MARMARA UNIVERSITY
INSTITUTE FOR GRADUATE STUDIES
IN PURE AND APPLIED SCIENCES

ROTATING GLOBAL MONOPOLE AND TOPOLOGICAL DEFECTS IN HORNDESKI THEORY

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Ph.D. THESIS

Department of Physics

Thesis Supervisor

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ACKNOWLEDGEMENTS

I would like to express my heartfelt gratitude to my thesis advisor, Prof. Dr. Özgür Delice, Department of Physics, Marmara University. He gave me the freedom to make this thesis truly my own, while always guiding me in the right direction whenever I needed it. His patience, trust, and support have played a major role in shaping me both as a researcher and as a scientist.

I would also like to thank my dearest friends and colleagues Alperen Kaya, Doğukan Açıkgöz, Emre Sezen, Muratcan Öner, Okan Atakişi, Okan Duman, and Sinan Güyer for standing by me, sharing their time, and offering support throughout this journey.

Finally, I would like to thank my parents and siblings Münevver and Dürdane Seçuk, Cemre, and Kerem for their endless love, patience, and unwavering support. They have been my strength and motivation throughout my years of study and during the research and writing of this thesis. This work would not have been possible without them. I am deeply grateful for their support.

I dedicate this thesis to my late grandmother, Gülendam Seçuk, whom I miss deeply and carry with me always. Her love, kindness, and quiet strength continue to guide and inspire me every day.

This thesis is supported by Marmara University Scientific Research Projects Committee (Project Code: FDK-2021-10432).

January, 2026 **M. Haluk SEÇUK**

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ÖZET

DÖNEN GLOBAL MONOPOL VE HORNDESKI TEORİDE TOPOLOJİK KİRLİLİKLER

Bu tezde ilk olarak, literatürde Janis–Newman yöntemi kullanılarak elde edilmiş olan global monopol içeren dönen karadelik çözümlerini inceledik. Bu çözümlerin kullanılan Boyer–Lindquist koordinat dönüşümlerinin integrallenemezliği nedeniyle geçersiz olduğunu gösterdik. Daha sonra, söz konusu integrallenemezlik problemi olmayan Janis–Newman–Azreg-Aïnou yöntemini kullanarak, aynı karadelik için bir aday çözüm türettik. Ancak, elde ettiğimiz çözümün global monopole karşılık gelen enerji-momentum tensörü ile uyumsuz olduğunu Einstein alan denklemlerini sağlayamaması dolayısıyla ortaya koyduk. Ayrıca, bu sistemin yavaş dönme limitindeki çözümü, dönen global monopol içeren karadelik çözümlerinde Janis–Newman yöntemi ve genelleştirmelerinin uygulanamazlığını da ima etmektedir.

Sonra, dört boyutta ikinci mertebeden alan denklemlerine sahip en genel skaler-tensör kuramı olan Horndeski kuramında, skaler alanın kütlesiz veya kütleli olması durumları için, zayıf alan limitinde yeni global monopol çözümleri elde ettik. Bu çözümlerin konformal yapısı, skaler alan kütlesinin perdeleme etkisi, test parçacıklarının jeodezik hareketleri, ve bunların galaksi dönme eğrilerine olası etkileri gibi çeşitli fiziksel özelliklerini de tartıştık. Sonuç olarak bu monopollerin, Genel Görelilikteki karşılıklarına göre kütleçekimsel olarak etkin olduklarını gözlemledik.

Tezin son kısmı, bir başka tür topolojik kusur olan kozmik sicimlerin Horndeski kuramında incelenmesine ayrılmıştır. Hem kütlesiz hem de kütleli Horndeski kuramında düz veya kıvrımlı kozmik sicimlere karşılık gelen yeni lineer çözümler elde ettik. Her iki tür kozmik sicim çözümünün çeşitli fiziksel özelliklerini ve geodezik yapısını kuramın kütleli veya kütlesiz durumu için detaylı biçimde inceledik. Skaler alan kütlesinin perdeleme etkisi, bu kuramları ayırmakta ve hem global monopollerin hem de kozmik sicimlerin asimptotik davranışını kütlesizlerden farklılaştırmaktadır.

ABSTRACT

ROTATING GLOBAL MONOPOLE AND TOPOLOGICAL DEFECTS IN HORNDESKI THEORY

In this thesis, we first analyze rotating black hole solutions with global monopoles reported in the literature derived using the Janis–Newman method. We show that these solutions are invalid due to the non-integrability of the employed Boyer–Lindquist coordinate transformations. We then derive a candidate solution for such a black hole using the Janis–Newman–Azreg-Aïnou method, which does not have the aforementioned non-integrability problem. However, we show that the resulting solution is also not compatible with the energy–momentum tensor of a global monopole since the Einstein field equations cannot be satisfied. The slow rotation limit of this system also implies the non-applicability of the Janis–Newman method and its extensions for generating global monopoles in rotating black holes.

Next, we obtain new exact solutions of global monopoles in Horndeski theory, the most general scalar-tensor theory with second–order field equations in four dimensions, for the case where the scalar field is massless or massive, in the weak field limit. Various physical properties of these solutions were discussed, such as the conformal structure, screening effect of the mass of the scalar field, geodesic motion of test particles, and their possible effects on the galaxy rotation curves. We see that these monopoles are gravitationally active compared to ones in General Relativity.

The last part of this thesis is devoted to studying another type of topological defect, cosmic strings, in Horndeski theory. Novel linearized solutions corresponding to straight or wiggly cosmic strings are presented in both the massless and the massive Horndeski theory. Several physical properties and the geodesic structure of these solutions are also studied in detail for both types of cosmic strings for massive and massless cases of this theory. The screening effect of the mass of the scalar field distinguishes these theories and affects the asymptotic behavior of both global monopoles and cosmic strings from massless ones.

YENİLİK BEYANI

DÖNEN GLOBAL MONOPOL VE HORNDESKI TEORİDE TOPOLOJİK KİRLİLİKLER

Ben, Muhammed Haluk SEÇUK, Marmara Üniversitesi Fen Bilimleri Enstitüsü Fizik Anabilim Dalı doktora programı öğrencisi olarak hazırladığım "ROTATING GLOBAL MONOPOLE AND TOPOLOGICAL DEFECTS IN HORNDESKI THEORY" başlıklı bu tezin, bilimsel ahlak ve geleneklere uygun biçimde tarafımdan özgün olarak yapıldığını beyan ederim.

Bu tezde sunulan çalışmanın bir kısmı, tez kapsamında elde edilen sonuçlara dayalı olarak, tarafımdan ve tez danışmanı Prof. Dr. Özgür DELİCE ile birlikte hazırlanmış ve daha önce hakemli dergilerde yayımlanmış ya da yayımlanma aşamasında olan makalelerden oluşmaktadır. Söz konusu yayınlar, tez kaynakçasında eksiksiz biçimde gösterilmiştir.

Bu kapsam dışında kalan kısımlarda:

-Başkalarının eserlerinden yararlanılan yerlerde, bilimsel atıf kurallarına uygun biçimde kaynak gösterildiğini,

-Bu tezde yer alan çalışmanın, daha önce aynı veya başka bir unvan almak amacıyla herhangi bir yükseköğretim kurumuna sunulmadığını,

-Araştırma ve yazım sürecinde Marmara Üniversitesi'nin bilimsel etik ve dürüstlük ilkelerine uygun davrandığımı beyan ederim.

Muhammed Haluk SEÇUK **Ocak, 2026**

SYMBOLS

$\boldsymbol{G_{ab}}$: Einstein tensor

$\boldsymbol{\eta_{ab}}$: Minkowski metric tensor

$\boldsymbol{T_{ab}}$: Energy-momentum tensor

$\boldsymbol{d\Omega^2}$: Line element of a two-sphere

$\boldsymbol{g_{ab}}$: Metric tensor

$\boldsymbol{g^{ab}}$: Inverse metric tensor

$\Box$: d'Alembert operator

$\Box_{\boldsymbol{\eta}}$: d'Alembert operator in Minkowski spacetime

$\boldsymbol{h_{ab}}$: Metric perturbation tensor

$\boldsymbol{\theta_{ab}}$: Auxiliary perturbation tensor

$\boldsymbol{\eta}$: Symmetry–breaking scale of the global monopole

$\boldsymbol{\mu}$: Energy per unit length of the string

$\tilde{\boldsymbol{\mu}}$: Effective mass per unit length of the wiggly string

$\tilde{\boldsymbol{T}}$: Effective tension per unit length of the wiggly string

ABBREVIATIONS

A–dS : (Anti)–de Sitter

BD : Brans–Dicke

BL : Boyer–Lindquist

BV : Barriola and Vilenkin

EF : Eddington–Finkelstein

GR : General Relativity

JN : Janis–Newman

JNA : Janis–Newman–Azreg-Aïnou

NP : Newman-Penrose

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1. INTRODUCTION

Global monopoles are very peculiar, exotic objects with quite interesting properties. Their possible production in the early universe is discussed in Grand Unified Theories [1, 2] when a global $O(3)$ symmetry is spontaneously broken to $U(1)$ [3] during the symmetry–breaking phase transitions. In General Relativity (GR), these monopoles have distinct properties, such as the spacetime around a global monopole has a solid angle deficit, and they possess no active gravitational mass [3], despite the tiny negative mass of the core [4]. They are believed to contribute to structure formation in the early universe [5], along with other topological defects, but observations revealed that they cannot be the primary mechanism and their contribution is rather limited, yet not completely ruled out [6–8]. They also have interesting cosmological outcomes since they are very dense objects and should dominate the early universe after being formed. The inflation mechanism can resolve this monopole problem by the exponential expansion of the universe during this phase, which dilutes away the monopoles. For a review of topological defects and their astrophysical and cosmological implications, we refer to [9]. It is worth mentioning that there are numerous interesting physical applications of global monopoles [10–16]. Understanding the effects of global monopoles and other topological defects, and their interaction with the surrounding matter and other fields, is an important topic concerning their cosmology and gravitational implications.

