Asian-European Journal of Mathematics
Vol. 14, No. 6 (2021) 2150101 (10 pages)
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DOI: 10.1142/S1793557121501011

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On a new family of gauss k-Lucas numbers and their polynomials

Engin Özkan

Department of Mathematics, Faculty of Arts and Sciences
Erzincan Binali Yıldırım University, Erzincan, Turkey
eozkanmath@gmail.com
eozkan@erzincan.edu.tr

Merve Taştan

Graduate School of Natural and Applied Sciences,
Erzincan Binali Yıldırım University, Erzincan, Turkey
mervetastan24@gmail.com

Communicated by I. Inam
Received April 9, 2020
Accepted May 22, 2020
Published September 5, 2020

In this paper, we define a new family of Gauss k-Lucas numbers. We give the relationships between the family of the Gauss k-Lucas numbers and the known Gauss Lucas numbers. We also define the generalized polynomials for these numbers. We obtain some interesting properties of the polynomials. We also give the relationships between the generalized Gauss k-Lucas polynomials and the known Gauss Lucas polynomials. Furthermore, we find new generalizations of these families and the polynomials in matrix representation. Then we prove Cassini's identities for the families and their polynomials.

Keywords: Lucas numbers; Lucas polynomials; Gauss Lucas numbers; Gauss Lucas polynomials; Cassini's identity.

AMS Subject Classification: 11B39, 11B83, 11C08

1. Introduction

Fibonacci numbers have exciting properties and many applications in many branches of mathematics [4, 6, 8, 10, 13]. In [5], Mikkawy and Sogabe gave a new family of k-Fibonacci numbers. In [9], Özkan *et al.* defined a new family of k-Lucas numbers and give some properties the family of these numbers. There are some works on polynomials of the families of k-Fibonacci numbers and k-Lucas numbers [7, 12, 15]. In [3], Falcőn and Plaza defined general k-Fibonacci numbers and showed that properties of these numbers were related with elementary matrix

algebra. In [2], Bolat and Kőse found some important properties of k-Fibonacci number. Finally, in [14], Taş presented k-Gauss Fibonacci numbers and gave some properties related to these numbers.

In Sec. 2, we recall some known definitions and properties. We organize Sec. 3 into two subsections. In Sec. 3.1, we define a new family of Gauss k-Lucas numbers. We obtain some exciting properties of the family. We give the relationships between the family of the Gauss k-Lucas numbers and the known Gauss Lucas numbers. Moreover, we find the new generalization of these family in matrix representation. Then we prove Cassini's identities for the family.

In Sec. 3.2, we define the generalized polynomials for these numbers. We also obtain some interesting properties of the polynomials. We also give the relationships between the generalized Gauss k-Lucas polynomials and the known Gauss Lucas polynomials. Furthermore, we find the new generalizations of these polynomials in matrix representation. Then we prove Cassini's identities for the polynomials.

2. Material and Methods

In this section, we introduce the Gauss Fibonacci numbers GF_n, the Gauss Lucas numbers, the Gauss Fibonacci polynomials $GF_n(x)$, the Gauss Lucas polynomials $GL_n(x)$, the Gauss Lucas polynomial matrix $gl_n(x)$, k-Fibonacci numbers $F_n^{(k)}$ and k-Gaussian Fibonacci numbers $GF_n^{(k)}$.

Definition 2.1. The Gaussian Fibonacci numbers $\{GF_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation:

$$GF_{n+1} = GF_n + GF_{n-1}, \quad n \geq 1 \tag{2.1}$$

with initial conditions $GF_0 = i$ and $GF_1 = 1$ [1].

Binet formulas for GF_n are given as follows:

$$GF_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}, \tag{2.2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and$\beta = \frac{1-\sqrt{5}}{2}$ [4].

Definition 2.2. The Gaussian Lucas numbers $\{GL_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation:

$$GL_{n+1} = GL_n + GL_{n-1}, \quad n \geq 1 \tag{2.3}$$

with initial conditions $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$ [1].

Binet formulas for GF_n are given as follows:

$$GL_n = \alpha^n + \beta^n, \tag{2.4}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ [4].

