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Algebraic, Differential and Fractional Properties of Balancing-like Polynomials

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Abstract

This study investigates the algebraic and differential properties of balancing-like polynomials, which constitute a polynomial extension of balancing-like sequences. Several fundamental properties are established, including matrix representations, addition and subtraction formulas, and polynomial analogues of the Cassini, Catalan, D'Ocagne, and Vajda identities. A product factorization is obtained, which yields an explicit description of the zeros of these polynomials. Moreover, a direct connection with the Chebyshev polynomials of the second kind is established, and the ordinary generating function of the balancing-like polynomials is derived. The derivative sequences associated with these polynomials are then systematically investigated. Explicit formulas, recurrence relations, and a self-convolution identity are obtained, together with a relation recovering the original polynomials from their neighboring derivative terms. Finally, the analysis is extended to higher-order derivatives, and a generating-function representation and a general recurrence relation for the resulting differential hierarchy are derived. The differential framework is further extended to non-integer orders by means of the Caputo fractional derivative. An explicit fractional derivative formula and a generating-function representation are obtained.

Keywords: balancing-like polynomials; chebyshev polynomials; caputo fractional derivative; generating functions; recurrence relations; fractional calculus; gegenbauer polynomials; orthogonal polynomials

1. Introduction

The study of integer sequences constitutes one of the central themes in number theory, with the Fibonacci sequence being perhaps the most renowned and extensively analyzed example [1,2]. Over the years, a multitude of integer sequences have been introduced and investigated across mathematical literature, each revealing intriguing structural and arithmetic properties. Among these, the balancing sequence occupies a special position due to its rich algebraic behavior and its connection with triangular numbers and other Diophantine relationships. The balancing sequence is defined through the following recurrence relation [3,4]:

$$B_{n+1} = 6B_n - B_{n-1} \quad (n \geq 1),$$

with $B_0 = 0$ and $B_1 = 1$. For example 6, 35 and 204 are balancing numbers [3].

The Binet formula for balancing sequence is given as [3]:

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$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}}.$$

Here $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

Balancing numbers exhibit numerous fascinating identities. For instance, they satisfy the following relations [5]:

$$\begin{aligned} B_n^2 - 1 &= B_{n+1}B_{n-1}, \\ B_{m+n}B_{m-n} &= B_m^2 - B_n^2, \\ B_1 + B_3 + \cdots + B_{2n-1} &= B_n^2, \\ B_2 + B_4 + \cdots + B_{2n} &= B_n B_{n+1}. \end{aligned}$$

In 2012, Panda and Rout [6] extended the concept of balancing numbers by introducing balancing-like sequences, which preserve many of the essential algebraic properties of the original balancing sequence.

The balancing-like sequences are defined by the following recurrence relation [6,7]:

$$x_{m+1} = Ax_m - x_{m-1}$$

with $x_0 = 0$ and $x_1 = 1$. Here $A > 2$ is any integer. For example $A, A^2 - 1, \cdots$ are balancing-like numbers.

The Binet formula for balancing-like sequences is [8]

$$x_m = \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2}$$

with $\lambda_1 = \frac{A+\sqrt{A^2-4}}{2}$ and $\lambda_2 = \frac{A-\sqrt{A^2-4}}{2}$.

Over time, several generalizations of the balancing sequence have been proposed. In 2016, Ray [9] introduced the k-balancing sequence, which depends on a real parameter k, and studied various algebraic properties of it [10]. Ray further defined balancing polynomials, which naturally extend k-balancing numbers to the polynomial domain and derived numerous related identities. A related polynomial generalization, arising from a different recurrence structure, has recently been studied in the context of k-Oresme and k-Oresme–Lucas sequences [11].

For any positive number k, the k-balancing sequence is defined by [9]:

$$B_{k,n+1} = 6kB_{k,n} - B_{k,n-1}$$

with $B_{k,0} = 0$ and $B_{k,1} = 1$. For example $6k, 36k^2 - 1, 216k^3 - 12k, \cdots$ are k-balancing numbers [12]. Its Binet formula is:

$$B_{k,n} = \frac{\theta_1^n - \theta_2^n}{\theta_1 - \theta_2}$$

with $\theta_1 = 3k + \sqrt{9k^2 - 1}$ and $\theta_2 = 3k - \sqrt{9k^2 - 1}$.

Balancing polynomials, which further generalize k-balancing numbers, are defined recursively as follows:

$$B_{n+1}(x) = \begin{cases} 1 & n = 0, \\ 6x & n = 1, \\ 6xB_n(x) - B_{n-1}(x) & n > 1. \end{cases}$$

The first few balancing polynomials are $\{0, 1, 6x, 36x^2 - 1, 216x^3 - 12x, \cdots\}$. Their Binet representation is:

$$B_n(x) = \frac{\mu^n(x) - \mu^{-n}(x)}{\mu(x) - \mu^{-1}(x)}$$

with $\mu(x) = 3x + \sqrt{9x^2 - 1}$ and $\mu^{-1}(x) = 3x - \sqrt{9x^2 - 1}$.

More recently, in 2024, Patra and Panda [13] introduced the k-balancing-like sequences, defined for $k \in \mathbb{Z}^+$ by,

$$x_{k,m+1} = Akx_{k,m} - x_{k,m-1} \quad (m \geq 1)$$

with $x_{k,0} = 0$ and $x_{k,1} = 1$. For example $Ak, A^2k^2 - 1, A^3k^3 - 2Ak, \cdots$ are k-balancing-like numbers. The Binet formula for k-balancing-like sequences are [13]

$$x_{k,m} = \frac{\gamma_1^m - \gamma_2^m}{\gamma_1 - \gamma_2}$$

with $\gamma_1 = \frac{Ak + \sqrt{A^2k^2 - 4}}{2}$ and $\gamma_2 = \frac{Ak - \sqrt{A^2k^2 - 4}}{2}$.

In this framework, balancing-like polynomials were defined as follows [14]:

$$x_{m+1}(y) = Ayx_m(y) - x_{m-1}(y) \quad (m \geq 1) \tag{1}$$

with $x_0(y) = 0$ and $x_1(y) = 1$. The first few balancing-like polynomials are $\{0, 1, Ay, A^2y^2 - 1, \cdots\}$. The Binet formula for balancing-like polynomials is given by

$$x_m(y) = \frac{\beta_1^m(y) - \beta_2^m(y)}{\beta_1(y) - \beta_2(y)} \tag{2}$$

with $\beta_1(y) = \frac{Ay + \sqrt{A^2y^2 - 4}}{2}$ and $\beta_2(y) = \frac{Ay - \sqrt{A^2y^2 - 4}}{2}$.

The companion Lucas-balancing-like polynomials were investigated in [15].

Lemma 1. *The characteristic roots of* (1) *satisfy*

$$\beta_1(y) + \beta_2(y) = Ay, \qquad \beta_1(y)\beta_2(y) = 1, \qquad \beta_2(y) = \beta_1^{-1}(y),$$

$$\beta_1(y) - \beta_2(y) = \sqrt{A^2y^2 - 4}.$$

Proof. β_1, β_2 are the roots of $\beta^2 - Ay\beta + 1 = 0$; the claims are Viète's formulas. □

Remark 1. *For every $m \geq 0$, $x_m(y)$ is a polynomial in y with integer coefficients in $\mathbb{Z}[A][y]$, hence it is defined for all $y \in \mathbb{C}$. The Binet representation is used for $A^2y^2 - 4 \neq 0$, where $\beta_1(y) \neq \beta_2(y)$; the excluded values $y = \pm 2/A$ are removable, since*

$$\lim_{y \to \pm 2/A} \frac{\beta_1^m(y) - \beta_2^m(y)}{\beta_1(y) - \beta_2(y)} = (\pm 1)^{m-1}m = x_m(\pm 2/A).$$

Consequently every identity proved via Binet formula is a polynomial identity and therefore remains valid for all y, including $y = \pm 2/A$ and complex y, by continuity (equivalently, because two polynomials agreeing on an infinite set are identical).

Throughout, A is regarded as a formal parameter. All identities in Sections 2 and 3 are polynomial identities in $\mathbb{Z}[A, y]$ and hence hold for every A. The restriction $A > 2$, $A \in \mathbb{Z}$, inherited from the theory of balancing-like sequences, is needed only for the growth and positivity statements and for the arithmetic interpretation of $x_m(1)$; it is not assumed

elsewhere. In particular the specialization $A = 1$ used in Section 3 is legitimate at the level of polynomial identities.

The theory of generalized polynomial sequences has also been developed through connections with classical special functions, finite operators, matrix representations, and polynomial factorization methods. In this direction, Kızılateş et al. [16] investigated (p,q)-Chebyshev polynomials and obtained explicit formulas, generating functions, derivative identities, and determinant representations. Their work provides a broader perspective on Chebyshev-type polynomial generalizations, which is particularly relevant to the representation of balancing-like polynomials in terms of Chebyshev polynomials of the second kind. Furthermore, Kızılateş [17] studied Horadam numbers associated with finite operators and established recurrence relations, generating functions, and tridiagonal determinant representations. These results illustrate the role of operator and matrix methods in the investigation of generalized recurrence sequences. More recently, Shi and Kızılateş [18] developed a complex factorization method for bi-periodic Fibonacci and Lucas polynomials using trigonometric identities and tridiagonal matrices. This approach is closely related to the factorization and zero distribution considered in the present study.

In addition to these algebraic developments, connections between special polynomials and fractional calculus have attracted considerable attention. Özmen and Yılmaz [19] investigated Mittag-Leffler polynomials and their deformations, including various generating-function identities. Moreover, Shahwan and Bin-Saad [20] studied Mittag-Leffler–Konhauser polynomials using operational methods and investigated their connections with fractional calculus. The generalized Mittag-Leffler function introduced by Prabhakar [21] plays a central role in this connection, and recently introduced orthogonal families of Appell type, such as the telephone polynomials connecting Hermite and Laguerre polynomials [22], further illustrate the interplay between special polynomials and their structural properties. These studies provide a broader context for extending classical polynomial structures through fractional operators. Motivated by this perspective, the present work investigates the Caputo fractional derivatives of balancing-like polynomials and derives explicit finite-sum and generating-function representations. Our approach therefore connects the classical Chebyshev structure of balancing-like polynomials with a unified framework for ordinary and fractional differentiation.