In this regard, an important question arose regarding global monopoles at the start of this dissertation: Do global monopoles have a rotating counterpart? This question is especially intriguing, since rotation of a body typically alters the causal and geometric structure of spacetime, modifies horizon properties, and influences the interaction of the body with surrounding matter and fields. This question is appears to be answered in [17], where it is shown that the solution is obtainable with the Janis–Newman (JN) method. The JN method is a well-known approach for generating rotating solutions in General Relativity, often used to obtain the rotating black hole solutions from the static ones. It was first introduced in [18] to generate

the massive, electrically charged rotating black hole solution. This method has been applied to global monopoles to explore their possible rotating configurations. However, upon further analysis, we found that previous attempts using this method did not yield valid results for rotating global monopoles. Our study reevaluates these solutions, aiming to derive physically meaningful rotating configurations. There are many attempts in the literature to construct rotating configurations using the JN method and its variants; however, several of the resulting metrics have unclear validity. We refer to the review [19] for a detailed analysis of this subject. The non-trivial topological nature of global monopoles, particularly their scalar field configuration, complicates the straightforward extension of spherically symmetric solutions to rotating ones, making it essential to carefully examine the assumptions behind these methods and the resulting spacetime structures. This motivated us to make a thorough re-evaluation of whether a rotating global monopole solution can truly exist within GR or its modifications, and if so, what form such a solution must take to be physically meaningful and mathematically consistent. Our findings, both positive and negative, on this matter are compiled and presented in Section 2.

Having addressed the question regarding the rotating global monopoles, we now turn our attention to modified theories of gravity and their implications for topological defects, specifically focusing on global monopoles and cosmic strings. We have already mentioned the global monopoles and their importance and implications. Cosmic strings [3, 9, 20], on the other hand, are linear topological defects which may be produced [1] during symmetry-breaking phase transitions in the early universe, similar to global monopoles. The very peculiar properties of these exotic objects make them appealing to study and investigate their cosmological and astrophysical consequences. When they formed in the early universe, a cosmic string network would have been composed of either infinite strings or string loops [1, 21]. Infinite strings may produce wakes in the cosmic background when they move [22]. It is thought that the density fluctuations leading to structure formation in the early universe may have originated from the evolution of cosmic string networks. The aforementioned network emits electromagnetic, gravitational, and other forms

of radiation when it evolves and relaxes. This evolution is rather complex and model-dependent, requiring numerical simulations to understand it. However, data accumulated in the late 20th century revealed that the primary source of density fluctuations has a quantum origin. These observations may reduce the role played by topological defects in structure formation in the early universe. Although the contribution of topological defects, including cosmic strings, to structure formation is not completely ruled out, it has been shown that it is very limited [6–8], if these objects really exist.

Nevertheless, the gravitational properties of cosmic strings are still a subject of interest. For example, in GR, the gravitational potential of an infinitely long straight cosmic string vanishes. Therefore, the spacetime around such a string is locally flat. It is globally conical, however, since the line element involves an angle deficit proportional to the tension of the string. This global effect produces observable astrophysical effects, such as double images of galaxies or other distant objects, and imprints on the cosmic microwave background radiation (CMB). The astrophysical, cosmological, and gravitational wave observations set limits on the physical parameters of the hypothetical cosmic strings; one such parameter is the string tension.

Numerical and theoretical investigations regarding cosmic string networks imply that small-scale perturbations such as wiggles and kinks may be realized in the cosmic string network [2, 23]. For a distant observer who cannot resolve these perturbations on a long string, the string will appear smooth [9, 23]. However, the effect of these perturbations will be realized in the mass per unit length and tension of the string [2]. These changes will have consequences for the gravitational and cosmological properties of wiggly strings [9], which motivates further investigation of their gravitational and cosmological implications [24–26].

Theories alternative to or modifying General Relativity are still an active research topic for various observational and theoretical reasons. We refer to the excellent reviews [27–31] on the modified gravity theories for these motivations and recent

developments. One way to distinguish such theories is to consider the same physical setup for the alternative theories, identify their outcomes, and then use observations to rule out or constrain the theories that conflict with the observations. Here, we will study global monopoles and cosmic strings. We will obtain their various properties for the theory we consider, and compare them with GR. There are various studies considering straight [32–42], and wiggly [43] cosmic strings in alternative theories in the last few decades.

Our main focus will be Horndeski theory [44,45] in the linearized framework. It is the most general scalar-tensor theory leading to second–order field equations in four dimensions. Any scalar-tensor theory having this property can be obtained from Horndeski theory. As a result, it generalizes General Relativity, Brans–Dicke theory (BD) [46], $f(R)$ gravity, among others. Thus, Horndeski theory is a framework that covers an important subclass of the modified gravity theories [47]. In this work, we will investigate the gravitational field of the global monopoles and straight or wiggly cosmic strings in Horndeski theory. Since the complexity of the field equations of this theory and also the fact that even in simpler theories, the corresponding global monopole solution is usually found in the linearized field equations for BD [33] and a scalar-tensor theory, [48], we also keep our discussion at the linearized level. A global monopole solution [49] is discussed for massless or massive dilaton gravity, where the scalar field is nonminimally coupled to matter-energy Lagrangian. Recently, a global monopole solution in the framework of extended Gauss-Bonnet theory with a positive ADM mass is presented in [50]. There are many other works [51–54] obtaining global monopole solutions in various modified gravity theories.

Despite restricting ourselves to the linearized level, several phenomenological questions can be answered, such as the behavior of the spacetime outside the core of the monopole and the behavior of test particles around such monopoles. The generality of this theory will lead to a rich phenomenology depending on the properties of the scalar field. It could involve a scalar potential that might change the local and global behavior of the underlying spacetime. In other words, the fact that the scalar field can be massless or massive in Horndeski theory leads to differ-

ent characteristics for the solutions obtained. Therefore, global monopole solutions and their physical properties reveal the effects of massive theory compared to massless one. The mass of the scalar field, originated from a scalar potential function, has the property that it generally leads to a massive theory [55–57], rendering the scalar field short–range. Resulting massive theory will have dramatically different local and asymptotic behavior, such as showing a screening effect [58–60] where the scalar field is frozen beyond a specific range determined by the mass of the scalar field, compared to the massless theories where the scalar field has a long range as in original BD theory [46]. Therefore, the obtained solutions might be another example of the realization of such a mechanism.

The thesis is organized as follows: In Section 2, we review the solutions given in [17, 61], and show that the resulting line elements for the gravitational field of a rotating global monopole in GR and in $f(R)$ theory are faulty. We further show, by extension, that the rotating cloud of strings solution in GR is also wrong. We propose a rectification for this problem by employing a modified version of the JN method, namely the Janis-Newman-Azreg-Aïnou (JNA) method. However, we showed that the resulting line element does not satisfy Einstein's field equations for the rotating global monopole. Then, we compare our results with the slow rotation limit of the rotating global monopole [62]. We argue about the arising contradictions in regards to obtaining the solution for the rotating global monopole with known methods. Our direct derivation attempts are also presented in this section. In Section 3, we first review the weak–field equations and the energy–momentum tensor of a global monopole in Horndeski theory. Then, we present solutions of the linearized field equations of Horndeski theory and employ them to obtain the solutions for the global monopole for both the massless and massive cases. Later in this section, we will show some applications of these newly found spacetimes. In Section 4, we use the weak-field equations of Horndeski theory discussed in Section 3, to derive solutions for straight cosmic strings in massless and massive Horndeski theory. In that section, we will also obtain several physical implications of these solutions. We discuss the solutions of wiggly cosmic strings and their properties in the fifth

section. Note that the solutions we will obtain for massless Horndeski theory resemble the form of the corresponding solutions for BD theory for both straight [32, 33] and wiggly [43] strings. However, the corresponding solutions in massive Horndeski theory are, to our knowledge, derived here for the first time. In Section 6, we investigate the unbounded motion of particles passing near straight and wiggly strings in massless/massive Horndeski theory. We explicitly show the contribution to the velocity gain of the particle in the direction perpendicular to the motion due to the gravitational force of the string in Horndeski theory. For the straight cosmic string, the velocity gain is originally due to global topology in GR. The last section is devoted to the conclusion. Unless stated otherwise, we use geometric units, that is $c = G = 1$, throughout this thesis. This dissertation aims to resolve the contradictions in previous attempts to describe rotating global monopoles in GR and to provide novel insights into the properties of global monopoles and cosmic strings within the context of Horndeski theory.

2. IS THERE A ROTATING GLOBAL MONOPOLE BLACK HOLE?

There are many misuses of the JN method in the literature, especially when the global topology of the spacetime is non-trivial. One major example is [17], where it is claimed that the solution for the rotating global monopole black hole is found with the JN method [18]. Here, we will show that the resulting line element is not in the proper Boyer–Lindquist (BL) coordinates. In other words, we will show that the solution presented in [17], whose explicit form will be presented in Eq. (2.32) below in BL coordinates, cannot correspond to a global monopole swallowed by a rotating Kerr black hole.

