U)) \cap cl(W) = \emptyset$ and hence $G(f)$ is b - θ -closed in $X \times Y$. $\square$

Theorem 30. *Let $f : X \rightarrow Y$ be a θ .b.c. function having the b - θ -closed graph $G(f)$. If f is injective, then X is b - T_2 .*

Proof. Let x_1 and x_2 be two distinct points of X . Since f is injective, $f(x_1) \neq f(x_2)$ and $(x_1, f(x_2)) \notin G(f)$. Since $G(f)$ is b - θ -closed, there exist $U \in BO(X, x_1)$ and an open set V of Y such that $(x_1, f(x_2)) \in U \times V$ and $f(bcl(U)) \cap cl(V) = \emptyset$. Since f is θ .b.c., there exists a $W \in BO(X, x_2)$ such that $f(bcl(W)) \subset cl(V)$. Therefore, we have $f(U) \cap f(W) = \emptyset$ and hence $U \cap W = \emptyset$. This shows that X is b - T_2 . $\square$

6 Covering properties

Definition 31. *A subset K of a space X is said to be H -set [23] or quasi H -closed relative to X [20] if for every cover $\{V_\alpha : \alpha \in I\}$ of K by open sets of X , there exists a finite subset I_0 of I such that $K \subset \cup\{cl(V_\alpha) : \alpha \in I_0\}$.*

Definition 32. *A topological space (X, τ) is said to be*

- (a) almost compact [15] or quasi H -closed [20] if every cover of X by open sets has a finite subcover whose closures cover of X ;
- (b) lightly compact [4] if every countable cover of X by open sets has a finite subcover whose closures cover of X .

Definition 33. *A topological space (X, τ) is said to be*

- (a) b -closed [19] if every cover of X by b -open sets has a finite subcover whose b -closures cover X ;

- (b) countably b -closed [19] if every countable cover of X by b -open sets has a finite subcover whose b -closures cover X .
- (c) C -compact [24] if for each closed subset $A \subset X$ and each cover $\{U_\alpha : \alpha \in I\}$ of A by open sets of X , there exists a finite subset I_0 of I such that $A \subset \cup\{cl(U_\alpha) : \alpha \in I_0\}$.

Definition 34. A subset K of a space X is said to be b -closed relative to X [19] if for every cover $\{V_\alpha : \alpha \in I\}$ of K by b -open sets of X , there exists a finite subset I_0 of I such that $K \subset \cup\{bcl(V_\alpha) : \alpha \in I_0\}$.

Theorem 35. If a function $f : X \rightarrow Y$ is θ - b -c. and K is b -closed relative to X , then $f(K)$ is quasi H -closed relative to Y .

Proof. Suppose that $f : X \rightarrow Y$ is θ - b -continuous and K is b -closed relative to X . Let $\{V_\alpha : \alpha \in I\}$ be a cover of $f(K)$ by open sets of Y . For each point $x \in K$, there exists $\alpha(x) \in I$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ - b -c., there exists a $U_x \in BO(X, x)$ such that $f(bcl(U_x)) \subset cl(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by b -open sets of X and hence there exists a finite subset K_0 of K such that $K \subset \bigcup_{x \in K_0} bcl(U_x)$. Therefore, we obtain $f(K) \subset \bigcup_{x \in K_0} cl(V_{\alpha(x)})$. This shows that $f(K)$ is quasi H -closed relative to Y . $\square$

Corollary 36. Let $f : X \rightarrow Y$ be a θ - b -c. surjection. Then the following properties hold:

- (a) If X is b -closed, then Y is almost compact.
- (b) If X is countably b -closed, then Y is lightly compact.

Proof. (a) Let $\{V_\alpha : \alpha \in I\}$ be a cover of Y by open sets of Y . For each point $x \in X$, there exists $\alpha(x) \in I$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ - b -c., there exists a b -open set U_x of X containing x such that $f(bcl(U_x)) \subset cl(V_{\alpha(x)})$. The family $\{U_x : x \in X\}$ is a cover of X by b -open sets of X and hence there exists a finite subset X_0 of X such that $X \subset \bigcup_{x \in X_0} bcl(U_x)$. Therefore, we obtain $Y = f(X) \subset \bigcup_{x \in X_0} cl(V_{\alpha(x)})$. This shows that Y is almost compact.
 (b) Similar to (a). $\square$

Theorem 37. If a function $f : X \rightarrow Y$ has the b - θ -closed graph, then $f(K)$ is θ -closed in Y for each subset K which is b -closed relative to X .