The objective of the present work is to examine and characterize the algebraic structure of balancing-like polynomials. We derive several classical identities, including the Catalan, Cassini, and Vajda identities—within this generalized framework. Moreover, using matrix representations as developed in [23], we provide new perspectives on these polynomials and their derivative sequences. The study also extends to the differential properties of the polynomials, exploring the relationships between the original sequence and its successive derivatives.

The main contributions of this paper can be summarized as follows. First, several fundamental algebraic properties of balancing-like polynomials are established, including matrix representations, addition and subtraction formulas, and polynomial analogues of the Cassini, Catalan, D'Ocagne, and Vajda identities. A product factorization is also obtained, providing an explicit description of the zeros of these polynomials. Second, the derivative sequences associated with balancing-like polynomials are systematically investigated. Explicit formulas and recurrence relations for these derivative sequences are derived, together with their asymptotic behavior. In particular, a self-convolution identity connecting the first derivative sequence with the original polynomial sequence is established. Furthermore, a direct relation that recovers the original balancing-like polynomials from their neighboring derivative terms is obtained. Finally, the analysis is extended to higher-order derivatives, leading to a general recurrence relation that describes

the resulting differential hierarchy. Furthermore, a direct connection with the Chebyshev polynomials of the second kind is established, and the ordinary generating function of the balancing-like polynomial sequence is derived.

Finally, the differential framework is extended to non-integer orders through the Caputo fractional derivative. We obtain an explicit finite-sum representation for the fractional derivative of an arbitrary balancing-like polynomial and derive the corresponding generating function. The ordinary derivative hierarchy is recovered as the integer-order specialization of this fractional construction.

2. Some Properties of Balancing-like Polynomials

In this section, we investigate a range of algebraic and structural properties of balancing-like polynomials. These polynomials are constructed as generalizations of k-balancing-like numbers, and several of their fundamental identities are derived. The presented results parallel many well-known properties of balancing and Fibonacci-type sequences, yet they exhibit distinct polynomial behavior under parameter transformations.

The addition and subtraction formulas, the Cassini, Catalan, D'Ocagne, and Vajda identities, the matrix representations, the product factorization, and the characterization of zeros presented in this section can be interpreted as specializations or rescaled forms of classical Chebyshev polynomial properties. These results are included to provide a coherent algebraic foundation for the subsequent analysis rather than being claimed as independent new identities.

The principal contribution of this work lies in the systematic treatment of the derivative sequences associated with balancing-like polynomials and their extension to fractional orders. In particular, Section 3 develops explicit formulas, generating functions, convolution and inversion relations, and recurrence formulas for successive derivatives. The connection with Gegenbauer polynomials is used to unify the higher-order derivative hierarchy and to obtain further structural consequences. Section 4 extends this framework through the Caputo fractional derivative, providing an explicit finite-sum representation and a corresponding generating function. Together, these results establish a unified algebraic, differential, and fractional framework for the balancing-like polynomial family.

It is noteworthy that when $A = 6$, the balancing-like polynomial sequence reduces to the classical balancing polynomials. Similarly, by setting $y = 1$, one recovers the original balancing-like numerical sequence. These reductions confirm the consistency of the generalized form with its predecessors.

Proposition 1. *Let $U_n(x)$ denote the Chebyshev polynomial of the second kind. Then the balancing-like polynomials satisfy*

$$x_m(y) = U_{m-1}\left(\frac{Ay}{2}\right), \qquad m \geq 1.$$

Proof. Recall (see, e.g., [24]) that the Chebyshev polynomials of the second kind satisfy

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

with

$$U_0(x) = 1, \qquad U_1(x) = 2x.$$

Setting

$$x = \frac{Ay}{2},$$

we obtain

$$U_{n+1}\left(\frac{Ay}{2}\right) = Ay\,U_n\left(\frac{Ay}{2}\right) - U_{n-1}\left(\frac{Ay}{2}\right).$$

On the other hand, the balancing-like polynomials satisfy

$$x_{m+1}(y) = Ay\,x_m(y) - x_{m-1}(y),$$

with

$$x_1(y) = 1, \qquad x_2(y) = Ay.$$

Since

$$U_0\left(\frac{Ay}{2}\right) = 1 = x_1(y)$$

and

$$U_1\left(\frac{Ay}{2}\right) = Ay = x_2(y),$$

the two polynomial sequences satisfy the same recurrence relation and the same initial conditions. Therefore,

$$x_m(y) = U_{m-1}\left(\frac{Ay}{2}\right)$$

for every $m \geq 1$. Moreover, adopting the standard convention $U_{-1}(z) \equiv 0$, the representation is also valid for $m = 0$, since

$$x_0(y) = U_{-1}\left(\frac{Ay}{2}\right) = 0.$$

Thus, the Chebyshev representation is consistent with both initial conditions of the balancing-like polynomial sequence. □

Remark 2. *The above representation places the balancing-like polynomials within the classical family of Chebyshev polynomials of the second kind. In particular, several structural properties of $x_m(y)$, including the location and simplicity of its zeros, can also be interpreted through the corresponding properties of $U_{m-1}(x)$.*

It should be emphasized that Proposition 1 identifies the balancing-like polynomials with a rescaled family of Chebyshev polynomials of the second kind,

$$x_m(y) = U_{m-1}\left(\frac{Ay}{2}\right).$$

Consequently, several algebraic properties obtained in Section 2 may also be viewed as scaled forms of classical identities for Chebyshev polynomials. These results are included in order to establish a complete structural framework for the balancing-like polynomial family.

The main contribution of the present work lies instead in the systematic development of the differential hierarchy associated with this family. Using the Chebyshev representation together with generating-function methods, we derive explicit formulas for successive derivatives, self-convolution identities, inversion relations, recurrence formulas, and endpoint evaluations. The higher-order derivative sequence is further expressed in terms of Gegenbauer polynomials, which allows several structural consequences to be obtained in a unified manner.

Finally, we extend the differential framework to non-integer orders by means of the Caputo fractional derivative and derive explicit fractional derivative formulas together with a corresponding generating-function representation.

The following lemma provides a closed-form summation expression for the $(m+1)$ th balancing-like polynomial, facilitating the computation of terms without recursive dependence.

Lemma 2. *Let $m \geq 0$,*

$$x_{m+1}(y) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m-i}{i} (Ay)^{m-2i}. \tag{3}$$

Proof. We proceed by strong induction on m, using Pascal's rule $C(n,k) = C(n-1,k) + C(n-1,k-1)$. Denote $a_m = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i C(m-i,i)(Ay)^{m-2i}$. For $m = 0$, the sum reduces to the single term $i = 0$, giving $a_0 = C(0,0)(Ay)^0 = 1 = x_1(y)$. For $m = 1$, $a_1 = C(1,0)(Ay)^1 = Ay = x_2(y)$. Thus the formula holds for the two base cases required by the second-order recurrence $x_{m+1}(y) = Ayx_m(y) - x_{m-1}(y)$. Assume $a_{m-1} = x_m(y)$ and $a_m = x_{m+1}(y)$ for some $m \geq 1$; we show $a_{m+1} = x_{m+2}(y)$, i.e., that $a_{m+1} = Aya_m - a_{m-1}$, where a_{m+1} is defined by the same closed-form sum with m replaced by $m+1$. Compare the coefficient of $(Ay)^{m-2i+1}$ on both sides of $Aya_m = a_{m+1} + a_{m-1}$. On the left, this coefficient is $(-1)^i C(m-i,i)$. On the right, the term of a_{m+1} with the same power of (Ay) corresponds to index $j = i$, contributing $(-1)^i C(m+1-i,i)$; the term of a_{m-1} with the same power corresponds to index $k = i-1$ (valid for $i \geq 1$), contributing $(-1)^{i-1} C(m-i,i-1)$. Hence the right-hand coefficient equals $(-1)^i[C(m+1-i,i) - C(m-i,i-1)]$. By Pascal's rule, $C(m+1-i,i) = C(m-i,i) + C(m-i,i-1)$, so $C(m+1-i,i) - C(m-i,i-1) = C(m-i,i)$, which matches the left-hand coefficient exactly. The boundary case $i = 0$ is immediate, since $C(m-i,i-1) = 0$ by convention. Consequently $a_{m+1} = Aya_m - a_{m-1} = Ayx_{m+1}(y) - x_m(y) = x_{m+2}(y)$, completing the induction. □

The following result provides a method for computing balancing-like polynomials with negative indices.

Corollary 1. *Let $m \in \mathbb{Z}^+$,*

$$x_{-m}(y) = -x_m(y). \tag{4}$$

Proof. Since $\beta_1(y)\beta_2(y) = 1$ we get

$$\begin{aligned} x_{-m}(y) &= \frac{\beta_1^{-m}(y) - \beta_2^{-m}(y)}{\beta_1(y) - \beta_2(y)} \\ &= \frac{((\beta_1(y))^{-1})^m - ((\beta_2(y))^{-1})^m}{\beta_1(y) - \beta_2(y)} \\ &= -x_m(y). \quad \square \end{aligned}$$

Thus, the extension to negative indices is antisymmetric with respect to the index, namely $x_{-m}(y) = -x_m(y)$.

The following theorem establishes the limit of the ratio of two consecutive terms of the balancing-like polynomials.