2.1. Global Monopoles and Their Energy–Momentum Tensor

The Lagrangian giving rise to a global monopole is constructed from a scalar field triplet Φ^A, $A = 1, 2, 3$, given by

$$L_{\text{mon}}[\Phi^A] = \frac{1}{2}\partial^\mu \Phi^A \partial_\mu \Phi^A - \frac{\lambda}{4}\left(\Phi^A \Phi^A - \eta^2\right)^2, \tag{2.1}$$

having a global $O(3)$ symmetry. Through a spontaneous symmetry–breaking mechanism, global monopoles are formed with the energy scale η. The usual monopole ansatz [2] is given by,

$$\Phi^A = \frac{\eta f(r) x^A}{r}. \tag{2.2}$$

For spherical symmetry the coordinates are chosen as,

$$x^1 = r\sin\theta\cos\phi, \quad x^2 = r\sin\theta\sin\phi, \quad x^3 = r\cos\theta, \tag{2.3}$$

with $x^A x^A = r^2$, and considering the fact that $f(r) \simeq 1$ outside the core of the monopole [2], the nonzero components of the energy–momentum tensor become

$$T_{00}^{\text{mon}} \approx -T_{rr}^{\text{mon}} \approx \frac{\eta^2}{r^2}. \tag{2.4}$$

The exact solution of Einstein field equations corresponding to a global monopole with this energy–momentum tensor was presented by Barriola and Vilenkin (BV) [3] as

$$ds^2 = -\left(1 - \kappa\,\eta^2 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \kappa\,\eta^2 - \frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.5}$$

where $\kappa = 8\pi G/c^4 = 8\pi$ (in the natural units $G = c = 1$) and M is either the core mass of the monopole itself or the mass of the black hole where the monopole is swallowed by. This black hole line element will be our starting point to demonstrate the derivation attempt of a rotating global monopole black hole solution. Now, we proceed with a brief review of the method that is used in the literature to obtain rotating counterparts of the static black holes, namely, the Janis–Newman (JN) method.

2.2. Janis-Newman Method

Historically, the JN method was first introduced in [18] to obtain the solution for a massive, charged rotating black hole, now known as the Kerr-Newman black hole. Here, we will briefly introduce the method, which we later use in our calculations and demonstrate how one can obtain the solution for a massive rotating black hole, i.e. Kerr black hole. Following the prescription for the JN method given in [19], we start with a Schwarzschild-type line element of the form,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.6}$$

Then employing retarded Eddington-Finkelstein (EF) coordinate transformation,

$$du = dt - f(r)^{-1}dr, \tag{2.7}$$

brings Eq. (2.6) to

$$ds^2 = -f(r)dt^2 - 2drdu + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.8}$$

To proceed with the method, we transform Eq. (2.8) to its contravariant form. This yields a metric:

$$g^{\mu\nu}\partial_\mu\partial_\nu = f\partial_r^2 - 2\partial_u\partial_r + \frac{1}{r^2}\left(\partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2\right). \tag{2.9}$$

At this point the JN method utilizes the Newman-Penrose (NP) formalism. Hence, we define the tetrad basis,

$$\begin{aligned} k^\mu = \delta^\mu_r, \quad l^\mu = \delta^\mu_u - \frac{f(r)}{2}\delta^\mu_r, \quad m^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\phi\right), \\ \bar{m}^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(\delta^\mu_\theta - \frac{i}{\sin\theta}\delta^\mu_\phi\right). \end{aligned} \tag{2.10}$$

where $\bar{m}$ represents the complex conjugate of m. Note that the JN method allows the coordinates u and r to be complex in this part of the process. With this basis, the metric tensor reads,

$$g^{\mu\nu} = -k^\mu l^\nu - k^\nu l^\mu + m^\mu\bar{m}^\nu + m^\nu\bar{m}^\mu. \tag{2.11}$$

Making the complex coordinate transformation,

$$u \to u + ia\cos\theta, \quad r \to r - ia\cos\theta \tag{2.12}$$

brings the tetrad basis to the new form of,

$$\begin{aligned} k^\mu &= \delta^\mu_r, \\ l^\mu &= \delta^\mu_u - \frac{\mathcal{F}}{2}\delta^\mu_r, \\ m^\mu &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left[\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\phi - ia\sin\theta(\delta^\mu_u - \delta^\mu_r)\right], \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left[\delta^\mu_\theta - \frac{i}{\sin\theta}\delta^\mu_\phi + ia\sin\theta(\delta^\mu_u - \delta^\mu_r)\right], \end{aligned} \tag{2.13}$$

where $\mathcal{F}$ is transformed version of the metric function $f(r)$ of the desired system according to JN algorithm. The type of this transformation depends on the polynomial degree of the radial coordinate r. For example, a second degree term transforms

as, $r^2 \to |r|^2$ with the transformation (2.12). Inverting back the newly transformed metric tensor to its covariant form yields,

$$ds^2 = -\mathcal{F}du^2 - 2dudr + 2a\sin^2\theta\,(\mathcal{F}-1)\,dud\phi + 2a\sin^2\theta drd\phi + \rho^2 d\theta^2 + \sin^2\theta\left[\rho^2 + a^2\sin^2\theta(2-\mathcal{F})\right]d\phi^2, \tag{2.14}$$

where $\rho^2 = |r|^2 = r^2 + a^2\cos^2\theta$. Now we define BL coordinate transformation in their differentiated form as,

$$du = dt - g\,dr, \quad d\phi = d\varphi - h\,dr, \tag{2.15}$$

which brings the metric (2.14) to,

$$\begin{aligned} ds^2 = &\left[-g^2\mathcal{F} + 2g + 2gha\sin^2\theta\,(\mathcal{F}-1) - 2ha\sin^2\theta + h^2\rho^2\sin^2\theta\right. \\ &\left.+h^2a^2\sin^4\theta\,(2-\mathcal{F})\right]dr^2 + \rho^2 d\theta^2 + \left\{\rho^2\sin^2\theta + a^2\sin^4\theta\,(2-\mathcal{F})\right\}d\varphi^2 \\ &+ 2a\sin^2\theta\,(\mathcal{F}-1)\,dtd\varphi + \left\{2g\mathcal{F} - 2ha\sin^2\theta\,(\mathcal{F}-1) - 2\right\}dtdr \\ &+ 2a\sin^2\theta\left[-(\mathcal{F}-1)g - \frac{\rho^2 h}{a} - ha\sin^2\theta(2-\mathcal{F}) + 1\right]drd\varphi - \mathcal{F}dt^2. \end{aligned} \tag{2.16}$$

An important and crucial note here: The functions h and g are meant to only depend on the radial coordinate, r. Otherwise, the BL transformations given in (2.15) are not integrable. This is a crucial step of the JN method since these BL transformations are utilized to understand the resulting metric structure. Without a proper BL form, one cannot directly determine whether the obtained line element indeed represents a rotating black hole or not. Now, using the condition $g_{r\varphi} = g_{tr} = 0$ we have,

$$-(\mathcal{F}-1)g - \frac{\rho^2 h}{a} + ha\sin^2\theta(\mathcal{F}-2) + 1 = 0, \tag{2.17}$$

$$g\mathcal{F} - ha\sin^2\theta(\mathcal{F}-1) - 1 = 0, \tag{2.18}$$

as a result,

$$g = \frac{h}{a}\left[\rho^2 + a^2\sin^2\theta\right]. \tag{2.19}$$

Exploiting this in (2.17) and (2.18) yields,

$$h = \frac{a}{\mathcal{F}\rho^2 + a^2 \sin^2\theta}, \quad g = \frac{r^2 + a^2}{\mathcal{F}\rho^2 + a^2 \sin^2\theta}. \tag{2.20}$$

Now to obtain the rotating counterpart of a static solution, one first determines the metric function of (2.6), then transforms this function to its $\mathcal{F}$ version. Utilizing this function with (2.20) in (2.16), one obtains the desired solution.

2.2.1. An Example: Schwarzschild to Kerr Black Hole

In order to obtain the Kerr black hole we start with the metric function of the Schwarzschild black hole given as,

$$f(r) = 1 - \frac{2M}{r}. \tag{2.21}$$

Transformation of this function gives,

$$\mathcal{F}_S = 1 - \frac{2Mr}{\rho^2}, \tag{2.22}$$

therefore the BL coordinate transformation functions for this system are,

$$h(r) = \frac{a}{\Delta_S}, \quad g(r) = \frac{r^2 + a^2}{\Delta_S}, \quad \Delta_S(r) = r^2 + a^2 - 2Mr. \tag{2.23}$$

As one can see, there are no coordinate dependencies other than r, as wanted. Using these in (2.16) yields,

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\Delta_S}{\rho^2} dr^2 + \rho^2 d\theta^2 \\ & + \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2, \end{aligned} \tag{2.24}$$

which is widely known Kerr metric in BL coordinates, that describes a rotating black hole with mass M [63].

In the following section, we will show the misuse of the JN method on Schwarzschild-monopole black hole to show how this method can become obsolete in non-trivial topologies.

2.3. Is There a Rotating Global Monopole Black Hole?

In the previous two subsections, we introduce the global monopole spacetime and review the JN method, and as an example, show how it can be used on the Schwarzschild black hole to obtain the Kerr black hole. Now, we change our focus to the Schwarzschild-monopole black hole and its supposed rotating counterpart solution, which is the focal point of this section. In this case, the metric function is,

$$f_{SM}(r) = 1 - 8\pi\eta^2 - \frac{2m}{r}. \tag{2.25}$$

Transformation of this function to $\mathcal{F}$ is given by,

$$\mathcal{F}_{SM} = 1 - 8\pi\eta^2 - \frac{2mr}{\rho^2}, \tag{2.26}$$

with BL coordinate transformation functions,

$$h(r,\theta) = \frac{a}{\Delta_{SM} - 8\pi\eta^2\rho^2}, \quad g(r,\theta) = \frac{r^2 + a^2}{\Delta_{SM} - 8\pi\eta^2\rho^2}, \tag{2.27}$$
$$\Delta_{SM}(r) = r^2 + a^2 - 2mr.$$

As we remarked earlier, these transformation functions cannot depend on a coordinate other than r, otherwise the BL coordinate transformation is not integrable. However, as we can easily see, due to $\rho^2 = r^2 + a^2\cos^2\theta$ having a θ dependence, the functions g and h also depend on θ, which contradicts Eq. (2.15). Hence, we conclude that one cannot use the JN formalism to find the rotating global monopole spacetime in BL coordinates.