Definition 2.3. The Gaussian Fibonacci polynomials $\{GF_n(x)\}_{n=0}^{\infty}$ are defined in the following recurrence relation:

$$GF_{n+1}(x) = xGF_n(x) + GF_{n-1}(x), \quad n \geq 2 \tag{2.5}$$

with initial conditions $GF_1(x) = 1$ and $GF_2(x) = x + i$.

For $n > 2$, we have

$$GF_n(x) = F_n(x) + iF_{n-1}(x) \tag{2.6}$$

Binet formulas for $GF_n(x)$ are given as follows:

$$GF_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i\frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)}, \tag{2.7}$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$.

The Gauss Fibonacci polynomial matrix $gf_n(x)$ is defined by

$$gf_n(x) = \begin{bmatrix} GF_{n+1}(x) & GF_n(x) \\ GF_n(x) & GF_{n-1}(x) \end{bmatrix}, \quad n \geq 0 \text{ [11].}$$

Definition 2.4. The Gaussian Lucas polynomials $\{GL_n(x)\}_{n=0}^{\infty}$ are defined in the following recurrence relation:

$$GL(x) = xGL_n(x) + GL_{n-1}(x), \quad n \geq 2 \tag{2.8}$$

with initial conditions $GL_1(x) = x + 2i$ and $GF_2(x) = x^2 + 2 + ix$.

For $n > 2$, we have

$$GL_n(x) = L_n(x) + iL_{n-1}(x). \tag{2.9}$$

Binet formulas for $GL_n(x)$ are given as follows:

$$GL_n(x) = \alpha^n + \beta^n + i(\alpha^{n-1} + \beta^{n-1}) \tag{2.10}$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$ [11].

The Gauss Lucas polynomial matrix $gl_n(x)$ is defined by

$$gl_n(x) = \begin{bmatrix} GL_{n+1}(x) & GL_n(x) \\ GL_n(x) & GL_{n-1}(x) \end{bmatrix}, \quad n \geq 0 \text{ [11].}$$

Definition 2.5. Let n and $k \neq 0$ be natural numbers, then there exist unique numbers m and r such that $n = mk + r\,(0 \leq r < k)$. Using these parameters, we define generalized k-Fibonacci numbers $F_n^{(k)}$ by

$$F_n^{(k)} = \frac{1}{(\sqrt{5})^k}(\alpha^{m+1} - \beta^{m+1})^{k-r}(\alpha^{m+2} - \beta^{m+2})^r, \quad n = mk + r \text{ [5].}$$

Definition 2.6. Let n and $k \neq 0$ be natural numbers, then there exist unique numbers m and r such that $n = mk + r (0 \leq r < k)$. Using these parameters, we define generalized k-Gauss Fibonacci numbers $GF_n^{(k)}$ by

$$GF_n^{(k)} := \left[\left(\frac{\sqrt{5}}{5} + \left(\frac{5-\sqrt{5}}{10}\right)i\right)\alpha^m + \left(-\frac{\sqrt{5}}{5} + \left(\frac{5+\sqrt{5}}{10}\right)i\right)\beta^m\right]^{k-r} \times \left[\left(\frac{\sqrt{5}}{5} + \left(\frac{5-\sqrt{5}}{10}\right)i\right)\alpha^{m+1} + \left(-\frac{\sqrt{5}}{5} + \left(\frac{5+\sqrt{5}}{10}\right)i\right)\beta^{m+1}\right]^r, \quad n = mk + r$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ [14].

3. Main Results

3.1. *The generalized* k*-Lucas numbers*

Definition 3.1. There are unique numbers, m and r such that $n = mk + r$ where m, k $(\neq 0)$ natural numbers and $0 \leq r < k$. The generalized Gauss k-Lucas numbers $GL_n^{(k)}$ are defined by

$$GL_n^{(k)} := ((\alpha^m + \beta^m)) + i(\alpha^{m-1} + \beta^{m-1})^{k-r}((\alpha^{m+1} + \beta^{m+1}) + i(\alpha^m + \beta^m))^r, \quad n = mk + r$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Also, we can find the generalized Gauss k-Lucas numbers by the matrix methods. Indeed, it is clear that

$$GL_n^{k-1} gl_n = \begin{bmatrix} GL_{kn+1}^{(k)} & GL_{kn}^{(k)} \\ GL_{kn}^{(k)} & GL_{kn-1}^{(k)} \end{bmatrix}$$

where $gl_n = \left[\begin{smallmatrix} GL_{n+1} & GL_n \\ GL_n & GL_{n-1} \end{smallmatrix}\right]$ and $n > 0$.