Theorem 1. *Let $A > 0$ and $y > \frac{2}{A}$. Then,*

$$\lim_{m \to \infty} \frac{x_{m+1}(y)}{x_m(y)} = \beta_1(y). \tag{5}$$

Proof. By using the Binet formula,

$$\lim_{m\to\infty}\frac{x_{m+1}(y)}{x_m(y)} = \lim_{m\to\infty}\frac{\beta_1^{m+1}(y)-\beta_2^{m+1}(y)}{\beta_1^m(y)-\beta_2^m(y)} = \lim_{m\to\infty}\frac{\beta_1(y)-\left(\frac{\beta_2(y)}{\beta_1(y)}\right)^m\beta_2(y)}{1-\left(\frac{\beta_2(y)}{\beta_1(y)}\right)^m}$$

For $y > \frac{2}{A}$ we have $Ay > \sqrt{A^2y^2-4} > 0$, so $\beta_1(y) > \beta_2(y) > 0$. Since $\beta_1(y)\beta_2(y) = 1$, this gives $0 < \beta_2(y) < 1 < \beta_1(y)$, and hence $0 < \beta_2(y)/\beta_1(y) = \beta_2^2(y) < 1$. Therefore

$$\lim_{m\to\infty}\left(\frac{\beta_2(y)}{\beta_1(y)}\right)^m = 0$$

and this yields the desired result. Hence, the dominant root $\beta_1(y)$ governs the asymptotic growth of the sequence, just as in other linear recurrence relations such as the Fibonacci and Lucas families. □

Corollary 2. *Let $\{x_m(y)\}_{m\geq 0}$ be the sequence of balancing-like polynomials defined by*

$$x_{m+1}(y) = Ay\,x_m(y) - x_{m-1}(y),$$

with $x_0(y) = 0$ and $x_1(y) = 1$. Then the following statements hold:

1. *If $y = \frac{2}{A}$, then*

$$x_m\left(\frac{2}{A}\right) = m,$$

and consequently

$$\lim_{m\to\infty}\frac{x_{m+1}\left(\frac{2}{A}\right)}{x_m\left(\frac{2}{A}\right)} = 1.$$

2. *If $y = -\frac{2}{A}$, then*

$$x_m\left(-\frac{2}{A}\right) = (-1)^{m-1}m,$$

and consequently

$$\lim_{m\to\infty}\frac{x_{m+1}\left(-\frac{2}{A}\right)}{x_m\left(-\frac{2}{A}\right)} = -1.$$

3. *If $-\frac{2}{A} < y < \frac{2}{A}$, the characteristic roots form a complex conjugate pair of modulus one, and the ratio*

$$\frac{x_{m+1}(y)}{x_m(y)}$$

does not converge in general.

Proof. For $y = \frac{2}{A}$, the recurrence relation becomes

$$x_{m+1} = 2x_m - x_{m-1},$$

with $x_0 = 0$ and $x_1 = 1$. Hence, by induction,

$$x_m = m.$$

Therefore,

$$\lim_{m\to\infty}\frac{x_{m+1}}{x_m}=\lim_{m\to\infty}\frac{m+1}{m}=1.$$

Similarly, for $y=-\dfrac{2}{A}$, the recurrence becomes

$$x_{m+1}=-2x_m-x_{m-1}.$$

Using the initial conditions, one obtains

$$x_m=(-1)^{m-1}m.$$

Thus,

$$\lim_{m\to\infty}\frac{x_{m+1}}{x_m}=\lim_{m\to\infty}\left(-\frac{m+1}{m}\right)=-1.$$

Finally, if $-\dfrac{2}{A}<y<\dfrac{2}{A}$, write $Ay=2\cos\theta$ with $\theta\in(0,\pi)$. Then $\beta_{1,2}(y)=e^{\pm i\theta}$ are complex conjugates of modulus one, and the Binet formula gives $x_m(y)=\sin(m\theta)/\sin\theta$. Hence, whenever $x_m(y)\neq 0$,

$$\frac{x_{m+1}(y)}{x_m(y)}=\frac{\sin((m+1)\theta)}{\sin(m\theta)}=\cos\theta+\sin\theta\,\cot(m\theta).$$

If this ratio converged to a finite limit, then $c_m=\cot(m\theta)$ would converge to some $c\in\mathbb{R}$. The addition formula for the cotangent gives $c_{m+1}(c_m+\cot\theta)=c_m\cot\theta-1$, and letting $m\to\infty$ yields $c^2=-1$, which is impossible. Hence the ratio of consecutive terms does not converge. □

Lemma 3. *For $1\le r\le m-1$,*

$$x_{m+1}(y)=x_r(y)x_{m-(r-2)}(y)-x_{r-1}(y)x_{m-(r-1)}(y). \tag{6}$$

Proof. Write $\beta_1=\beta_1(y),\beta_2=\beta_2(y)$, $D=\beta_1-\beta_2$, and recall $\beta_1\beta_2=1$ (Lemma 1). By the Binet Formula (2), $D^2[x_rx_{m-r+2}-x_{r-1}x_{m-r+1}]=(\beta_1^r-\beta_2^r)(\beta_1^{m-r+2}-\beta_2^{m-r+2})-(\beta_1^{r-1}-\beta_2^{r-1})(\beta_1^{m-r+1}-\beta_2^{m-r+1})$. Expanding both products and using the identity $\beta_1^a\beta_2^b=\beta_1^{a-b}$ (valid since $\beta_2=\beta_1^{-1}$), the four cross-terms simplify to β_1^{2r-m-2} and β_1^{m-2r+2}, each occurring once with a plus sign and once with a minus sign, so they cancel in pairs. What remains is $D^2[x_rx_{m-r+2}-x_{r-1}x_{m-r+1}]=\beta_1^{m+2}+\beta_2^{m+2}-\beta_1^m-\beta_2^m$. On the other hand, using $Dx_{m+1}=\beta_1^{m+1}-\beta_2^{m+1}$ together with $\beta_1\beta_2=1$ (so $\beta_1\beta_2^{m+1}=\beta_2^m$ and $\beta_2\beta_1^{m+1}=\beta_1^m$), $D^2x_{m+1}(y)=D(\beta_1-\beta_2)(\beta_1^{m+1}-\beta_2^{m+1})/D=\beta_1^{m+2}-\beta_1\beta_2^{m+1}-\beta_2\beta_1^{m+1}+\beta_2^{m+2}=\beta_1^{m+2}+\beta_2^{m+2}-\beta_1^m-\beta_2^m$, which coincides exactly with the expression obtained above. Dividing by $D^2\neq 0$ gives $x_{m+1}(y)=x_r(y)x_{m-r+2}(y)-x_{r-1}(y)x_{m-r+1}(y)$, as claimed. Since both sides are polynomials in y, the identity extends by continuity to $y=\pm 2/A$ as well (cf. Remark 1). □

Lemma 4. *For $m,s\in\mathbb{Z}^+$,*

$$x_{m+s}(y)=x_m(y)x_{s+1}(y)-x_{m-1}(y)x_s(y). \tag{7}$$

Proof. Write $\beta_1=\beta_1(y),\beta_2=\beta_2(y)$, $D=\beta_1-\beta_2$, and recall $\beta_1\beta_2=1$ (Lemma 1). By the Binet formula (2), $D^2[x_m(y)x_{s+1}(y)-x_{m-1}(y)x_s(y)]=(\beta_1^m-\beta_2^m)(\beta_1^{s+1}-\beta_2^{s+1})-(\beta_1^{m-1}-\beta_2^{m-1})(\beta_1^s-\beta_2^s)$. Expanding both products and using $\beta_1^a\beta_2^b=\beta_1^{a-b}$ (since $\beta_2=\beta_1^{-1}$), the four cross-terms simplify as follows: $\beta_1^m\beta_2^{s+1}=\beta_1^{m-s-1}$ and $\beta_1^{m-1}\beta_2^s=\beta_1^{m-1-s}$, which are equal, and occur with opposite signs (one from each product), so they cancel. Likewise

$\beta_2^m\beta_1^{s+1} = \beta_1^{s+1-m}$ and $\beta_2^{m-1}\beta_1^s = \beta_1^{s-m+1}$, which are equal and also occur with opposite signs, so they too cancel. What remains is $D^2[x_m(y)x_{s+1}(y) - x_{m-1}(y)x_s(y)] = \beta_1^{m+s+1} + \beta_2^{m+s+1} - \beta_1^{m+s-1} - \beta_2^{m+s-1}$. On the other hand, for every integer n, a direct computation using $Dx_n = \beta_1^n - \beta_2^n$ and $\beta_1\beta_2 = 1$ (so $\beta_1\beta_2^n = \beta_2^{n-1}$ and $\beta_2\beta_1^n = \beta_1^{n-1}$) gives $D^2x_n(y) = D(\beta_1 - \beta_2)(\beta_1^n - \beta_2^n)/D = \beta_1^{n+1} - \beta_1\beta_2^n - \beta_2\beta_1^n + \beta_2^{n+1} = \beta_1^{n+1} + \beta_2^{n+1} - \beta_1^{n-1} - \beta_2^{n-1}$. Taking $n = m + s$, the right-hand side becomes exactly $\beta_1^{m+s+1} + \beta_2^{m+s+1} - \beta_1^{m+s-1} - \beta_2^{m+s-1}$, which coincides with the expression obtained above for $D^2[x_mx_{s+1} - x_{m-1}x_s]$. Dividing by $D^2 \neq 0$ (and extending to $y = \pm 2/A$ by continuity, as in Remark 1) gives

$$x_{m+s}(y) = x_m(y)x_{s+1}(y) - x_{m-1}(y)x_s(y). \quad \square$$

Lemma 3 and 4 form the foundation for several classical identities adapted to the balancing-like context, including Cassini, Catalan, and Vajda-type relations. They also serve as essential tools for proving index reduction formulas and matrix representations later in the paper.

By applying Lemma 4, we obtain the following result.

Corollary 3. *For $m \in \mathbb{Z}^+$,*

$$(i) \quad x_{2m-1}(y) = x_m^2(y) - x_{m-1}^2(y), \tag{8}$$

$$(ii) \quad x_{2m}(y) = x_m(y)[x_{m+1}(y) - x_{m-1}(y)] \quad \textit{or equivalently} \tag{9}$$

$$x_{2m}(y) = \frac{x_{m+1}^2(y) - x_{m-1}^2(y)}{Ay}, \qquad y \neq 0. \tag{10}$$

Let us now define the matrix,

$$P = \begin{bmatrix} Ay & -1 \\ 1 & 0 \end{bmatrix}.$$

This matrix representation provides a compact algebraic form for generating balancing-like polynomials and facilitates the derivation of several recurrence-based identities.

Theorem 2. *For any integer t,*

$$P^t = \begin{bmatrix} x_{t+1}(y) & -x_t(y) \\ x_t(y) & -x_{t-1}(y) \end{bmatrix} \tag{11}$$

Proof. For $t = 1$,

$$P = \begin{bmatrix} x_2(y) & -x_1(y) \\ x_1(y) & -x_0(y) \end{bmatrix} = \begin{bmatrix} Ay & -1 \\ 1 & 0 \end{bmatrix}$$

For $t = 0$ the right-hand side is the identity matrix, since $x_1(y) = 1$, $x_0(y) = 0$ and $x_{-1}(y) = -1$. If the claim holds for $t = k$, then

$$P^{k+1} = P^kP = \begin{bmatrix} Ay\,x_{k+1}(y) - x_k(y) & -x_{k+1}(y) \\ Ay\,x_k(y) - x_{k-1}(y) & -x_k(y) \end{bmatrix} = \begin{bmatrix} x_{k+2}(y) & -x_{k+1}(y) \\ x_{k+1}(y) & -x_k(y) \end{bmatrix},$$

so it holds for $t = k + 1$. Similarly, multiplying by $P^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & Ay \end{bmatrix}$ and using $x_{k-1}(y) = Ay\,x_k(y) - x_{k+1}(y)$ shows that the claim for $t = k$ implies the claim for $t = k - 1$. Hence the formula holds for every integer t. $\square$

In particular, P^0 corresponds to the identity matrix, emphasizing that the entire family of balancing-like polynomials can be generated via powers of P.