To understand the problem with the results of [17] more clearly, we start again with the Schwarzschild-monopole black hole metric defined in [17] as follows

$$ds^2 = -\left(1 - 8\pi\eta^2 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1 - 8\pi\eta^2 - \frac{2m}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Here, η^2 encodes the contribution from the global monopole term. Now we define

$b^2 = 1 - 8\pi\eta^2$ and rescale the coordinates as, $t \to t/b, r \to br$ and obtain,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + b^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

where $M = m/b^3$. At this point, we apply the JN prescription that we presented earlier to this metric to obtain the modified version of (2.14).This modification arises due to the coordinate rescaling. The resulting metric is

$$\begin{aligned} ds^2 = &-f(r,\theta)du^2 - 2dudr + 2a\sin^2\theta\left[f(r,\theta) - 1\right] dud\phi + 2a\sin^2\theta drd\phi + \rho^2 b^2 d\theta^2 \\ &+ \sin^2\theta\left[\rho^2 b^2 + a^2\sin^2\theta(2 - f(r,\theta))\right] d\phi^2, \end{aligned} \tag{2.28}$$

with $f(r,\theta)$ being the transformed metric function according to the complexification rules of the polynomial degree of r in JN prescription [19]. It is given by,

$$f(r,\theta) = 1 - \frac{2Mr}{\rho^2}, \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{2.29}$$

In [17], the following transformation is used to bring (2.28) to the BL coordinates. Note that in [17] a metric signature of $(+,-,-,-)$ is used, therefore there is a sign change in these coordinate transformations compared to ones we defined earlier. We presented them as is for accuracy,

$$t \to u - r - M\ln(r^2 - 2Mr - a^2) - \frac{2M^2\arctan(\frac{r-M}{\sqrt{a^2-M^2}})}{\sqrt{a^2 - M^2}}, \tag{2.30}$$

$$\phi \to \phi + \frac{a\arctan(\frac{r-M}{\sqrt{a^2-M^2}})}{\sqrt{a^2 - M^2}}. \tag{2.31}$$

This results in the rotating global monopole solution that exists in the literature by Filho-Bezerra [17], where the resulting metric tensor is given as,

$$\begin{aligned} g_{tt} =& 1 - \frac{2Mr}{\rho^2}, \quad g_{\theta\theta} = -b^2\rho^2, \quad g_{t\phi} = -\frac{2Mar\sin^2\theta}{\rho^2}, \\ g_{rr} =& -\left\{\frac{r^2 - a^2[(1-b^2)\sin^2\theta - \cos^2\theta]}{r^2 - 2Mr + a^2}\right. \\ &\left. -(1-b^2)\frac{a^2\sin^2\theta[2Mr - a^2(1-\sin^4\theta)]}{(r^2 - 2Mr + a^2)^2}\right\}, \end{aligned}$$

$$g_{\phi\phi} = -\frac{\sin^2\theta}{\rho^2}\left\{b^2r^4 + a^2r^2[1-(1-2b^2)\cos^2\theta] + 2Ma^2r\sin^2\theta \right. \\ \left. + a^4\cos^2\theta[b^2\cos^2\theta + \sin^2\theta]\right\},$$
$$g_{r\phi} = -\frac{(1-b^2)a[r^2\sin^2\theta - a^2\cos^2\theta(1+\cos^2\theta)]}{r^2 - 2Mr + a^2}. \tag{2.32}$$

A key issue is that the metric contains a nonzero $g_{r\phi}$ term, which is problematic. Since we already established that, in the JN method, one of the functions of the BL coordinate transformations is obtained by setting this term to zero, i.e., $g_{r\phi} = 0$. It is not clear how these transformations, Eqs. (2.31) and (2.30), were found in order to obtain such a metric tensor (2.32). Yet, we can try to find the correct versions of these transformations by exploiting (2.15) in (2.28), then the conditions of $g_{tr} = g_{r\varphi} = 0$ give,

$$\begin{aligned} g(r,\theta) &= \frac{b^2\rho^2 + a^2\sin^2\theta}{f(r,\theta)b^2\rho^2 + a^2\sin^2\theta} \\ &= \frac{r^2b^2 + a^2}{\Delta - 8\pi\eta^2a^2\cos^2\theta} - \frac{8\pi\eta^2a^2\cos^2\theta}{\Delta - 8\pi\eta^2a^2\cos^2\theta}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} h(r,\theta) &= \frac{a}{f(r,\theta)b^2\rho^2 + a^2\sin^2\theta} \\ &= \frac{a}{\Delta - 8\pi\eta^2a^2\cos^2\theta}, \end{aligned} \tag{2.34}$$

where $\Delta = (r^2 - 2Mr)b^2 + a^2$. Resulting functions h and g clearly depend on both r and θ. Therefore, one cannot use them to obtain the line element for the rotating global monopole in BL coordinates. As a result, the metric tensor given by Eq. (2.32) cannot describe the rotating global monopole black hole. One important note arises when we take $f(r,\theta) = 1$ by neglecting the mass parameter M, ergo

$$g(r,\theta) = 1, \quad h(r,\theta) = \frac{a}{r^2 + a^2 - 8\pi\eta^2\rho^2}. \tag{2.35}$$

It can be seen that these functions satisfy the requirement $g_{tr} = g_{r\varphi} = 0$. One can check this by inserting Eq. (2.35) into Eq. (2.16) given earlier. Regardless, $h(r,\theta)$ still has a θ dependence; therefore, one is unable to use them as the BL transformation for pure rotating global monopole spacetime.

To sum up, observe that the transformations given in (2.30) and (2.31) are not in agreement with our results (2.33) and (2.34); however, the line element presented by Eq. (2.32) includes a $g_{r\varphi}$ term which creates a contradiction. This is because, in order to find the functions g and h, one imposes the conditions such that $g_{r\varphi} = g_{tr} = 0$. Then, as a result, one obtains two equations with two unknown functions. By solving these equations for g and h, one obtains the BL transformations of their system. Furthermore, one *has to* confirm that these functions depend *only* on r due to the fact that transformations given by (2.15) need to be integrable. Otherwise, one, by any means, uses incorrect transformations or cannot transform the line element to BL coordinates at all.

This incorrect coordinate transformation has been carried out with ease to the 'rotating' Letelier spacetime in [64]. Since pure monopole spacetime also describes a Letelier spacetime, which is a configuration where an ensemble of radially distributed straight cosmic strings intersecting at a common point. Algebraically, all one needs to do is to change the monopole term $(1-8\pi\eta^2)$ by $(1-a)$[1]. Then perform the same calculation done in [17] for Letelier spacetime, as the authors did in [64]. They use the BL coordinate transformation given by Eq. (53-56) in [64], which is the same transformation defined by (2.30) and (2.31) except they change the monopole term b^2 with $(1-a)$. Hence, the metric tensor given by Eq. (57) in [64] is also not valid.

The third example of this mistake in global monopole spacetime is made in [61] where the authors exploited the JN algorithm to find the rotating counterpart of global monopole in $f(R)$ theory by following the prescription given in Drake and Szekeres [65]. Here, the transformed metric function is,

$$f_R(r,\theta) = 1 - \frac{2Mr}{\rho^2} - \Psi_0 r, \quad \Psi_0 = \frac{\psi_0}{b}, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad M = \frac{m}{b^3}, \tag{2.36}$$

where the parameter ψ_0 represents the deviation from the static global monopole solution in GR with rotation parameter a and $b^2 = 1-\eta^2$ encodes the monopole contribution as in the previous case. Now, if one uses the $f_R(r,\theta)$ in Eq. (2.28), one obtains the metric, in EF coordinates, given by Eq. (18) of [61]. We present this

[1]Here a represents the string cloud parameter in [64], not the rotation parameter we used earlier.

metric as is for ease of reading,

$$\begin{aligned} g_{uu} =& 1 - \frac{2Mr}{\rho^2} - \Psi_0 r, \quad g_{\theta\theta} = -b^2\rho^2, \quad g_{ur} = 1, \\ g_{u\phi} =& \frac{2Mar\sin^2\theta}{\rho^2} + a\Psi_0 \sin^2\theta\, r, \quad g_{r\phi} = -a\,\sin^2\theta, \\ g_{\phi\phi} =& -\sin^2\theta \left[b^2 r^2 + \frac{2Ma^2 r \sin^2\theta}{\rho^2} + a^2(b^2\cos^2\theta + \sin^2\theta) \right]. \end{aligned} \tag{2.37}$$

In order to follow the prescription of [65], this metric was written in the form of [61],

$$\begin{aligned} ds^2 =& -e^{2\Phi(r,\theta)}du^2 - 2e^{\lambda(r,\theta)+\Phi(r,\theta)}dudr + 2a\sin^2\theta e^{\Phi(r,\theta)}[e^{\Phi(r,\theta)} - e^{\lambda(r,\theta)}]dud\phi \\ &+ 2a\sin^2\theta e^{\lambda(r,\theta)+\Phi(r,\theta)}drd\phi + \Sigma\, d\theta^2 \\ &+ \sin^2\theta \left[\Sigma + a^2\sin^2\theta e^{\Phi(r,\theta)}(2e^{\lambda(r,\theta)} - e^{\Phi(r,\theta)})\right] d\phi^2. \end{aligned} \tag{2.38}$$

For Eq. (2.38) the BL transformation functions can be written as [65],

$$g(r) = \frac{e^{\lambda(r,\theta)}(\Sigma + a^2\sin^2\theta e^{\lambda(r,\theta)+\Phi(r,\theta)})}{e^{\Phi(r,\theta)}(\Sigma + a^2\sin^2\theta e^{2\lambda(r,\theta)})}, \tag{2.39}$$

$$h(r) = \frac{ae^{2\lambda(r,\theta)}}{\Sigma + a^2\sin^2\theta e^{2\lambda(r,\theta)}}. \tag{2.40}$$