Proof. Let K be b -closed relative to X and $y \in Y \setminus f(K)$. Then for each $x \in K$, we have $(x, y) \in (X \times Y) \setminus G(f)$ and there exists $U_x \in BO(X, x)$ and an open set V_x of Y containing y such that $f(bcl(U_x)) \cap cl(V_x) = \emptyset$. The family $\{U_x : x \in K\}$ is a cover of K by b -open sets of X . Since K is b -closed relative to X , there exists a finite subset K_0 of K such that $K \subset \cup\{bcl(U_x) : x \in K_0\}$. Put $V = \cap_{x \in K_0} V_x$, then V is an open set containing y and $f(K) \cap cl(V) \subset \bigcup_{x \in K_0} [f(bcl(U_x)) \cap cl(V)] \subset \bigcup_{x \in K_0} [f(bcl(U_x)) \cap cl(V_x)] = \emptyset$. This shows that $f(K)$ is θ -closed in Y . $\square$

Theorem 38. *Let X be a submaximal extremally disconnected space. If a function $f : X \rightarrow Y$ has the b - θ -closed graph, then $f^{-1}(K)$ is θ -closed in X for each H -set K in Y .*

Proof. Let K be an H -set of Y and $x \notin f^{-1}(K)$. For each $y \in K$, we have $(x, y) \in (X \times Y) \setminus G(f)$ and there exist a b -open set U_y of X containing x and an open set V_y of Y containing y such that $f(bcl(U_y)) \cap cl(V_y) = \emptyset$ and hence $bcl(U_y) \cap f^{-1}(cl(V_y)) = \emptyset$. The family $\{V_y : y \in K\}$ is a cover of K by open sets of Y . Since K is an H -set in Y , there exists a finite subset K_0 of K such that $K \subset \cup\{cl(V_y) : y \in K_0\}$. Since X is submaximal extremally disconnected, each U_y is open in X and $bcl(U_y) = cl(U_y)$. Set $U = \cap_{y \in K_0} U_y$, then U is an open set containing x and $f(cl(U)) \cap K \subset \bigcup_{y \in K_0} [f(cl(U)) \cap cl(V_y)] \subset \bigcup_{y \in K_0} [f(bcl(U_y)) \cap cl(V_y)] = \emptyset$. Therefore, we have $cl(U) \cap f^{-1}(K) = \emptyset$ and hence $f^{-1}(K)$ is θ -closed in X . $\square$

Corollary 39. *Let X be a submaximal extremally disconnected space and Y be a compact Urysohn space. Then the following properties are equivalent for a function $f : X \rightarrow Y$:*

- (a) f is θ .b.c.;
- (b) $G(f)$ is b - θ -closed in $X \times Y$;
- (c) f is strongly θ -continuous;
- (d) f is θ -continuous.

Proof. (a) $\Rightarrow$ (b): It follows from Theorem 29.

(b) $\Rightarrow$ (c): Since Y is compact Urysohn space, Y is C -compact. Let A be a closed set in Y . Then A is an H -set and $f^{-1}(A)$ is θ -closed in X according to Theorem 38. Therefore f is strongly θ -continuous by [8, Theorem 1].

(c) $\Rightarrow$ (d): This is clear.

(d) $\Rightarrow$ (a): Since Y is regular, it follows from Corollary 23. $\square$

Definition 40. A space X is said to be b -connected [5] (resp. b - θ -connected) if it can not be written as the union of the two nonempty disjoint b -open (resp. b - θ -open) sets.

Theorem 41. A space X is b - θ -connected if and only if it is b -connected.

Proof. We prove only necessity, sufficiency can be seen similarly. Suppose that X is not b -connected. There exists nonempty b -open sets V and W of X such that $V \cup W = X$ and $V \cap W = \emptyset$. Then V and W are b -open and b -closed and so b -regular in X from the definition of b -regularness. This gives V and W are b - θ -open and b - θ -closed by [19, Theorem 3.8]. Then X is not b - θ -connected. $\square$

Theorem 42. If $f : X \rightarrow Y$ is θ .b.c. surjective function and X is b -connected, then Y is connected.

Proof. Let $f : X \rightarrow Y$ be a θ .b.c. surjective function and assume X is b -connected. Then f is weakly b -continuous by Theorem 18. Since the image of a b -connected space under a w .b.c. surjective function is connected [21], the proof is complete. $\square$

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