Corollary 4. *For any integer t,*

$$P^{-t} = \begin{bmatrix} -x_{t-1}(y) & x_t(y) \\ -x_t(y) & x_{t+1}(y) \end{bmatrix}. \tag{12}$$

Since $\det P = 1$, it follows that $\det P^t = (\det P)^t = 1$, yielding the important identity

$$x_t^2(y) - x_{t+1}(y)x_{t-1}(y) = 1,$$

which serves as Cassini formula for balancing-like polynomials.

Theorem 3. *For any $m, n \in \mathbb{Z}$,*

$$x_{m+n+1}(y) = x_{m+1}(y)x_{n+1}(y) - x_m(y)x_n(y) \tag{13}$$

Proof. Multiplying the corresponding matrix representations P^m and P^n and equating the entries leads directly to the result. □

Corollary 5. *For any $m, n \in \mathbb{Z}$,*

$$(i) - x_{m+n}(y) = -x_{m+1}(y)x_n(y) + x_{n-1}(y)x_m(y)$$
$$(ii) - x_{m+n-1}(y) = -x_m(y)x_n(y) + x_{m-1}(y)x_{n-1}(y)$$

Theorem 4. *For any $m, n \in \mathbb{Z}$,*

$$x_{m-n+1}(y) = -x_{m+1}(y)x_{n-1}(y) + x_m(y)x_n(y). \tag{14}$$

This identity parallels the subtraction form of the Fibonacci relation, extending it to polynomial arguments.

Proof. Let $m, n \in \mathbb{Z}$,

$$\begin{aligned} P^m P^{-n} &= \begin{bmatrix} x_{m+1}(y) & -x_m(y) \\ x_m(y) & -x_{m-1}(y) \end{bmatrix} \begin{bmatrix} -x_{n-1}(y) & x_n(y) \\ -x_n(y) & x_{n+1}(y) \end{bmatrix} \\ &= \begin{bmatrix} x_{m-n+1}(y) & -x_{m-n}(y) \\ x_{m-n}(y) & -x_{m-n-1}(y) \end{bmatrix} \end{aligned}$$

By using the equality of the two matrices, we derive the desired result. □

Corollary 6. *For any $m, n \in \mathbb{Z}$,*

$$i) - x_{m-n}(y) = x_{m+1}(y)x_n(y) - x_m(y)x_{n+1}(y)$$
$$ii) - x_{m-n-1}(y) = x_m(y)x_n(y) - x_{m-1}(y)x_{n+1}(y)$$
$$iii) x_{m-n}(y) = x_{m-1}(y)x_n(y) - x_m(y)x_{n-1}(y).$$

Theorem 5 (Catalan Identity)**.** *Let $n, r \in \mathbb{Z}^+$,*

$$x_{n+r}(y)x_{n-r}(y) - x_n^2(y) = -x_r^2(y). \tag{15}$$

Proof.

$$\begin{aligned} x_{n+r}(y)x_{n-r}(y) - x_n^2(y) &= \frac{\left(\beta_1^{n+r}(y) - \beta_2^{n+r}(y)\right)\left(\beta_1^{n-r}(y) - \beta_2^{n-r}(y)\right)}{\left(\beta_1(y) - \beta_2(y)\right)^2} - \frac{\left(\beta_1^n(y) - \beta_2^n(y)\right)^2}{\left(\beta_1(y) - \beta_2(y)\right)^2} \\ &= \frac{-\left(\beta_1^{2r}(y) + \beta_2^{2r}(y) - 2\right)}{\left(\beta_1(y) - \beta_2(y)\right)^2} \\ &= -x_r^2(y) \quad \square \end{aligned}$$

This generalization of the Catalan identity highlights the structural parallelism between balancing-like polynomials and other well-known recursive polynomial systems.

Theorem 6.

$$(i) \quad x_1(y) + x_3(y) + \cdots + x_{2m-1}(y) = x_m^2(y), \tag{16}$$

$$(ii) \quad x_2(y) + x_4(y) + \cdots + x_{2m}(y) = x_m(y)x_{m+1}(y). \tag{17}$$

Proof. Both statements can be proven by mathematical induction, with the base case $m = 1$ is trivial. The inductive step follows from identity (8) and (9), ensuring consistency across even and odd indices. $\square$

Theorem 7 (D'Ocagne's Identity)**.** *Let* $m, r \in \mathbb{Z}$,

$$x_{m+1}(y)x_r(y) - x_m(y)x_{r+1}(y) = -x_{m-r}(y). \tag{18}$$

Proof. Using the Binet formula, we obtain

$$\begin{aligned} &\frac{\left(\beta_1^{m+1}(y) - \beta_2^{m+1}(y)\right)\left(\beta_1^r(y) - \beta_2^r(y)\right)}{\left(\beta_1(y) - \beta_2(y)\right)^2} - \frac{\left(\beta_1^m(y) - \beta_2^m(y)\right)\left(\beta_1^{r+1}(y) - \beta_2^{r+1}(y)\right)}{\left(\beta_1(y) - \beta_2(y)\right)^2} = \\ &= \frac{-\left(\beta_1^{m-r}(y) - \beta_2^{m-r}(y)\right)}{\beta_1(y) - \beta_2(y)} \\ &= -x_{m-r}(y). \quad \square \end{aligned}$$

Theorem 8 (Vajda Identity)**.** *Let* $n, i, j \in \mathbb{Z}$,

$$x_{n+i}(y)x_{n+j}(y) - x_n(y)x_{n+i+j}(y) = x_i(y)x_j(y). \tag{19}$$

Proof. Write $\beta_1 = \beta_1(y)$, $\beta_2 = \beta_2(y)$ and $D = \beta_1 - \beta_2$. By Lemma 1, $\beta_1\beta_2 = 1$. Then

$$\begin{aligned} &D^2\left(x_{n+i}x_{n+j} - x_n x_{n+i+j}\right) \\ &= (\beta_1^{n+i} - \beta_2^{n+i})(\beta_1^{n+j} - \beta_2^{n+j}) - (\beta_1^n - \beta_2^n)(\beta_1^{n+i+j} - \beta_2^{n+i+j}) \\ &= -\beta_1^{n+i}\beta_2^{n+j} - \beta_1^{n+j}\beta_2^{n+i} + \beta_1^n\beta_2^{n+i+j} + \beta_1^{n+i+j}\beta_2^n \\ &= (\beta_1\beta_2)^n(\beta_1^{i+j} + \beta_2^{i+j} - \beta_1^i\beta_2^j - \beta_1^j\beta_2^i) \\ &= (\beta_1^i - \beta_2^i)(\beta_1^j - \beta_2^j) = D^2 x_i x_j. \end{aligned}$$

Dividing by $D^2 \neq 0$ gives the claim for $A^2y^2 \neq 4$; since both sides are polynomials in y, the identity holds for all y. $\square$

Both theorems represent key polynomial analogues of classical integer sequence identities, extending D'Ocagne's and Vajda's formulations to a wider family of balancing-like structures.

By applying the Vajda identity, we obtain the following result.

Theorem 9 (General bilinear index-reduction formula)**.** *For $r, a, b, c, d \in \mathbb{Z}$ with $a + b = c + d$ then,*

$$x_a(y)x_b(y) - x_c(y)x_d(y) = x_{a-r}(y)x_{b-r}(y) - x_{c-r}(y)x_{d-r}(y) \tag{20}$$

Proof. By using the Binet Formula (2) we obtain

$$x_{n+h}(y)x_{n+k}(y) - x_n(y)x_{n+h+k}(y) = x_h(y)x_k(y). \tag{21}$$

Setting $n = c, h = a - c, k = b - c$ in the equality (21), we get

$$x_a(y)x_b(y) - x_c(y)x_{a+b-c}(y) = x_{a-c}(y)x_{b-c}(y). \tag{22}$$

Similarly setting $n = c - r, h = a - c, k = b - c$ in the equality (21) we find

$$x_{a-r}(y)x_{b-r}(y) - x_{c-r}(y)x_{a+b-c-r}(y) = x_{a-c}(y)x_{b-c}(y). \tag{23}$$

When these two identities are compared and using $a + b - c = d$ we complete the proof of the theorem. □

By taking $a = n + 1, b = m, c = n, d = m + 1, r = n - 1$ in the general bilinear index-reduction formula and using the initial conditions of the balancing-like polynomials, the following result is obtained.

Corollary 7.

$$x_{m-n}(y) = x_{n+1}(y)x_m(y) - x_n(y)x_{m+1}(y)$$

In the next theorem a product formula for the balancing-like polynomials is derived.