Comparison of Eq. (2.28) and (2.38) with (2.36), the functions $\Phi(r,\theta)$, $\lambda(r,\theta)$ and Σ are found to be,

$$\Phi(r,\theta) = -\lambda(r,\theta) = \frac{1}{2}\ln(1 - \frac{2Mr}{\rho^2} - \Psi_0 r), \quad \Sigma = b^2\left(r^2 + a^2\cos^2\theta\right). \tag{2.41}$$

Hence, similar to (2.33) and (2.34) we obtain,

$$\begin{aligned} g(r,\theta) &= \frac{b^2\rho^2 + a^2\sin^2\theta}{f_R(r,\theta)b^2\rho^2 + a^2\sin^2\theta} \\ &= \frac{r^2b^2 + a^2}{\Delta_R - (8\pi\eta^2 + \Psi_0 r b^2)a^2\cos^2\theta} - \frac{8\pi\eta^2 a^2\cos^2\theta}{\Delta_R - (8\pi\eta^2 + \Psi_0 r b^2)a^2\cos^2\theta}, \end{aligned} \tag{2.42}$$

$$h(r,\theta) = \frac{a}{f_R(r,\theta)b^2\rho^2 + a^2\sin^2\theta} = \frac{a}{\Delta_R - (8\pi\eta^2 + \Psi_0 r b^2)a^2\cos^2\theta}, \tag{2.43}$$

where $\Delta_R = (r^2 - 2Mr - \Psi_0 r^3)b^2 + a^2$. Same as the previous case, functions g and h have a dependence not only on r but on θ as well. Therefore, they cannot be used to describe the BL system of a rotating global monopole in $f(R)$ theory.

In [61] however, these transformations presented as,

$$g(r) = \frac{a^2 + r^2}{\Psi_0\, r\, a^2 \cos^2\theta + \Psi_0 r^3 + 2Mr - r^2 - a^2} \tag{2.44}$$

$$h(r) = \frac{a}{\Psi_0\, r\, a^2 \cos^2\theta + \Psi_0 r^3 + 2Mr - r^2 - a^2} \tag{2.45}$$

We can see that these transformations do not match the ones we found in Eqs. (2.42) and (2.43). The mistake in [61] was made when comparing Eq. (2.37) with Eq. (2.38) and took $\Sigma = r^2 + a^2\cos^2\theta$; however, as one can see from (2.41), this is not the case. Note here that, even if we remove the monopole contribution from Eqs. (2.42) and (2.43) by taking $b^2 = 1$, the above functions would still have a θ dependence since the monopole term is not the only reason for the θ dependence in the functions (2.42) and (2.43).

The mathematical issue concerning the θ dependence of the functions g and h can be the complexification of the coordinate r, or in our case the lack of it, with the prescription given by the JN algorithm [18, 19]. The reason being that the monopole term η in the static metric function does not depend on r and did not change under the complexification process, as a result, the BL transformation functions g and h remain θ dependent. In addition, we also know that the JN method fails to produce a rotating counterpart when the spacetime contains a cosmological constant [19], namely in the de Sitter (dS) spacetime. One of the distinct features of dS spacetime is that the asymptotic region is not flat, which is a common point between dS and global monopole spacetimes. Since, global monopoles also constitute an asymptotically non-flat spacetime as a result of the deficit angle. Therefore, the more potent reason of why the JN method fails to produce these spacetimes could be the global topology of the spacetime. As a result of these three examples, we conclude that one could not produce the rotating global monopole system(s) in the BL coordinates using the JN algorithm and the validity of the analyses made on these line elements is also questionable.

2.3.1. A candidate solution using the JNA Method

There is an extended version of the JN method, presented by Azreg-Aïnou in [66]. In this work, it is shown that the rotating counterpart of a regular static black hole can be obtained without any complexification on the coordinate r. This is a modified version of the JN method called the JNA method, which we demonstrated in the previous Section 2.2. Here, for clarity, we will not repeat this modified version of the JN method and present the core results. We refer to [66] for the details of the calculation process. The BL transformation functions for a static global monopole-black hole are now in the form,

$$g(r) = \frac{r^2 + a^2}{f_\eta(r) r^2 + a^2}, \quad h(r) = \frac{a}{f_\eta(r) r^2 + a^2}, \quad f_\eta(r) = 1 - \frac{2m}{r} - 8\pi\eta^2, \tag{2.46}$$

which are *independent* of θ therefore the BL transformation (2.15) is integrable. The line element is now given by,

$$ds^2 = -\frac{\Delta_\eta}{\rho^2}\left[dt - a\sin^2\theta d\varphi\right]^2 + \frac{\rho^2}{\Delta_\eta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(a^2 + r^2)d\varphi - a dt\right]^2,$$
$$\Delta_\eta = r^2 + a^2 - 2mr - 8\pi\eta^2 r^2, \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{2.47}$$

It can be seen that by neglecting the monopole contribution η, we obtain the Kerr metric. On the other hand, by setting $a = M = 0$ and keeping $\eta \neq 0$, the line element reduces to the monopole spacetime. Moreover, contrary to Eq. (16) in [17], line element (2.47) *does not contain* the off-diagonal $g_{r\varphi}$ term as it should be in a proper BL coordinate system. However, it should be noted that since the static Schwarzschild-monopole spacetime, similar to Schwarzschild-(Anti)-de Sitter spacetime, is not *asymptotically flat*, the formalism given in [66] fails to produce Kerr-(Anti)-de Sitter spacetime. Based on this fact, the solution (2.47) may not reflect the true form of the rotating counterpart of the global monopole black hole spacetime. In the next subsection, we will share our attempts to confirm the validity of line element (2.47) by trying to find the exact solution for a rotating global monopole spacetime.

2.3.2. Solution Attempts

In order to find a solution for a rotating global monopole we consider, among others, the following general line element involving six unknown functions,

$$ds^2 = -p^2(r,\theta)[dt - X(r,\theta)d\phi]^2 + g^2(r,\theta)d\theta^2 + h^2(r,\theta)dr^2 + q^2(r,\theta)[Y(r,\theta)d\phi - adt]^2, \tag{2.48}$$

which is in the Boyer–Lindquist like coordinates. We define the vielbein as,

$$\vartheta^0 = p(r,\theta)[dt - X(r,\theta)\,d\phi], \quad \vartheta^1 = h(r,\theta)dr, \quad \vartheta^2 = g(r,\theta)d\theta, \quad \vartheta^3 = q(r,\theta)[adt - Y(r,\theta)\,d\phi]. \tag{2.49}$$

We calculated the connection one–forms using [67],

$$\omega_{ab} = \iota_a\iota_b(d\vartheta^c \wedge \vartheta_c) - \iota_a d\vartheta_b + \iota_b d\vartheta_a, \tag{2.50}$$

where ι_a is the interior product defined as the antiderivation on the exterior algebra with degree -1. In other words, it is a linear mapping that sends a $p-$form to $(p-1)-$form satisfying,

$$\iota_a(\alpha \wedge \beta) = (\iota_a\alpha) \wedge \beta + (-1)^p\alpha \wedge \iota_a\beta, \tag{2.51}$$

where β is a $q-$form. Here the exterior derivative of the one–forms are very lengthy, so we will proceed with compact notation and examples of functions. For example,

$$d\vartheta^0 = a_0\vartheta^{10} + b_0\vartheta^{20} + c_0\vartheta^{13} + d_0\vartheta^{23}, \tag{2.52}$$

$$d\vartheta^3 = a_3\vartheta^{10} + b_3\vartheta^{20} + c_3\vartheta^{13} + d_3\vartheta^{23}. \tag{2.53}$$

where $\vartheta^{ab} = \vartheta^a \wedge \vartheta^b$ and,

$$a_0(r,\theta) = \frac{p^{(1,0)}(r,\theta)}{h(r,\theta)p(r,\theta)} - \frac{aX^{(1,0)}(r,\theta)}{h(r,\theta)\rho(r,\theta)}, \qquad a_3(r,\theta) = \frac{aq(r,\theta)Y^{(1,0)}(r,\theta)}{h(r,\theta)p(r,\theta)\rho(r,\theta)},$$
$$b_0(r,\theta) = \frac{p^{(0,1)}(r,\theta)}{g(r,\theta)p(r,\theta)} - \frac{aX^{(0,1)}(r,\theta)}{g(r,\theta)\rho(r,\theta)}, \qquad b_3(r,\theta) = \frac{aq(r,\theta)Y^{(0,1)}(r,\theta)}{g(r,\theta)p(r,\theta)\rho(r,\theta)},$$

$$b_0(r,\theta)=\frac{p^{(0,1)}(r,\theta)}{g(r,\theta)p(r,\theta)}-\frac{aX^{(0,1)}(r,\theta)}{g(r,\theta)\rho(r,\theta)},\quad b_3(r,\theta)=\frac{aq(r,\theta)Y^{(0,1)}(r,\theta)}{g(r,\theta)p(r,\theta)\rho(r,\theta)},$$
$$c_0(r,\theta)=-\frac{p(r,\theta)X^{(1,0)}(r,\theta)}{h(r,\theta)q(r,\theta)\rho(r,\theta)},\qquad c_3(r,\theta)=\frac{q^{(1,0)}(r,\theta)}{h(r,\theta)q(r,\theta)}+\frac{Y^{(1,0)}(r,\theta)}{h(r,\theta)\rho(r,\theta)}$$
$$d_0(r,\theta)=-\frac{p(r,\theta)X^{(0,1)}(r,\theta)}{g(r,\theta)q(r,\theta)\rho(r,\theta)},\qquad d_3(r,\theta)=\frac{q^{(0,1)}(r,\theta)}{g(r,\theta)q(r,\theta)}+\frac{Y^{(0,1)}(r,\theta)}{g(r,\theta)\rho(r,\theta)},$$

with definitions $p^{(1,0)}(r,\theta):=\partial_r p(r,\theta)$, $p^{(0,1)}(r,\theta):=\partial_\theta p(r,\theta)$, $p^{(0,2)}(r,\theta):=\partial_\theta^2 p(r,\theta)$ etc. Now we will calculate the Einstein's field equations via [67],

$$\star G_c=-\frac{1}{2}\Omega_{ab}\wedge\star\vartheta_c{}^{ab}, \tag{2.54}$$

where $\star G_a=G_{ab}\star\vartheta^b$ and $\star G_a$ is called the Einstein three-form, and the operator $\star$ is defined as a linear operator which sends any p–form α, in n–dimensional manifold, to $(n-p)$–form given by the following,

$$\star\alpha=\frac{1}{p!(n-p)!}\epsilon_{\nu_1\ldots\nu_p\mu_1\ldots\mu_{(n-p)}}\,\alpha^{\nu_1\ldots\nu_p}\,\vartheta^{\mu_1}\wedge\ldots\wedge\vartheta^{\mu_{(n-p)}} \tag{2.55}$$

with volume form defined via the Levi-Civita symbol ε and the determinant of metric tensor $detg_{\mu\nu}=g$ as,

$$\varepsilon=\frac{1}{n!}\sqrt{|g|}\,\epsilon_{\mu_1\ldots\mu_n}\,\vartheta^{\mu_1}\wedge\ldots\wedge\vartheta^{\mu_n}. \tag{2.56}$$