Let us give some values for the Gauss k-Lucas numbers in Table 1.

From (2.4) and Definition 3.1, we obtain the following relation between the generalized Gauss k-Lucas and the Lucas numbers.

$$GL_n^{(k)} := (GL_m)^{k-r}(GL_{m+1})^r, \quad n = mk + r \tag{3.1}$$

If $k = 1$ in the last equation, we have that $m = n$ and $r = 0$ so $GL_n^{(1)} = GL_n$.

Throughout this paper, let $km \in \{1, 2, 3, \ldots\}$.

Theorem 3.1.2. *For k and m, we have*

$$[(GL_{m+1})^k - (GL_m)^k] = [GL_{(m+1)k}^{(k)} - GL_{mk}^{(k)}].$$

Table 1. The generalized Gauss k-Lucas numbers $GL_n^{(k)}$ for some k and n.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$GL_0^{(k)}$	$2-i$	$3-4i$	$2-11i$	$-7-24i$	$-38-41i$	$-117-44i$
$GL_1^{(k)}$	$1+2i$	$4+3i$	$11+2i$	$24-7i$	$41-38i$	$117+44i$
$GL_2^{(k)}$	$3+i$	$-3+4i$	$-2+11i$	$7+24i$	$38+41i$	$-44+117i$
$GL_3^{(k)}$	$4+3i$	$1+7i$	$-11-2i$	$-24+7i$	$-41+38i$	$-117-44i$
$GL_4^{(k)}$	$7+4i$	$8+6i$	$-13+9i$	$-7-24i$	$-38-41i$	$44-117i$
$GL_5^{(k)}$	$11+7i$	$9+13i$	$-4+22i$	$-31-17i$	$41-38i$	$117+44i$
$GL_6^{(k)}$	$18+11i$	$7+24i$	$18+26i$	$-48+14i$	$3-79i$	$161-73i$
$GL_7^{(k)}$	$29+18i$	$16+37i$	$8+48i$	$-34+62i$	$-76-82i$	$88-234i$
$GL_8^{(k)}$	$47+28i$	$33+56i$	$-3+79i$	$28+96i$	$-158-6i$	$-146-322i$
$GL_9^{(k)}$	$76+47i$	$49+93i$	$-44+117i$	$-6+158i$	$-164+152i$	$-468-176i$
$GL_{10}^{(k)}$	$123+76i$	$72+154i$	$-47+196i$	$-88+234i$	$-12+316i$	$-644+292i$

Proof.

$$\begin{aligned} GL_{(m+1)k}^{(k)} - GL_{mk}^{(k)} &= [(GL_m)^{k-k}(GL_{m+1})^k] - [(GL_m)^{k-0}(GL_{m+1})^0] \\ &= (GL_{m+1})^k - (GL_m)^k \end{aligned}$$

□

For $k=2,3,4$ and n, we have the following interesting properties between these numbers:

(i) $GL_{2n}^{(2)} = GL_n^2$,
(ii) $GL_{2n+1}^{(2)} = GL_{2n}^{(2)} + GL_{2n-1}^{(2)}$,
(iii) $GL_{3n}^{(3)} = GL_n^3$,
(iv) $GL_{3n+1}^{(3)} = GL_n^2 GL_{n+1}$,
(v) $GL_{3n+2}^{(3)} = GL_n GL_{n+1}^2$,
(vi) $GL_{3n+1}^{(3)} = GL_{3n}^{(3)} + GL_{3n-1}^{(3)}$,
(vii) $GL_{4n}^{(4)} = GL_n^4$,
(viii) $GL_{4n+1}^{(4)} = GL_n^3 GL_{n+1}$,
(ix) $GL_{4n+2}^{(4)} = GL_n^2 GL_{n+1}^2$,
(x) $GL_{4n+3}^{(4)} = GL_n GL_{n+1}^3$,
(xi) $GL_{4n+1}^{(4)} = GL_{4n}^{(4)} + GL_{4n-1}^{(4)}$.