Theorem 10. *If $x_m(y)$ are the m*th *balancing-like polynomials, then*

$$x_m(y) = \prod_{1 \le k \le m-1} \left[Ay - 2\cos\frac{k\pi}{m} \right]. \tag{24}$$

Proof. Both sides are polynomials in y, so it suffices to prove the identity for $y \in (-2/A, 2/A)$. Write $Ay = 2\cos\theta$ with $\theta \in (0, \pi)$. Then $\beta_{1,2}(y) = e^{\pm i\theta}$ and Binet formula gives

$$x_m(y) = \frac{e^{im\theta} - e^{-im\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin m\theta}{\sin\theta}.$$

Hence $x_m(y) = 0$ iff $\sin m\theta = 0$ and $\sin\theta \neq 0$, i.e., $\theta = k\pi/m$ for $k = 1, \ldots, m-1$, which yields the $m - 1$ distinct roots $y_k = \frac{2}{A}\cos\left(\frac{k\pi}{m}\right)$. Since $\deg x_m = m - 1$ with leading coefficient A^{m-1} (Lemma 2), these are all the roots and each is simple, so

$$x_m(y) = A^{m-1}\prod_{k=1}^{m-1}(y - y_k) = \prod_{k=1}^{m-1}\left(Ay - 2\cos\frac{k\pi}{m}\right). \quad \square$$

Corollary 8. *Assume $A > 0$. For $m \ge 2$, all zeros of the balancing-like polynomial $x_m(y)$ are real and simple. More precisely, they are given by*

$$y_k = \frac{2}{A}\cos\left(\frac{k\pi}{m}\right), \qquad k = 1, 2, \ldots, m-1.$$

In particular, all zeros lie in the interval

$$\left(-\frac{2}{A}, \frac{2}{A}\right).$$

Theorem 11. *The ordinary generating function of the balancing-like polynomials* $\{x_m(y)\}_{m\geq 0}$ *is given by*

$$G(t,y) = \sum_{m=0}^{\infty} x_m(y)t^m = \frac{t}{1 - Ayt + t^2}.$$

Proof. Let

$$G(t,y) = \sum_{m=0}^{\infty} x_m(y)t^m.$$

Using the recurrence relation

$$x_{m+1}(y) = Ay\, x_m(y) - x_{m-1}(y), \qquad m \geq 1,$$

we have

$$\sum_{m=1}^{\infty} x_{m+1}(y)t^{m+1} = Ay \sum_{m=1}^{\infty} x_m(y)t^{m+1} - \sum_{m=1}^{\infty} x_{m-1}(y)t^{m+1}.$$

Since $x_0(y) = 0$ and $x_1(y) = 1$, this becomes

$$G(t,y) - t = Ayt\, G(t,y) - t^2 G(t,y).$$

Hence,

$$G(t,y)\left(1 - Ayt + t^2\right) = t,$$

and therefore

$$\boxed{G(t,y) = \frac{t}{1 - Ayt + t^2}.} \quad \square$$

Corollary 9. *For* $y = 1$*, the product relation established in Theorem 10 implies closed-form product evaluation for* $x_m(y)$*. For example,*

$$x_4 = \prod_{1\leq k\leq 3}\left[A - 2cos\frac{k\pi}{4}\right] = A^3 - 2A.$$

3. Balancing-like Polynomials: Derivative Sequences

In this section, we analyze the sequences generated by successive differentiation of the balancing-like polynomials with respect to the variable y. These derivative sequences reveal additional algebraic and combinatorial structures that parallel the properties of the original polynomials. By differentiating the recurrence relations and using previously established identities, we derive several formulas describing the behavior, recurrence, and asymptotic nature of these derivative sequences.

Proposition 2. *Let* $(m \geq 1)$*. The balancing-like polynomial* $(x_m(y))$ *satisfies the second-order differential equation*

$$(4 - A^2y^2)x_m''(y) - 3A^2yx_m'(y) + A^2(m^2 - 1)x_m(y) = 0.$$

Proof. By Proposition 1, the balancing-like polynomials can be expressed in terms of the Chebyshev polynomials of the second kind as

$$x_m(y) = U_{m-1}\left(\frac{Ay}{2}\right).$$

It is well known that the Chebyshev polynomials of the second kind satisfy the differential equation

$$(1-z^2)U_n''(z) - 3zU_n'(z) + n(n+2)U_n(z) = 0.$$

Setting $(n = m-1)$ and $(z = Ay/2)$, we obtain

$$\left(1-\frac{A^2y^2}{4}\right)U_{m-1}''\left(\frac{Ay}{2}\right) - \frac{3Ay}{2}U_{m-1}'\left(\frac{Ay}{2}\right) + (m^2-1)U_{m-1}\left(\frac{Ay}{2}\right) = 0.$$

On the other hand, differentiating the relation $(x_m(y) = U_{m-1}(Ay/2))$ with respect to (y) yields

$$x_m'(y) = \frac{A}{2}U_{m-1}'\left(\frac{Ay}{2}\right)$$

and

$$x_m''(y) = \frac{A^2}{4}U_{m-1}''\left(\frac{Ay}{2}\right).$$

Substituting these expressions into the preceding differential equation and multiplying by (A^2), we obtain

$$(4-A^2y^2)x_m''(y) - 3A^2yx_m'(y) + A^2(m^2-1)x_m(y) = 0.$$

Thus, the balancing-like polynomials satisfy the stated second-order differential equation. □

Upon differentiating the initial balancing-like polynomials with respect to y, the following results are obtained:

$$x_0'(y) = 0, x_1'(y) = 0, x_2'(y) = A, x_3'(y) = 2A^2y, x_4'(y) = 3A^3y^2 - 2A, \cdots$$

Applying differentiation a second time gives:

$$x_0''(y) = 0, x_1''(y) = 0, x_2''(y) = 0, x_3''(y) = 2A^2, x_4''(y) = 6A^3y, x_5''(y) = 12A^4y^2 - 6A^2, \cdots$$

Evaluation of these derivatives at integer arguments produces distinct numerical sequences, many of which correspond to entries in the On-Line Encyclopedia of Integer Sequences (OEIS). These evaluations provide a bridge between symbolic polynomial formulations and concrete numerical patterns. For the first derivative we obtain, for instance:

$$\{x_m'(1)\} = \{0, 0, A, 2A^2, 3A^3 - 2A, 4A^4 - 6A^2, 5A^5 - 12A^3 + 3A, \cdots\} \quad (25)$$

$$\{x_m'(2)\} = \{0, 0, A, 4A^2, 12A^3 - 2A, 32A^4 - 12A^2, 80A^5 - 48A^3 + 3A, \cdots\} \quad (26)$$

$$\{x_m'(3)\} = \{0, 0, A, 6A^2, 27A^3 - 2A, 108A^4 - 18A^2, \cdots\} \quad (27)$$

$$\{x_m'(4)\} = \{0, 0, A, 8A^2, 48A^3 - 2A, 256A^4 - 24A^2, \cdots\} \quad (28)$$

For $A = 1$, the nonzero terms obtained from the specialization in (26), up to an index shift, coincide with OEIS A000292 [25]. Similarly, the specialization $A = 1$ in (27) is related, up to an index shift, to OEIS A001871.

For the second derivative we obtain:

$$\{x_m''(1)\} = \{0, 0, 0, 2A^2, 6A^3, 12A^4 - 6A^2, \cdots\}$$
$$\{x_m''(2)\} = \{0, 0, 0, 2A^2, 12A^3, 48A^4 - 6A^2, \cdots\}$$
$$\{x_m''(3)\} = \{0, 0, 0, 2A^2, 18A^3, 108A^4 - 6A^2, \cdots\}$$
$$\{x_m''(4)\} = \{0, 0, 0, 2A^2, 24A^3, 192A^4 - 6A^2, \cdots\}$$

Theorem 12. *Let $y \neq \pm\frac{2}{A}$. Assuming that $x_m'(y)$ denotes the derivative of the mth balancing-like polynomial, we have*

$$x_m'(y) = \frac{Amx_{m+1}(y) - A^2yx_m(y) - Amx_{m-1}(y)}{A^2y^2 - 4}. \tag{29}$$

Proof. Since $\beta_1(y) = \frac{Ay+\sqrt{A^2y^2-4}}{2}$, its derivative $\beta_1'(y) = \frac{A^2y+A\sqrt{A^2y^2-4}}{2\sqrt{A^2y^2-4}}$. On the other hand we get

$$\frac{A\beta_1(y)}{\beta_1(y) - \beta_2(y)} = \beta_1'(y)$$

and

$$1 + \beta_1^{-2}(y) = \frac{Ay}{\beta_1(y)}.$$

Using Binet's formula for balancing-like polynomials,

$$x_m'(y) = \frac{m\beta_1'(y)\left(\beta_1^{m-1}(y) + \beta_2^{m+1}(y)\right)(\beta_1(y) - \beta_2(y)) - (\beta_1^m(y) - \beta_2^m(y))\beta_1'(y)(1 + \beta_2^2(y))}{(\beta_1(y) - \beta_2(y))^2}$$
$$= Am\frac{\beta_1^m(y) + \beta_2^m(y)}{(\beta_1(y) - \beta_2(y))^2} - \frac{A^2yx_m(y)}{(\beta_1(y) - \beta_2(y))^2}.$$

On the other hand,

$$x_{m+1}(y) - x_{m-1}(y) = \frac{\beta_1^{m+1}(y) - \beta_2^{m+1}(y) - \beta_1^{m-1}(y) + \beta_2^{m-1}(y)}{\beta_1(y) - \beta_2(y)}$$
$$= \frac{(\beta_1(y) - \beta_2(y))(\beta_1^m(y) + \beta_2^m(y))}{\beta_1(y) - \beta_2(y)}$$
$$= \beta_1^m(y) + \beta_2^m(y).$$

Thus we obtain

$$x_m'(y) = \frac{Amx_{m+1}(y) - A^2yx_m(y) - Amx_{m-1}(y)}{A^2y^2 - 4} \quad \square$$

Using the recurrence relation for the balancing-like polynomials, Equation (29) can therefore be expressed in the following form:

$$x_m'(y) = \frac{A^2(m-1)yx_m(y) - 2Amx_{m-1}(y)}{A^2y^2 - 4}$$

for $y \neq \pm\frac{2}{A}$.

Proposition 3. *Let*

$$G(t, y) = \sum_{m=0}^{\infty} x_m(y)t^m = \frac{t}{1 - Ayt + t^2}$$

be the generating function of the balancing-like polynomials. Then the generating function of the first derivative sequence is

$$\sum_{m=0}^{\infty} x'_m(y)t^m = \frac{At^2}{(1-Ayt+t^2)^2}.$$

Moreover,

$$\frac{\partial G(t,y)}{\partial y} = A\,G(t,y)^2.$$

Proof. Differentiating

$$G(t,y) = \frac{t}{1-Ayt+t^2}$$

with respect to y, we obtain

$$\frac{\partial G(t,y)}{\partial y} = \frac{At^2}{(1-Ayt+t^2)^2}.$$

On the other hand,

$$G(t,y)^2 = \frac{t^2}{(1-Ayt+t^2)^2}.$$

Hence,

$$\frac{\partial G(t,y)}{\partial y} = A\,G(t,y)^2.$$

Since

$$\frac{\partial G(t,y)}{\partial y} = \sum_{m=0}^{\infty} x'_m(y)t^m,$$

the desired result follows. □

Lemma 5. *For $m \geq 1$,*

$$x'_{m+1}(y) = \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m-i}{i} A^{m-2i}(m-2i)y^{m-2i-1}$$

with $x'_1(y) = 0$.