The operator defined in equation (2.55) is called the Hodge dual operator [67]. We also employed the Cartan's second structure equation defined as,

$$\Omega^a{}_b=d\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b, \tag{2.57}$$

to find the curvature two–forms, Ω_{ab}. In order to see how complicated these equations, (2.54), are observe,

$$G_{00}(r,\theta)=\gamma_{13}(r,\theta)+\delta_{23}(r,\theta)+\varepsilon_{12}(r,\theta), \tag{2.58}$$
$$G_{11}(r,\theta)=\beta_{02}(r,\theta)-\delta_{23}(r,\theta)+\sigma_{03}(r,\theta), \tag{2.59}$$
$$G_{22}(r,\theta)=\alpha_{01}(r,\theta)-\gamma_{13}(r,\theta)+\sigma_{03}(r,\theta), \tag{2.60}$$
$$G_{33}(r,\theta)=\alpha_{01}(r,\theta)+\beta_{02}(r,\theta)-\varepsilon_{12}(r,\theta), \tag{2.61}$$

$$G_{03}(r,\theta) = \beta_{23}(r,\theta) - \alpha_{13}(r,\theta), \tag{2.62}$$

$$G_{12}(r,\theta) = \gamma_{23}(r,\theta) - \alpha_{02}(r,\theta), \tag{2.63}$$

where we made the following definitions,

$$\begin{aligned}\gamma_{13}(r,\theta) =& -\frac{1}{2}c_0(r,\theta)\left[a_3(r,\theta) - c_0(r,\theta)\right] + \frac{1}{4}\left[c_0(r,\theta)^2 - a_3(r,\theta)^2\right] \\ & - \frac{{c_3}^{(1,0)}(r,\theta)}{h(r,\theta)} - c_3(r,\theta)^2 - \frac{d_3(r,\theta)h^{(0,1)}(r,\theta)}{g(r,\theta)h(r,\theta)},\end{aligned} \tag{2.64}$$

$$\begin{aligned}\delta_{23}(r,\theta) =& -\frac{1}{2}d_0(r,\theta)\left[b_3(r,\theta) - d_0(r,\theta)\right] + \frac{1}{4}\left[d_0(r,\theta)^2 - b_3(r,\theta)^2\right] \\ & - \frac{c_3(r,\theta)g^{(1,0)}(r,\theta)}{g(r,\theta)h(r,\theta)} - \frac{{d_3}^{(0,1)}(r,\theta)}{g(r,\theta)} - d_3(r,\theta)^2,\end{aligned} \tag{2.65}$$

$$\begin{aligned}\varepsilon_{12}(r,\theta) =& -\frac{1}{g(r,\theta)h(r,\theta)}\left[-\frac{g^{(0,1)}(r,\theta)h^{(0,1)}(r,\theta)}{g(r,\theta)^2} - \frac{g^{(1,0)}(r,\theta)h^{(1,0)}(r,\theta)}{h(r,\theta)^2}\right. \\ & \left. + \frac{g^{(2,0)}(r,\theta)}{h(r,\theta)} + \frac{h^{(0,2)}(r,\theta)}{g(r,\theta)}\right],\end{aligned} \tag{2.66}$$

$$\begin{aligned}\beta_{02}(r,\theta) =& \frac{a_0(r,\theta)g^{(1,0)}(r,\theta)}{g(r,\theta)h(r,\theta)} - \frac{1}{2}b_3(r,\theta)\left[b_3(r,\theta) - d_0(r,\theta)\right] + \frac{1}{4}\left[d_0(r,\theta)^2 - b_3(r,\theta)^2\right] \\ & + \frac{{b_0}^{(0,1)}(r,\theta)}{g(r,\theta)} + b_0(r,\theta)^2,\end{aligned} \tag{2.67}$$

$$\begin{aligned}\sigma_{03}(r,\theta) =& \frac{1}{4}\left[a_3(r,\theta) - c_0(r,\theta)\right]^2 + a_0(r,\theta)c_3(r,\theta) + \frac{1}{4}\left[b_3(r,\theta) - d_0(r,\theta)\right]^2 \\ & + b_0(r,\theta)d_3(r,\theta),\end{aligned} \tag{2.68}$$

$$\begin{aligned}\alpha_{01}(r,\theta) =& -\frac{1}{2}a_3(r,\theta)\left[a_3(r,\theta) - c_0(r,\theta)\right] + \frac{1}{4}\left[c_0(r,\theta)^2 - a_3(r,\theta)^2\right] + \frac{{a_0}^{(1,0)}(r,\theta)}{h(r,\theta)} \\ & + a_0(r,\theta)^2 + \frac{b_0(r,\theta)h^{(0,1)}(r,\theta)}{g(r,\theta)h(r,\theta)},\end{aligned} \tag{2.69}$$

$$\begin{aligned}\beta_{23}(r,\theta) =& \frac{g^{(1,0)}(r,\theta)\left[a_3(r,\theta) - c_0(r,\theta)\right]}{2g(r,\theta)h(r,\theta)} + \frac{{b_3}^{(0,1)}(r,\theta) - {d_0}^{(0,1)}(r,\theta)}{2g(r,\theta)} \\ & + \frac{1}{2}b_0(r,\theta)\left[b_3(r,\theta) - d_0(r,\theta)\right] - \frac{1}{2}b_0(r,\theta)\left[b_3(r,\theta) + d_0(r,\theta)\right] \\ & + b_3(r,\theta)d_3(r,\theta),\end{aligned} \tag{2.70}$$

$$\begin{aligned}\alpha_{13}(r,\theta) =& \frac{{a_3}^{(1,0)}(r,\theta) - {c_0}^{(1,0)}(r,\theta)}{2h(r,\theta)} + \frac{1}{2}a_0(r,\theta)\left[a_3(r,\theta) - c_0(r,\theta)\right] \\ & - \frac{1}{2}a_0(r,\theta)\left[a_3(r,\theta) + c_0(r,\theta)\right] + a_3(r,\theta)c_3(r,\theta) \\ & + \frac{h^{(0,1)}(r,\theta)\left(b_3(r,\theta) - d_0(r,\theta)\right)}{2g(r,\theta)h(r,\theta)},\end{aligned} \tag{2.71}$$

$$\gamma_{23}(r,\theta) = -\frac{1}{4}\left[a_3(r,\theta)+c_0(r,\theta)\right]\left[b_3(r,\theta)-d_0(r,\theta)\right] - \frac{d_3{}^{(1,0)}(r,\theta)}{h(r,\theta)} - \frac{1}{2}c_0(r,\theta)\left[b_3(r,\theta)-d_0(r,\theta)\right] - c_3(r,\theta)d_3(r,\theta) + \frac{c_3(r,\theta)h^{(0,1)}(r,\theta)}{g(r,\theta)h(r,\theta)}, \tag{2.72}$$

$$\alpha_{02}(r,\theta) = -\frac{1}{4}\left[a_3(r,\theta)+c_0(r,\theta)\right]\left[(b_3(r,\theta)-d_0(r,\theta)\right] - \frac{a_0(r,\theta)h^{(0,1)}(r,\theta)}{g(r,\theta)h(r,\theta)} - \frac{1}{2}a_3(r,\theta)\left[b_3(r,\theta)-d_0(r,\theta)\right] + a_0(r,\theta)b_0(r,\theta) + \frac{b_0{}^{(1,0)}(r,\theta)}{h(r,\theta)}. \tag{2.73}$$

Now we will calculate the energy–momentum three-form, $\star T_a$, given by,

$$\star T_a[\Phi^A] = \frac{1}{2}\left[\left(\iota_a d\Phi^A\right)\star d\Phi^A + d\Phi^A \wedge \left(\iota_a \star d\Phi^A\right)\right] \tag{2.74}$$

for the scalar triplet ϕ^A of the global monopole in orthonormal basis defined in (2.49). We chose the following spheroidal coordinates for scalar triplet,

$$x^1 = \sqrt{a^2+r^2}\sin\theta\cos\phi, \quad x^2 = \sqrt{a^2+r^2}\sin\theta\sin\phi, \quad x^3 = r\cos\theta, \tag{2.75}$$

with $x^A x^A = r^2 + a^2\sin^2\theta$. The reason for this choice is the fact that, for the Kerr black hole, when the mass parameter taken to be zero, that is $M \to 0$, the metric reduces to the Minkowski spacetime in spheroidal coordinates and we want this solution to yield the same result whence global monopole term vanishes, i.e. $\eta \to 0$ alongside with the black hole mass M. As a result the scalar field triplet is written as,

$$\Phi^1 = \frac{\eta\sqrt{a^2+r^2}\sin\theta\cos\phi}{\sqrt{a^2\sin^2\theta+r^2}}, \quad \Phi^2 = \frac{\eta\sqrt{a^2+r^2}\sin\theta\sin\phi}{\sqrt{a^2\sin^2\theta+r^2}}, \quad \Phi^3 = \frac{\eta\, r\cos\theta}{\sqrt{a^2\sin^2\theta+r^2}}.$$