Theorem 3.1.4. *For n, we have the relation*

$$GL_{kn+1}^{(k)} = GL_{kn}^{(k)} + GL_{kn-1}^{(k)}.$$

Proof. By using (3.1),

$$\begin{aligned} GL_{kn}^{(k)} + GL_{kn-1}^{(k)} &= GL_n^k + GL_{n-1}GL_n^{k-1} \\ &= GL_n^{k-1}(GL_n + GL_{n-1}) \\ &= GL_n^{k-1}GL_{n+1} \\ &= GL_{kn+1}^{(k)}. \end{aligned}$$

□

Theorem 3.1.6 (Cassini's identity). *Let $GL_n^{(k)}$ be the generalized Gauss k-Lucas numbers. For $n, k \geq 2$, Cassini's identity for $GL_n^{(k)}$ is as follows*:

$$GL_{kn+t}^{(k)}GL_{kn+t-2}^{(k)} - (GL_{kn+t-1}^{(k)})^2 = \begin{cases} GL_n^{2k-2}(-1)^{n+1}5(2-i), & t = 1, \\ 0, & t \neq 1. \end{cases}$$

Proof. By using (3.1), we get

$$\begin{aligned} &GL_{kn+t}^{(k)}GL_{kn+t-2}^{(k)} - (GL_{kn+t-1}^{(k)})^2 \\ &\quad = (GL_n^{k-1}GL_{n+t})(GL_n^{k-1}GL_{n+t-2}) - (GL_n^{k-1}GL_{n+t-1})^2 \\ &\quad = (GL_n^{k-1})^2(GL_{n+t}GL_{n+t-2} - (GL_{n+t-1})^2) \\ &\quad = GL_n^{2k-2}((GL_{n+t}GL_{n+t-2} - (GL_{n+t-1})^2) \end{aligned}$$

For $t = 1$;

$$\begin{aligned} &= GL_n^{2k-2}((GL_{n+1}GL_{n-1} - (GL_n)^2) \\ &= GL_n^{2k-2}(-1)^{n+1}5(2-i). \end{aligned}$$

For $t \neq 1$, $t = m$;

$$\begin{aligned} &= GL_n^{2k-2}(GL_{n+m}GL_{n+m-2} - (GL_{n+m-1})^2) \\ &= GL_n^{2k-2}(GL_{2n+2m-2}^{(2)} - GL_{2n+2m-2}^{(2)}) \\ &= 0. \end{aligned}$$

□

For n, there are exciting relations between the generalized Gauss k-Lucas numbers and the known Gauss Lucas numbers

$$GL_{nk+t}^{(k)} = GL_n^{k-t}GL_{n+1}^t,$$

$t = 0, 1, 2, \ldots, k-1$.

3.2. *The generalized Gauss* k*-Lucas polynomials*

Definition 3.2.1. There are unique numbers m and r such that $n = mk + r$ where m, k ($\neq 0$) natural numbers and $0 \leq r < k$. The generalized Gauss k-Lucas polynomials $GL_n^{(k)}(x)$ are defined by

$$GL_n^{(k)}(x) := (\alpha^m(x) + \beta^m(x)) + ix(\alpha^{m-1}(x) + \beta^{m-1}(x))^{k-r}((\alpha^{m+1}(x) + \beta^{m+1}(x)) + ix(\alpha^m(x) + \beta^m(x))^r, \quad n = mk + r,$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$.

Also, we can find the generalized Gauss k-Lucas polynomials by the matrix methods. Indeed, it is clear that

$$GL_n^{k-1}(x)gl_n(x) = \begin{bmatrix} GL_{kn+1}^{(k)}(x) & GL_{kn}^{(k)}(x) \\ GL_{kn}^{(k)}(x) & GL_{kn-1}^{(k)}(x) \end{bmatrix}$$

where

$$gl_n(x) = \begin{bmatrix} GL_{n+1}(x) & GL_n(x) \\ GL_n(x) & GL_{n-1}(x) \end{bmatrix}, \quad n > 0.$$

Let us see some value of $GL_n^{(k)}(x)$ for some k and n in Table 2.

Table 2. The generalized Gauss k-Lucas polynomials $GL_n^{(k)}(x)$ for some k and n.