Proof. Differentiate the closed-form expansion of Lemma 2 term by term with respect to y-legitimate since it is a finite sum of monomials: $x_{m+1}(y) = \sum_{i=0}^{\lfloor m/2 \rfloor}(-1)^i C(m-i,i)A^{m-2i}y^{m-2i} \to x'_{m+1}(y) = \sum_{i=0}^{\lfloor m/2 \rfloor}(-1)^i C(m-i,i)A^{m-2i}(m-2i)y^{m-2i-1}$. It remains to check that the upper limit of summation reduces from $\lfloor m/2 \rfloor$ to $\lfloor (m-1)/2 \rfloor$. If m is odd, say $m = 2p+1$, then $\lfloor m/2 \rfloor = \lfloor (m-1)/2 \rfloor = p$ and no term is lost. If m is even, say $m = 2p$, then $\lfloor m/2 \rfloor = p$, but the term $i = p$ has exponent $m-2i = 0$, and the factor $(m-2i)$ in its derivative vanishes; hence this term contributes 0 and may be dropped, leaving exactly the terms$i = 0, \cdots, p-1 = \lfloor (m-1)/2 \rfloor$. In both cases the surviving terms are indexed by $i = 0, \cdots, \lfloor (m-1)/2 \rfloor$, which is the stated formula. Finally $x'_1(y) = 0$ since $x_1(y) = 1$ is constant, consistent with the empty sum obtained when $m = 0$. □

This closed form generalizes Lemma 2 to the derivative case and permits explicit computation of higher-order terms without recursive evaluation. Similarly, for the second derivative ($m \geq 2$),

$$x''_{m+1}(y) = \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} (-1)^i \binom{m-i}{i} A^{m-2i}(m-2i)(m-2i-1)y^{m-2i-2}$$

with $x_1''(y) = 0$.

The next result demonstrates that derivative sequences of balancing-like polynomials can be generated via a self-convolution of the original sequence, underscoring the deep combinatorial linkage between $x_m(y)$ and $x_m'(y)$.

Theorem 13. *For $m \geq 1$,*

$$x_m'(y) = \sum_{i=1}^{m-1} Ax_i(y)x_{m-i}(y) \tag{30}$$

with $x_1'(y) = 0$.

Proof. This theorem is established through the principle of mathematical induction. Since $x_2'(y) = Ax_1(y)x_1(y) = A$, the theorem holds for $m = 2$. Suppose that the assertion is valid for all polynomials $x_k'(y)$ corresponding to indices $k \leq m$. Differentiation of the recurrence relation for the balancing-like polynomials with respect to y together with the induction hypothesis, yields

$$\begin{aligned}
x_{m+1}'(y) &= Ax_m(y) + Ayx_m'(y) - x_{m-1}'(y) \\
&= Ax_m(y) + Ay\sum_{i=1}^{m-1} Ax_i(y)x_{m-i}(y) - \sum_{i=1}^{m-2} Ax_i(y)x_{m-1-i}(y) \\
&= Ax_m(y) + A^2yx_{m-1}(y) + \sum_{i=1}^{m-2} Ax_i(y)[Ayx_{m-i}(y) - x_{m-1-i}(y)] \\
&= Ax_m(y) + A^2yx_{m-1}(y) + \sum_{i=1}^{m-2} Ax_i(y)x_{m-i+1}(y) \\
&= A\sum_{i=1}^{m} x_i(y)x_{m+1-i}(y).
\end{aligned}$$

This concludes the proof of the theorem. □

Combining Equations (29) and (30) yields the following corollary.

Corollary 10. *For $m > 1$ and $y \neq \pm\frac{2}{A}$,*

$$\sum_{i=1}^{m-1} x_i(y)x_{m-i}(y) = \frac{A(m-1)yx_m(y) - 2mx_{m-1}(y)}{A^2y^2 - 4}.$$

Theorem 14. *For $m \geq 1$,*

$$x_{m+1}'(y) = \sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor} A(m-2i)x_{m-2i}(y). \tag{31}$$

Proof. The case $m = 1$ is trivial. Assume that the equation holds for all $k \leq m$. Suppose that

$$\sum_{i=0}^{\lfloor\frac{m-2}{2}\rfloor} A(m-1-2i)x_{m-1-2i}(y) = x_m'(y)$$

and

$$\sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor} A(m-2i)x_{m-2i}(y) = x_{m+1}'(y).$$

Differentiating the recurrence and using $Ay\, x_k(y) = x_{k+1}(y) + x_{k-1}(y)$ (note that the term with $i = \frac{m-1}{2}$ in the second sum below contains $x_0(y) = 0$ when m is odd), we obtain

$$\begin{aligned} x'_{m+2}(y) &= Ax_{m+1}(y) + Ayx'_{m+1}(y) - x'_m(y) \\ &= Ax_{m+1}(y) + Ay \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} A(m-2i)x_{m-2i} - \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} A(m-1-2i)x_{m-1-2i} \\ &= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} A(m+1-2i)x_{m+1-2i}. \quad \square \end{aligned}$$

Theorem 15. *For $m \in \mathbb{Z}^+$,*

$$x_m(y) = \frac{x'_{m+1}(y) - x'_{m-1}(y)}{Am} \tag{32}$$

Proof.

$$\begin{aligned} x_{m+1}(y) - x_{m-1}(y) &= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m-i}{i} (Ay)^{m-2i} - \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} (-1)^i \binom{m-2-i}{i} (Ay)^{m-2i-2} \\ &= (Ay)^m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \left[\binom{m-i}{i} + \binom{m-1-i}{i-1} \right] (Ay)^{m-2i} \\ &= (Ay)^m + m \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m-1-i}{i-1} \frac{1}{i} (Ay)^{m-2i}. \end{aligned}$$

Differentiating the above equation with respect to y we obtain

$$\begin{aligned} x'_{m+1}(y) - x'_{m-1}(y) &= Am(Ay)^{m-1} + Am \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m-1-i}{i-1} \frac{m-2i}{i} (Ay)^{m-2i-1} \\ &= Am \left[(Ay)^{m-1} + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m-1-i}{i-1} \frac{m-2i}{i} (Ay)^{m-2i-1} \right]. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{x'_{m+1}(y) - x'_{m-1}(y)}{Am} &= (Ay)^{m-1} + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m-1-i}{i-1} \frac{m-2i}{i} (Ay)^{m-1-2i} \\ &= \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m-1-i}{i} (Ay)^{m-1-2i} \\ &= x_m(y). \quad \square \end{aligned}$$

This result emphasizes the reversible correspondence between the polynomial family and its derivatives.

Proposition 4. *For every integer $r \geq 0$, the generating function of the rth derivative sequence is*

$$\sum_{m=0}^{\infty} x_m^{(r)}(y) t^m = \frac{r! A^r t^{r+1}}{(1 - Ayt + t^2)^{r+1}}.$$

Equivalently,

$$\frac{\partial^r G(t,y)}{\partial y^r} = r! A^r G(t,y)^{r+1}.$$

Proof. Since

$$G(t,y) = \frac{t}{1 - Ayt + t^2},$$

successive differentiation with respect to y gives

$$\frac{\partial^r G(t,y)}{\partial y^r} = \frac{r! A^r t^{r+1}}{(1 - Ayt + t^2)^{r+1}}.$$

Furthermore,

$$G(t,y)^{r+1} = \frac{t^{r+1}}{(1 - Ayt + t^2)^{r+1}},$$

and hence

$$\frac{\partial^r G(t,y)}{\partial y^r} = r! A^r G(t,y)^{r+1}. \quad \square$$

Theorem 16. *Let $C_n^{(\lambda)}$ denote the Gegenbauer (ultraspherical) polynomials. Then, for all $r \geq 0$ and $m \geq r+1$,*

$$x_m^{(r)}(y) = r!\, A^r\, C_{m-r-1}^{(r+1)}\Big(\frac{Ay}{2}\Big),$$

and $x_m^{(r)}(y) = 0$ for $m \leq r$.

Proof. The Gegenbauer polynomials are defined by the generating function $\sum_{n\geq 0} C_n^{(\lambda)}(x) t^n = (1 - 2xt + t^2)^{-\lambda}$. Taking $x = Ay/2$ and $\lambda = r+1$ and comparing with Proposition 4,

$$\sum_{m\geq 0} x_m^{(r)}(y) t^m = \frac{r! A^r t^{r+1}}{(1 - Ayt + t^2)^{r+1}} = r! A^r \sum_{n\geq 0} C_n^{(r+1)}\Big(\frac{Ay}{2}\Big) t^{n+r+1}.$$

Comparing the coefficients of t^m gives the assertion with $n = m - r - 1$. $\square$

Lemma 6. *Let $y > \frac{2}{A}$ and $\beta_1(y) = \frac{Ay + \sqrt{A^2y^2 - 4}}{2}$; then,*

$$\lim_{m\to\infty} \frac{x_{m+1}(y)}{x_m(y)} = \lim_{m\to\infty} \frac{x'_{m+1}(y)}{x'_m(y)} = \lim_{m\to\infty} \frac{x''_{m+1}(y)}{x''_m(y)} = \cdots = \beta_1(y).$$

Proof. By Theorem 16, $x_m^{(r)}(y) = r! A^r C_{m-r-1}^{(r+1)}(Ay/2)$ for $m \geq r+1$. Put $\xi = Ay/2 > 1$ and write $\xi = \cosh\tau$ with a unique $\tau > 0$, so that $e^\tau = \xi + \sqrt{\xi^2 - 1} = \beta_1(y)$. The generating function of the Gegenbauer polynomials factors as

$$\sum_{n\geq 0} C_n^{(\lambda)}(\xi) z^n = (1 - 2\xi z + z^2)^{-\lambda} = (1 - ze^\tau)^{-\lambda}(1 - ze^{-\tau})^{-\lambda}.$$

Both factors have positive Taylor coefficients, so $C_n^{(\lambda)}(\xi) > 0$ for all n. For $\lambda = r+1$ the dominant singularity is a pole of order $r+1$ at $z = e^{-\tau}$, where the second factor is analytic and equals $(1 - e^{-2\tau})^{-(r+1)} \neq 0$. Extracting coefficients from this pole gives

$$C_n^{(r+1)}(\xi) = \frac{(1 - e^{-2\tau})^{-(r+1)}}{r!}\, n^r e^{n\tau}\big(1 + o(1)\big) = K_r\, n^r \beta_1(y)^n \big(1 + o(1)\big), \qquad n \to \infty,$$

with a constant $K_r > 0$ independent of n. Hence, with $n = m - r - 1$,

$$\frac{x_{m+1}^{(r)}(y)}{x_m^{(r)}(y)} = \frac{C_{m-r}^{(r+1)}(\xi)}{C_{m-r-1}^{(r+1)}(\xi)} = \Big(\frac{m-r}{m-r-1}\Big)^r \beta_1(y)\big(1 + o(1)\big) \longrightarrow \beta_1(y)$$

as $m \to \infty$, for every fixed $r \geq 0$. □

Thus, each derivative sequence shares the same asymptotic growth factor $\beta_1(y)$ as the original polynomial sequence, indicating consistent exponential-type behavior under differentiation.