With this choice the triplet is also confined on a sphere such that $\phi^A\phi^A = \eta^2$ outside the monopole core. Diagonal elements of the energy–momentum form are,

$$T_{00} = R(r,\theta)\left[A(r,\theta)+B(r,\theta)+C(r,\theta)\right], \tag{2.76}$$

$$T_{11} = R(r,\theta)\left[A(r,\theta)-B(r,\theta)+C(r,\theta)\right], \tag{2.77}$$

$$T_{22} = R(r,\theta)\left[A(r,\theta)+B(r,\theta)-C(r,\theta)\right], \tag{2.78}$$

$$T_{33} = R(r,\theta)\left[A(r,\theta)-B(r,\theta)-C(r,\theta)\right], \tag{2.79}$$

where the following definitions were made for clarity,

$$R(r,\theta) = \frac{\eta^2}{2\left(a^2\sin^2\theta + r^2\right)^2}, \tag{2.80}$$

$$A(r,\theta) = \frac{(a^2+r^2)\sin^2\theta\left(a^2\sin^2\theta + r^2\right)\left[a^2q(r,\theta)^2 + p(r,\theta)^2\right]}{p(r,\theta)^2q(r,\theta)^2[Y(r,\theta) - aX(r,\theta)]^2}, \tag{2.81}$$

$$B(r,\theta) = \frac{r^2\left(a^2+r^2\right)}{g(r,\theta)^2}, \tag{2.82}$$

$$C(r,\theta) = \frac{a^4\sin^2(2\theta)}{4\left(a^2+r^2\right)h(r,\theta)^2}. \tag{2.83}$$

The off diagonal ones are,

$$T_{03} = \frac{a\,\eta^2\left(a^2+r^2\right)\sin^2\theta}{p(r,\theta)q(r,\theta)\left(a^2\sin^2\theta + r^2\right)\left[Y(r,\theta) - aX(r,\theta)\right]^2}, \tag{2.84}$$

$$T_{12} = -\frac{2a^2\eta^2 r\sin(2\theta)}{g(r,\theta)h(r,\theta)\left[a^2 + 2r^2 - a^2\cos 2\theta)\right]^2}. \tag{2.85}$$

In order to confirm the validity of the solution (2.47) obtained using the JNA method, we chose the metric functions as,

$$p^2(r,\theta) = \frac{r^2 + a^2 - 2Mr - \eta^2r^2}{r^2 + a^2\cos^2\theta}, \quad q^2(r,\theta) = \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}, \quad X(r,\theta) = a\sin^2\theta,$$
$$Y(r,\theta) = a^2 + r^2, \quad g^2(r,\theta) = r^2 + a^2\cos^2\theta, \quad h(r,\theta) = \frac{1}{p(r,\theta)}.$$

However these functions did not satisfy the field equations listed above. For example, the Einstein three-form components G_{03} and G_{12} vanishes, but the corresponding energy–momentum form components are non-zero, explicitly given as,

$$T_{03} = \frac{\eta^2\left(a^2+r^2\right)a\sin\theta}{\left(r^2 + a^2\cos^2\theta\right)\left(r^2 + a^2\sin^2\theta\right)\sqrt{r^2 + a^2 - 2Mr - \eta^2r^2}} \neq G_{03}, \tag{2.86}$$

$$T_{12} = -\frac{a^2\eta^2 r\sin 2\theta\sqrt{r^2 + a^2 - 2Mr - \eta^2r^2}}{2\left(r^2 + a^2\cos^2\theta\right)\left(r^2 + a^2\sin\theta\right)^2} \neq G_{12}. \tag{2.87}$$

Therefore, (2.47) cannot represent the solution of the massive, rotating global monopole black hole. Consequently, the modified JN method presented in [66] cannot also be used for obtaining the said solution. Yet, the (2.47) can be a solution to some energy–momentum distribution. What it is is an open question and is a task for future work.

In our search for the correct solution for the massive, rotating global monopole black hole we have also tried the following methods. Firstly, we tried a more general form of the line element than (2.48), involving eight arbitrary functions, given as,

$$\begin{aligned} \mathrm{ds}^2 = &-p^2(r,\theta)[dt - X(r,\theta)d\phi + Z(r,\theta)dr]^2 + g^2(r,\theta)d\theta^2 + h^2(r,\theta)dr^2 \\ &+ q^2(r,\theta)[Y(r,\theta)\mathrm{d}\phi - adt + R(r,\theta)dr]^2. \end{aligned} \tag{2.88}$$

As one can expect, the field equations for this line element are even more complicated and lengthy than the one given before. Hence, we do not list them here. At the end, the line element (2.88) also did not yield a proper, physically meaningful solution.

Secondly, in order to find a more suitable form of a line element to work on, we start with a Cartesian coordinate transformation on the line element of the Schwarzschild-monopole black hole (2.5) given by,

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos\left(\frac{z}{r}\right), \quad \phi = \arctan\left(\frac{y}{x}\right). \tag{2.89}$$

With further transformation defined as,

$$dt = d\tau + \frac{2M + \eta^2\, r}{2M - (1-\eta^2)\, r} dr, \tag{2.90}$$

yields,

$$ds^2 = ds_0^2 + \left(\kappa\eta^2 + \frac{2M}{r}\right) k_\mu k_\nu dx^\mu dx^\nu, \tag{2.91}$$

where ds_0^2 is the Minkowski metric and k_μ is the null vector with respect to both Minkowski and the general spacetime, i.e.,

$$\eta_{\alpha\beta} k^\alpha k^\beta = g_{\alpha\beta} k^\alpha k^\beta = 0, \quad k^\alpha = \eta^{\alpha\beta} k_\beta = g^{\alpha\beta} k_\beta, \tag{2.92}$$

and can be chosen as,

$$k_\alpha = \left[1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right]. \tag{2.93}$$

Eq. (2.91) is the Kerr-Schild form of the Schwarzschild-monopole black hole. Being able to obtain this form of the static line element led us to look for a rotating solution in the Kerr-Schild like coordinates. Yet again, we could not succeed.

2.3.3. Slowly Rotating (Phantom) Global Monopole Black Hole

As our work progressed, we became aware, albeit late, of [62], where the line element for a slowly rotating black hole with a phantom global monopole was presented. Phantom global monopoles differ from regular ones by their kinetic term in the Lagrangian, given as,

$$L_{\text{mon}}[\Phi^a] = \frac{\xi}{2}\partial^\mu\Phi^a\partial_\mu\Phi^a - \frac{\lambda}{4}\left(\Phi^a\Phi^a - \eta^2\right)^2. \tag{2.94}$$

In the regular global monopole Lagrangian (2.1), the coupling constant of the kinetic term, ξ, has a plus sign, whereas phantom global monopoles have a negative. Nevertheless, this does not change the form of the line element in the linear order. The line element ansatz chosen as [62],

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} - 2aF(r,\theta)dtd\phi + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.95}$$

where a is the rotation parameter, angular momentum per unit length. In this background, the diagonal components of the Einstein tensor and the scalar field equations outside the monopole core are trivially solved for,

$$f(r) = 1 - 8\pi\xi\eta^2 - \frac{2M}{r}, \quad F(r,\theta) = h(r)\sin^2\theta, \tag{2.96}$$

except the $t\phi$ component, when the equations are kept in linear order in a, by ignoring the term in $O(a^2)$ or higher. For this system, the field equations reduce to a single differential equation,

$$r^2 f(r)\frac{d^2h(r)}{dr^2} - 2h(r)f(r) - 16\pi\xi\eta^2 h(r) = 0, \tag{2.97}$$

which has the solution,

$$\begin{aligned} h(r) =& \left(\frac{\chi}{r}\right)^{\frac{\zeta-1}{2}} {}_2F_1\left[\frac{\zeta-3}{2}, \frac{\zeta+3}{2}, \zeta+1; \frac{\chi}{r}\right], \\ \chi =& \frac{2M}{1-b}, \quad \zeta = \sqrt{\frac{9-b}{1-b}}, \quad b = 8\pi\xi\eta^2. \end{aligned} \tag{2.98}$$

Here, ${}_2F_1[\alpha,\beta,\gamma;x]$ is the hypergeometric function [68]. One can easily see that neither our line element (2.47) nor the one given in (2.32) is reduced to this line

element in the slow-rotation limit. We see that even in the slow-rotation limit, the solution is very complex and includes a hypergeometric function, which in itself is not easy to work with or analyze. Therefore, we can expect that the exact solution of the rotating global monopole black hole is unlikely to be straightforward. A natural extension of this solution would be the global monopole swallowed by a slowly rotating Reissner-Nordström-(Anti)-de Sitter black hole.

Unfortunately, our endeavor to find the line element for the gravitational field of a rotating global monopole was not successful. Therefore, we focus on our other ideas to proceed with our work for the thesis. However, we should note that based on our work [69] and others, a very recent study shows that there is a no-go theorem for rotating global monopoles [70], which supports and amplifies the importance of our findings.