	$k=1$	$k=2$	$k=3$	$k=4$
$GL_0^{(k)}(x)$	2	4	8	16
$GL_1^{(k)}(x)$	$x+2i$	$2x+4i$	$4x+8i$	$8x+16i$
$GL_2^{(k)}(x)$	x^2+2+ix	$x^2-4+4ix$	$2x^2-8+8ix$	$4x^2-16+16ix$
$GL_3^{(k)}(x)$	x^3+3x $+ix^2+2i$	x^3+3ix^2+4i	x^3-12x $+6ix^2-8i$	$2x^3-24x$ $+12ix^2-16i$
$GL_4^{(k)}(x)$	x^4+4x^2+2 $+ix^3+3ix$	x^4+3x^2+4 $+2ix^3+4ix$	x^4-6x^2-8 $+5ix^3+4ix$	x^4-24x^2+16 $+8ix^3-32ix$
$GL_5^{(k)}(x)$	x^5+5x^3+5x $+ix^4+4ix^2+2i$	x^5+4x^3+4x $+2ix^4+7ix^2$ $+4i$	x^5-x^3-4x $+4ix^4+10ix^2$ $+8i$	x^5+16x^3-16x $+7ix^4+8ix^2$ $-16i$
$GL_6^{(k)}(x)$	$x^6+6x^4+9x^2+2$ $+ix^5+5ix^3+5ix$	$x^6+5x^4+5x^2-4$ $+2ix^5+10ix^3$ $+12ix$	$x^6+3x^4+6x^2+8$ $+3ix^5+11ix^3$ $+12ix$	$x^6-9x^4-24x^2$ $-16+6ix^5$ $+8ix^3$
$GL_7^{(k)}(x)$	$x^7+7x^5+14x^3$ $+7x+ix^6+6ix^4$ $+9ix^2+2i$	$x^7+6x^5+9x^3$ $+2ix^6+12ix^4$ $+19ix^2+4i$	$x^7+4x^5+5x^3$ $+4x+3ix^6$ $+15ix^4+22ix^2$ $+8i$	$x^7-3x^5-16x^3$ $-16x+5ix^6$ $+17ix^4+24ix^2$ $+6i$

The generalized Gauss k-Lucas polynomials and the known Gauss Lucas polynomials are related by

$$GL_n^{(k)}(x) := (GL_m(x))^{k-r}(GL_{m+1}(x))^r, \quad n = mk + r. \tag{3.2}$$

If $k = 1$ in last equation, we have that $m = n$ and $r = 0$, we get $GL_n^{(1)}(x) = GL_n(x)$.

Theorem 3.2.2. *For k and m, the generalized Gauss k-Lucas polynomials and the known Gauss Lucas polynomials satisfy*

$$[(GL_{m+1}(x))^k - (GL_m(x))^k] = [GL_{(m+1)k}^{(k)}(x) - GL_{mk}^{(k)}(x)].$$

Proof. By using (3.2), we have

$$\begin{aligned} GL_{(m+1)k}^{(k)}(x) - GL_{mk}^{(k)}(x) &= [(GL(x))^{k-k}(GL_{m+1}(x))^k] \\ &\quad - [(GL_m(x))^{k-0}(GL_{m+1}(x))^0] \\ &= (GL_{m+1}(x))^k - (GL_m(x))^k. \end{aligned}$$

□

For $k = 2, 3, 4$ and n, we have the following interesting properties:

(i) $GL_{2n}^{(2)}(x) = GL_n^2(x)$,
(ii) $GL_{2n+1}^{(2)}(x) = GL_n(x)GL_{n+1}(x)$,
(iii) $GL_{2n+1}^{(2)}(x) = xGL_{2n}^{(2)}(x) + GL_{2n-1}^{(2)}(x)$.
(iv) $GL_{3n}^{(3)}(x) = GL_n^3(x)$,
(v) $GL_{3n+1}^{(3)}(x) = GL_n^2(x)GL_{n+1}(x)$,
(vi) $GL_{3n+2}^{(3)}(x) = GL_n(x)GL_{n+1}^2(x)$,
(vii) $GL_{3n+1}^{(3)}(x) = xGL_{3n}^{(3)}(x) + GL_{3n-1}^{(3)}(x)$,
(viii) $GL_{4n}^{(4)}(x) = GL_n^4(x)$,
(ix) $GL_{4n+1}^{(4)}(x) = GL_n^3(x)GL_{n+1}(x)$,
(x) $GL_{4n+2}^{(4)}(x) = GL_n^2(x)GL_{n+1}^2(x)$,
(xi) $GL_{4n+3}^{(4)}(x) = GL(x)GL_{n+1}^3(x)$,
(xii) $GL_{4n+1}^{(4)}(x) = xGL_{4n}^{(4)}(x) + GL_{4n-1}^{(4)}(x)$.