Corollary 11. *For all $r \geq 0$ and $m \geq r+1$,*

$$x_m^{(r)}\Big(\frac{2}{A}\Big) = r!\,A^r\binom{m+r}{2r+1}, \qquad x_m^{(r)}\Big(-\frac{2}{A}\Big) = (-1)^{m-r-1}r!\,A^r\binom{m+r}{2r+1}.$$

In particular

$$x_m\Big(\frac{2}{A}\Big) = m, \qquad x_m'\Big(\frac{2}{A}\Big) = A\,\frac{m(m^2-1)}{6}, \qquad x_m''\Big(\frac{2}{A}\Big) = A^2\,\frac{m(m^2-1)(m^2-4)}{60}.$$

Proof. By Theorem 16 and the classical value $C_n^{(\lambda)}(\pm 1) = (\pm 1)^n\binom{n+2\lambda-1}{n}$ with $\lambda = r+1$, $n = m-r-1$. □

Theorem 17. *For $m \geq r+1$,*

$$x_m^{(r)}(y) = A^r \sum_{k=0}^{\lfloor (m-r-1)/2 \rfloor} (-1)^k \frac{(m-1-k)!}{k!\,(m-r-1-2k)!}\,(Ay)^{\,m-r-1-2k}.$$

Proof. Insert the explicit formula $C_n^{(\lambda)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor}(-1)^k\frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)\,k!\,(n-2k)!}(2x)^{n-2k}$ into Theorem 16 with $\lambda = r+1$, $n = m-r-1$, $x = Ay/2$, and simplify $\Gamma(m-k)/\Gamma(r+1)\cdot r! = (m-1-k)!$. □

Consequently, the recurrence relation governing the r th order derivatives of the balancing-like polynomials can be formulated as follows:

Theorem 18 (Higher-order derivative recurrence)**.**

$$x_{m+1}^{(r)}(y) = \begin{cases} 0 & m < r, \\ A^r r! & m = r, \\ \frac{1}{m-r}\Big[Amy\,x_m^{(r)}(y) - (m+r)x_{m-1}^{(r)}(y)\Big] & m > r. \end{cases}$$

Proof. Put $F_r(t) = \sum_{m\geq 0} x_m^{(r)}(y)t^m = r!A^r t^{r+1}D^{-(r+1)}$ with $D = 1 - Ayt + t^2$, by Proposition 4. Logarithmic differentiation in t gives

$$tD\,\partial_t F_r = (r+1)\big(D + Ayt - 2t^2\big)F_r = (r+1)(1-t^2)F_r.$$

Writing $a_m = x_m^{(r)}(y)$ and comparing the coefficients of t^{m+1} on both sides yields

$$(m+1)a_{m+1} - Ay\,m\,a_m + (m-1)a_{m-1} = (r+1)a_{m+1} - (r+1)a_{m-1},$$

that is $(m-r)\,a_{m+1} = Ay\,m\,a_m - (m+r)\,a_{m-1}$, which is the stated recurrence for $m > r$. For $m < r$ we have $\deg x_{m+1} = m < r$, so $a_{m+1} = 0$; and for $m = r$ the leading term $(Ay)^r$ of $x_{r+1}(y)$ gives $a_{r+1} = r!A^r$. □

In the following result, greatest common divisors in $\mathbb{Z}[A][y]$ are understood up to multiplication by a nonzero unit. We choose the normalization for which the greatest common divisor is identified with the corresponding balancing-like polynomial.

Theorem 19. *In $\mathbb{Z}[A][y]$ the sequence $(x_m(y))_{m\geq 0}$ is a strong divisibility sequence; that is, for all $m, n \geq 1$,*

$$\gcd\left(x_m(y), x_n(y)\right) = x_{\gcd(m,n)}(y).$$

In particular $x_m(y) \mid x_n(y)$ if and only if $m \mid n$.

Proof. By the Cassini identity, $x_m^2 - x_{m+1}x_{m-1} = 1$; hence, $\gcd(x_m, x_{m\pm 1}) = 1$ for every m. Since $\mathbb{Z}[A][y]$ is a unique factorization domain, we may argue as in $\mathbb{Z}$. By Lemma 4, $x_{m+s} = x_m x_{s+1} - x_{m-1}x_s$, so any common divisor of x_m and x_{m+s} divides $x_{m-1}x_s$ and, being coprime to x_{m-1}, divides x_s. Conversely, by the same identity any common divisor of x_m and x_s divides x_{m+s}. This gives $\gcd(x_m, x_{m+s}) = \gcd(x_m, x_s)$, and the Euclidean algorithm on the indices, which terminates with $\gcd(x_d, x_0) = x_d$ because $x_0 = 0$, yields the assertion. Finally, if $x_m \mid x_n$, then $x_{\gcd(m,n)} = \pm x_m$, and comparing degrees in y ($\deg x_k = k - 1$) gives $\gcd(m, n) = m$, i.e., $m \mid n$. □

Proposition 5 (Orthogonality)**.** *Assume $A > 0$. For $m, n \geq 1$,*

$$\int_{-2/A}^{2/A} x_m(y)x_n(y)\sqrt{4 - A^2y^2}\,dy = \frac{2\pi}{A}\delta_{mn}.$$

Moreover, for every fixed integer $r \geq 0$, the derivative family $\{x_m^{(r)}(y)\}_{m\geq r+1}$ is orthogonal on $[-2/A, 2/A]$ with respect to the weight

$$w_r(y) = \left(4 - A^2y^2\right)^{r+\frac{1}{2}}.$$

Proof. By Proposition 1,

$$x_m(y) = U_{m-1}\left(\frac{Ay}{2}\right).$$

Setting

$$z = \frac{Ay}{2}, \qquad dy = \frac{2}{A}\,dz,$$

gives

$$\sqrt{4 - A^2y^2} = 2\sqrt{1 - z^2}.$$

Therefore,

$$\int_{-2/A}^{2/A} x_m(y)x_n(y)\sqrt{4 - A^2y^2}\,dy = \frac{4}{A}\int_{-1}^{1} U_{m-1}(z)U_{n-1}(z)\sqrt{1 - z^2}\,dz.$$

Using the classical orthogonality relation

$$\int_{-1}^{1} U_p(z)U_q(z)\sqrt{1 - z^2}\,dz = \frac{\pi}{2}\delta_{pq},$$

the result follows. □

For the higher-order derivative family, Theorem 16 gives

$$x_m^{(r)}(y) = r!A^r C_{m-r-1}^{(r+1)}\left(\frac{Ay}{2}\right).$$

Using again the substitution

$$z = \frac{Ay}{2},$$

the weight

$$\left(4 - A^2y^2\right)^{r+\frac{1}{2}}$$

is transformed, up to a positive constant factor, into

$$(1 - z^2)^{r+\frac{1}{2}}.$$

The classical Gegenbauer orthogonality relation

$$\int_{-1}^{1} C_p^{(r+1)}(z) C_q^{(r+1)}(z)(1 - z^2)^{r+\frac{1}{2}}\, dz = 0, \qquad p \neq q,$$

therefore proves the asserted orthogonality of $\{x_m^{(r)}(y)\}_{m \geq r+1}$.

Proposition 6 (Interlacing). *Assume $A > 0$. Let*

$$\xi_k^{(m)} = \frac{2}{A} \cos\left(\frac{(m-k)\pi}{m}\right), \qquad k = 1, 2, \ldots, m-1,$$

denote the zeros of $x_m(y)$ arranged in increasing order. Then the zeros of $x_m(y)$ and $x_{m+1}(y)$ strictly interlace:

$$\xi_1^{(m+1)} < \xi_1^{(m)} < \xi_2^{(m+1)} < \xi_2^{(m)} < \cdots < \xi_{m-1}^{(m)} < \xi_m^{(m+1)}.$$

Proof. $\xi_k^{(m)} = \frac{2}{A} \cos\theta_k^{(m)}$ with $\theta_k^{(m)} = \pi\left(1 - \frac{k}{m}\right) \in (0, \pi)$; since $\theta_k^{(m)}$ decreases in k, the $\xi_k^{(m)}$ are indeed increasing. As the cosine is strictly decreasing on $(0, \pi)$ and $A > 0$, the inequalities $\xi_k^{(m+1)} < \xi_k^{(m)} < \xi_{k+1}^{(m+1)}$, $1 \leq k \leq m-1$, are equivalent to

$$\frac{k}{m+1} < \frac{k}{m} < \frac{k+1}{m+1}.$$

The left inequality holds because $k \geq 1$, and the right one is equivalent to $k(m+1) < m(k+1)$, i.e., $k < m$. This proves the strict interlacing. □

Proposition 7. *For every $m \geq 1$, $x_m(-y) = (-1)^{m-1} x_m(y)$; equivalently, $x_m(y)$ is an even polynomial when m is odd and an odd polynomial when m is even. Consequently the zero set of x_m is symmetric about the origin, and $x_m(0) = 0$ for even m, $x_m(0) = (-1)^{(m-1)/2}$ for odd m.*

Proof. We use strong induction on m, based on the recurrence $x_{m+1}(y) = Ayx_m(y) - x_{m-1}(y)$. Base cases: $x_1(-y) = 1 = (-1)^0 x_1(y)$, and $x_2(-y) = A(-y) = -Ay = (-1)^1 x_2(y)$; both agree with the claimed formula. Inductive step: suppose $x_k(-y) = (-1)^{k-1} x_k(y)$ holds for $k = m-1$ and $k = m$, for some $m \geq 2$. Then, replacing y by $-y$ in the recurrence, $x_{m+1}(-y) = A(-y)x_m(-y) - x_{m-1}(-y) = -Ay(-1)^{m-1} x_m(y) - (-1)^{m-2} x_{m-1}(y)$. Since $-(-1)^{m-1} = (-1)^m$ and $(-1)^{m-2} = (-1)^m$ (parity is unchanged by subtracting 2), this becomes $x_{m+1}(-y) = (-1)^m Ayx_m(y) - (-1)^m x_{m-1}(y) = (-1)^m [Ayx_m(y) - x_{m-1}(y)] = (-1)^m x_{m+1}(y)$, which is exactly the claimed formula for $m+1$, since $(-1)^{(m+1)-1} = (-1)^m$. By induction the identity holds for every $m \geq 1$. Finally, the values of $x_m(0)$ follow from Lemma 2: for $m = 2j+1$ the only constant term in the expansion of $x_{2j+1}(y)$ is the term $i = j$, which equals $(-1)^j \binom{j}{j} = (-1)^{(m-1)/2}$, while for even m the polynomial x_m is odd, so $x_m(0) = 0$. □

4. Fractional Calculus of Balancing-like Polynomials

In this section, we extend the differential properties of balancing-like polynomials to non-integer order derivatives by utilizing the Caputo fractional derivative operator [26,27], which is widely preferred in mathematical physics and applied analysis due to its physical compatibility with standard initial conditions.

Throughout this section, unless otherwise stated, we assume that $y > 0$ and $\alpha > 0$. This restriction avoids ambiguity arising from fractional powers of y in the Caputo fractional derivative formulas. The parameter A is assumed to be a nonzero real constant. Extensions to complex values of y require a prescribed branch of the corresponding fractional powers and are not considered here.

Definition 1. *Let $\alpha > 0$. The Caputo fractional derivative of order α with lower terminal 0 is defined as follows.*

If $\alpha \notin \mathbb{N}$ and $n = \lceil \alpha \rceil$, so that $n - 1 < \alpha < n$, then

$$ {}^{C}D^{\alpha}_{0+} f(y) = \frac{1}{\Gamma(n-\alpha)} \int_0^y (y-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau. $$

If $\alpha = n \in \mathbb{N}$, then

$$ {}^{C}D^{n}_{0+} f(y) = f^{(n)}(y). $$

Theorem 20. *Let $\alpha > 0$, let $n = \lceil \alpha \rceil$, and assume $y > 0$. For $m \geq n$,*

$$ {}^{C}D^{\alpha}_{0+} x_{m+1}(y) = \sum_{i=0}^{\lfloor \frac{m-n}{2} \rfloor} (-1)^i \binom{m-i}{i} A^{m-2i} \frac{\Gamma(m-2i+1)}{\Gamma(m-2i+1-\alpha)} y^{m-2i-\alpha}. $$

For $m < n$,

$$ {}^{C}D^{\alpha}_{0+} x_{m+1}(y) = 0. $$

Proof. By Lemma 2, the explicit expansion of $x_{m+1}(y)$ is given by

$$ x_{m+1}(y) = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-i}{i} (Ay)^{m-2i}. $$

Applying the linearity of the Caputo derivative operator along with the power rule

$$ {}^{C}D^{\alpha}_{0+}[y^p] = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} y^{p-\alpha}, & p \geq n, \\ 0, & p < n, \end{cases} $$

we differentiate $x_{m+1}(y)$ term by term:

$$ {}^{C}D^{\alpha}_{0+}[x_{m+1}(y)] = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-i}{i} A^{m-2i}\, {}^{C}D^{\alpha}_{0+}[y^{m-2i}]. $$

Since ${}^{C}D^{\alpha}_{y}[y^{m-2i}] = 0$ whenever $m - 2i < n$ (which corresponds to $i > \frac{m-n}{2}$, the upper limit of the summation reduces to $\lfloor \frac{m-n}{2} \rfloor$. Consequently, we arrive at

$$ {}^{C}D^{\alpha}_{0+}[x_{m+1}(y)] = \sum_{i=0}^{\lfloor \frac{m-n}{2} \rfloor} (-1)^i \binom{m-i}{i} A^{m-2i} \frac{\Gamma(m-2i+1)}{\Gamma(m-2i+1-\alpha)} y^{m-2i-\alpha}, $$

which completes the proof. □

Remark 3. *Theorem 20 shows that the Caputo fractional derivative preserves the finite-term structure inherited from the balancing-like polynomial expansion. In contrast with the ordinary derivative hierarchy, however, the fractional derivative introduces non-integer powers of y and Gamma function coefficients. Thus, for $\alpha \notin \mathbb{N}$, the resulting family is no longer polynomial in y, although it retains the finite combinatorial structure of the original balancing-like polynomials.*

Corollary 12. *Let $r \in \mathbb{N}$. Then the Caputo derivative of integer order r coincides with the ordinary rth derivative:*

$$^{C}D^{r}_{0+}x_m(y) = x^{(r)}_m(y).$$

Consequently, for $m \geq r+1$,

$$^{C}D^{r}_{0+}x_m(y) = r!A^r C^{(r+1)}_{m-r-1}\left(\frac{Ay}{2}\right),$$

where $C^{(\lambda)}_n$ denotes the Gegenbauer polynomial.

Proof. The first assertion is the case $\alpha = r \in \mathbb{N}$ of Definition 1; the second then follows from Theorem 16. □

Proposition 8. *Let $\alpha > 0$ and let $G(t,y) = \frac{t}{1-Ayt+t^2}$ be the ordinary generating function of the balancing-like polynomial sequence. The ordinary generating function for the sequence of Caputo fractional derivatives $\{^{C}D^{\alpha}_{y}[x_m(y)]\}_{m\geq 0}$ is given by*

$$\sum_{m=0}^{\infty} {}^{C}D^{\alpha}_{0+}[x_m(y)]t^m = {}^{C}D^{\alpha}_{0+}[G(t,y)] = \sum_{k=\lceil\alpha\rceil}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} \frac{A^k t^{k+1}}{(1+t^2)^{k+1}} y^{k-\alpha}.$$

Proof. Recall that the ordinary generating function $G(t,y)$ can be expanded into a series with respect to y as

$$G(t,y) = \frac{t}{1+t^2}\left(1 - \frac{Ayt}{1+t^2}\right)^{-1} = \sum_{k=0}^{\infty} \frac{A^k t^{k+1}}{(1+t^2)^{k+1}} y^k.$$

Applying the Caputo fractional derivative operator $^{C}D^{\alpha}_{y}$ term-wise with respect to y yields

$$^{C}D^{\alpha}_{0+}[G(t,y)] = \sum_{k=0}^{\infty} \frac{A^k t^{k+1}}{(1+t^2)^{k+1}} {}^{C}D^{\alpha}_{0+}[y^k].$$

By the definition of the Caputo derivative, $^{C}D^{\alpha}_{y}[y^k] = 0$ for $k < \lceil\alpha\rceil$, and for $k \geq \lceil\alpha\rceil$ we have

$$^{C}D^{\alpha}_{0+}[y^k] = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} y^{k-\alpha}.$$

Substituting this into the summation shifts the starting index to $k = \lceil\alpha\rceil$, which completes the proof. □

Remark 4. *When $\alpha = 1$ ($\lceil\alpha\rceil = 1$), noting that $\frac{\Gamma(k+1)}{\Gamma(k)} = k$, the fractional generating function in Proposition 8 reduces to*

$$^{C}D^{1}_{y}[G(t,y)] = \sum_{k=1}^{\infty} k \frac{A^k t^{k+1}}{(1+t^2)^{k+1}} y^{k-1} = \frac{At^2}{(1+t^2)^2} \sum_{k=1}^{\infty} k\left(\frac{Ayt}{1+t^2}\right)^{k-1}.$$

Utilizing the standard binomial identity $\sum_{k=1}^{\infty} kz^{k-1} = (1-z)^{-2}$ *for* $z = \frac{Ayt}{1+t^2}$, *we obtain*

$$^{C}D_y^1[G(t,y)] = \frac{At^2}{(1+t^2)^2} \frac{1}{\left(1-\frac{Ayt}{1+t^2}\right)^2} = \frac{At^2}{(1-Ayt+t^2)^2},$$

which recovers the first-derivative generating function established in Proposition 3.

5. Conclusions

In this paper, we have developed a unified algebraic, differential, and fractional framework for balancing-like polynomials. The identification

$$x_m(y) = U_{m-1}\left(\frac{Ay}{2}\right)$$

places this polynomial family naturally within the theory of Chebyshev polynomials of the second kind and explains several of its classical algebraic properties, including Cassini-, Catalan-, D'Ocagne-, and Vajda-type identities, matrix representations, factorization formulas, and the structure of its zeros.

The main contribution of the paper is the systematic study of the derivative sequences associated with balancing-like polynomials. Explicit formulas, generating functions, self-convolution identities, and recurrence relations have been obtained. Moreover, using the scaled Chebyshev representation and the classical Chebyshev–Gegenbauer differentiation relation, the rth derivative sequence admits the unified representation

$$x_m^{(r)}(y) = r!A^r C_{m-r-1}^{(r+1)}\left(\frac{Ay}{2}\right).$$

This correspondence provides a unified description of the entire higher-order differential hierarchy and yields, as consequences, orthogonality, recurrence, and asymptotic properties of the derivative sequences.

The differential theory has also been extended to non-integer orders using the Caputo fractional derivative. An explicit formula for $^{C}D_{0+}^{\alpha} x_m(y)$ has been derived, together with a generating-function representation. The ordinary higher-order derivatives are recovered naturally when the fractional order becomes an integer, showing that the fractional construction is consistent with the differential hierarchy developed earlier.

The results therefore establish a direct structural chain between balancing-like polynomials, Chebyshev polynomials, Gegenbauer polynomials, and fractional differential operators. Further research may investigate analogous fractional structures for other balancing-type polynomial families, differential equations satisfied by the fractional extensions, and possible applications of these families in approximation theory and fractional dynamical systems.

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Abbreviations

The following abbreviations are used in this manuscript:

OEIS The On-Line Encyclopedia of Integer Sequences

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