Theorem 3.2.3. *For n, we have the following relation*:

$$GL_{kn+1}^{(k)}(x) = xGL_{kn}^{(k)}(x) + GL_{kn-1}^{(k)}(x).$$

Proof. By using (3.2), we have

$$\begin{aligned} xGL_{kn}^{(k)}(x) + GL_{kn-1}^{(k)}(x) &= xGL_n^k(x) + GL_{n-1}(x)GL_n^{k-1}(x) \\ &= GL_n^{k-1}(x)(xGL_n(x) + GL_{n-1}(x)) \\ &= GL_n^{k-1}(x)GL_{n+1}(x) \\ &= GL_{kn+1}^{(k)}(x). \end{aligned}$$

□

Theorem 3.2.4 (Cassini's identity). *Let $GL_n^{(k)}(x)$ be the generalized Gauss k-Lucas polynomials. For $n, k \geq 2$, Cassini's identity for $GL_n^{(k)}(x)$ is as follows*:

$$\begin{aligned} &GL_{kn+t}^{(k)}(x)GL_{kn+t-2}^{(k)}(x) - (GL_{kn+t-1}^{(k)}(x))^2 \\ &\quad = \begin{cases} GL_n^{2k-2}(x)(-1)^{n+1}(x^2+4)(2-ix), & t=1, \\ 0, & t \neq 1. \end{cases} \end{aligned}$$

Proof. By using (3.2), we get

$$\begin{aligned} &GL_{kn+t}^{(k)}(x)GL_{kn+t-2}^{(k)}(x) - (GL_{kn+t-1}^{(k)}(x))^2 \\ &\quad = (GL_n^{k-1}(x)GL_{n+t}(x))(GL_n^{k-1}(x)GL_{n+t-2}(x)) - (GL_n^{k-1}(x)GL_{n+t-1}(x))^2 \\ &\quad = (GL_n^{k-1}(x))^2(GL_{n+t}(x)GL_{n+t-2}(x) - (GL_{n+t-1}(x))^2) \\ &\quad = GL_n^{2k-2}(x)(GL_{n+t}(x)GL_{n+t-2}(x) - (GL_{n+t-1}(x))^2). \end{aligned}$$

For $t = 1$;

$$\begin{aligned} &= GL_n^{2k-2}(x)(GL_{n+1}(x)GL_{n-1}(x) - (GL_n(x))^2) \\ &= GL_n^{2k-2}(x)(-1)^{n+1}(x^2+4)(2-ix). \end{aligned}$$

For $t \neq 1$, $t = m$;

$$\begin{aligned} &= GL_n^{2k-2}(x)(GL_{n+m}(x)GL_{n+m-2}(x) - (GL_{n+m-1}(x))^2) \\ &= GL_n^{2k-2}(x)(GL_{2n+2m-2}^{(2)}(x) - GL_{2n+2m-2}^{(2)}(x)) \\ &= 0. \end{aligned}$$

For any natural numbers n, there are the following interesting relations between the generalized Gauss k-Lucas polynomials and the known Gauss Lucas polynomials:

$$GL_{nk+t}^{(k)}(x) = GL_n^{k-t}(x)GL_{n+1}^t(x); \quad t = 0, 1, 2, \ldots, k-1.$$

□

4. Conclusion

In this paper, we defined a new family of Gauss k-Lucas numbers and their polynomials. We gave some relations between the family and the known Gauss Lucas numbers. We also give the relationships among their polynomials and the known Gauss Lucas polynomials. Furthermore, we find the new generalization of the family and the polynomials in matrix representation. Then we proved Cassini's identities for the family and their polynomials.

Acknowledgments

The authors are grateful to the editor and referees for helpful suggestions and comments.

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