3. GLOBAL MONOPOLES IN HORNDESKI THEORY

One of the major motivations to work on the modified theories of gravity lie in the discovery of the accelerated expansion of the universe [71,72]. Coupling a scalar field to gravity may explain this mechanism by extending the theory's degrees of freedom and may provide additional flexibility to investigate and explore alternative reasons for this expansion. Our choice was Horndeski theory, since it is the most general scalar-tensor theory with second–order field equations in four dimensions [47]. In the literature, the effects of the global monopole in some alternative theories of gravity are already presented [33, 48–53]. Here, however, by working on the global monopoles in Horndeski theory, we generalize and combine these results within a single framework. Our calculations are limited to the linearized level since the field equations of the complete theory are very complex. Nevertheless, it is still possible to draw meaningful, physical conclusions even in this limit. Hence, in this section, we first review the weak–field equations and the energy–momentum tensor of a global monopole in Horndeski theory. We then present solutions of the linearized field equations of Horndeski theory for the global monopole for both the massless and massive cases. Then we obtain and discuss several physical implications of these solutions, such as the angular deficit they possess, the force they apply on nonrelativistic test particles, and geodesic particle motion around such monopoles for massless and massive cases. The last part of this section is devoted to galaxy rotation curves, to which global monopoles might contribute to both cases. This section of the thesis is based on our work published in [73].

3.1. Field Equations

The Lagrangian of the action $S = \mathcal{L}\sqrt{|g|}d^4x$ of Horndeski theory [44], also known as the generalized Galileon theory [45], can be written as [47]:

$$\begin{aligned}\mathcal{L} &= K(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)\right] \\ &\quad + G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5,X}}{6}\left[(\Box\phi)^3 - 3\Box\phi\,(\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)\right. \\ &\quad \left. + 2(\nabla_\mu\nabla^\nu\phi)(\nabla_\nu\nabla^\lambda\phi)(\nabla_\lambda\nabla^\mu\phi)\right], \end{aligned} \tag{3.1}$$

where $X = -\nabla^\mu\phi\nabla_\mu\phi/2$ is a kinetic term, K, G_3, G_4 and G_5 are arbitrary functions of the scalar field ϕ and kinetic term X. R is the Ricci scalar and ∇ denotes a covariant derivative operator with respect to a metric compatible and torsion free connection. The comma denotes partial differentiation for the variable after the comma, such as $f_{,X} = \partial f/\partial X$. We can add an appropriate minimally coupled matter Lagrangian $\mathcal{L}_m$ to this Lagrangian to explore the behavior of the matter-energy distribution in this theory. The field equations of this theory are rather complex and we will refer to the work [74] for those equations.

3.1.1. The linearized field equations

In this work, we will consider the linearized field equations and their solutions of Horndeski theory. To achieve that, we assume a weak field expansion of the metric tensor and the scalar field as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad \phi = \phi_0 + \varphi, \tag{3.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $h_{\mu\nu}$ is the metric perturbation tensor with $|h_{\mu\nu}| \ll 1$, ϕ_0 a constant and φ is the scalar perturbation function. An arbitrary function of the scalar field and the kinetic term X can be expanded as

$$\begin{aligned} f(\phi, X) &= f(\phi_0, X_0) + f_{,\phi}(\phi_0, X_0)(\phi - \phi_0) + f_{,X}(\phi_0, X_0)(X - X_0) + \ldots \\ &= f(\phi_0, 0) + f_{,\phi}(\phi_0, 0)\varphi + O(\varphi^2) \end{aligned} \tag{3.3}$$

around $\phi = \phi_0$ and $X = X_0 = 0$. Hereafter we use the abbreviation $f(\phi_0, 0) = f(0)$ for clarity.

Reading up linear terms in [74] and adding matter-energy Lagrangian and corresponding energy–momentum tensor, and also setting $K(0) = K_{,\phi}(0) = 0$ to have an asymptotically flat background (characteristic deviations from asymptotic flatness due to monopole term in the energy–momentum tensor will be encoded into $h_{\mu\nu}$), since they can play the role of a cosmological constant [75], we have the following field equations in linear order in h and φ as [76, 77]:

$$G_4(0)G^{(1)}_{\mu\nu} + G_{4,\phi}(0)\left(\eta_{\mu\nu}\Box_\eta\varphi - \varphi_{,\mu\nu}\right) = \kappa\, T^{(1)}_{\mu\nu}, \tag{3.4}$$

$$K_{,\phi\phi}\,(0)\,\varphi + \left[K_{,X}(0) - 2G_{3,\phi}(0)\right]\Box_\eta\varphi + G_{4,\phi}(0)R^{(1)} = 0. \tag{3.5}$$

Substituting the Eq. (3.4) into Eq. (3.5) we bring the scalar field equation in the form as

$$(\Box_\eta - m^2)\varphi = \kappa'\, T^{(1)}, \tag{3.6}$$

with the mass of the scalar field, m, and coupling constant κ' are defined as

$$m^2 = -\frac{K_{,\phi\phi}(0)}{K_{,X}(0) - 2G_{3,\phi}(0) + 3G^2_{4,\phi}/G_4(0)}, \tag{3.7}$$

$$\kappa' = \frac{G_{4,\phi}(0)\,\kappa}{G_4(0)\left[K_{,X}(0) - 2G_{3,\phi}(0)\right] + 3G^2_{4,\phi}(0)}. \tag{3.8}$$

The equation (3.4) can also be put in a convenient form by expanding $G^{(1)}_{\mu\nu}$ in terms of $h_{\mu\nu}$ and then transforming the metric perturbation terms into a new perturbation tensor $\theta_{\mu\nu}$ defined as

$$\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \eta_{\mu\nu}\frac{G_{4,\phi}(0)}{G_4(0)}\varphi. \tag{3.9}$$

We also employ a Lorenz gauge

$$\theta^{\mu}{}_{\nu;\mu} = 0, \tag{3.10}$$

then the equation (3.4) takes a much simple form

$$\Box_\eta\theta_{\mu\nu} = -\frac{2\kappa}{G_4(0)}\,T^{(1)}_{\mu\nu} = -2\kappa_1\,T^{(1)}_{\mu\nu}, \tag{3.11}$$

where we have defined

$$\kappa_1 = \frac{\kappa}{G_4(0)} \tag{3.12}$$

for clarity. The equations (3.6) and (3.11), together with the gauge condition (3.10), are the main equations we will employ to obtain the desired solutions. In the following subsection, we will discuss the linearized energy–momentum tensor that corresponds to global monopoles. The above equations agree with linearized Horndeski field equations employed in the works [76, 77], mainly to the discussion of gravitational waves.

3.2. Global Monopoles in Linearized Horndeski Theory

In this section, we study linearized solutions. We have already re-derived the line element for the Schwarzschild-monopole black hole in Section 2 given by the equation (2.5). The weak field form of (2.5) will be required in determining integration constants for those solutions. Since a global monopole has spherical symmetry, it might be convenient to consider the Minkowski metric in spherical coordinates with $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$. Hence, from now on, in this section we use the Minkowski metric in spherical coordinates, where

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2\sin^2\theta), \tag{3.13}$$

unless stated otherwise. The reason for using these coordinates is the fact that expressing the energy–momentum tensor of a monopole in Cartesian coordinates will give more cumbersome equations to deal with. Hence, from now on, the differential operators assume their flat spherical counterparts with partial derivatives replaced with covariant ones in the relevant equations [78]. Hence, as a limiting case, we consider the weak field line element of a global monopole, Eq. (2.5), as

$$ds^2 = -\left(1 - \kappa\,\eta^2 - \frac{2M}{r}\right)dt^2 + \left(1 + \kappa\,\eta^2 + \frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.14}$$

with $M = 0$ is utilized for pure monopole spacetime. Another form used in the literature is obtained by setting $M = 0$ and rescaling time and radial coordinates [3], which results

$$ds^2 = -dt^2 + dr^2 + (1 - \kappa\eta^2)\, r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.15}$$

describing a spacetime with a global angle deficit generated by the global monopole. The weak field global monopole solutions in GR will serve us as a boundary condition to determine the remaining integration constants obtained by solving the linearized field equations of Horndeski theory. Note that the solution (3.15) also describes a configuration described by Letelier [79], where an ensemble of radially distributed straight cosmic strings intersecting at a common point, which is also called the gravitational hedgehogs [80, 81].

3.2.1. Massless case

Now let us derive the corresponding solution of a global monopole in the linearized Horndeski theory. Namely, we need first to solve the tensor part described by the equation (3.11) for the energy–momentum tensor (2.4). Already knowing particular forms (3.14,3.15) of the global monopole solution [3] in linearized GR, we expect to have a diagonal $h_{\mu\nu}$ which may require a diagonal $\theta_{\mu\nu}$, then we employ the following ansatz

$$\theta_{\mu\nu} = \text{diag}\left(\theta_{00}, \theta_{rr}, \theta_{\theta\theta}\, r^2, \theta_{\theta\theta}\, r^2 \sin^2\theta\right). \tag{3.16}$$

The tensor field equations (3.11), for a global monopole source (2.4), with prime being derivative with respect to the radial coordinate, leading to the following equations

$$\theta''_{00} + \frac{2}{r}\theta'_{00} = -2\kappa_1 \frac{\eta^2}{r^2}, \tag{3.17}$$

$$\theta''_{rr} + \frac{2}{r}\theta'_{rr} - 4\frac{\theta_{rr}}{r^2} + 4\frac{\theta_{\theta\theta}}{r^2} = 2\kappa_1 \frac{\eta^2}{r^2}, \tag{3.18}$$

$$\theta''_{\theta\theta} + 2\frac{\theta'_{\theta\theta}}{r} - 2\frac{\theta_{\theta\theta}}{r^2} + 2\frac{\theta_{rr}}{r^2} = 0. \tag{3.19}$$

Note that the equations (3.17-3.19) correspond to tensor Laplace equations in spherical coordinates and are more complicated than the usual scalar Laplace equations. Note also that the solutions of the equations (3.17-3.19) should satisfy the Lorenz gauge condition (3.10). Indeed, we will use the equation obtained by using this gauge to eliminate one of the variables from the equations (3.18,3.19). Namely, the Lorenz gauge (3.10) gives

$$\theta'_{rr} + 2\frac{\theta_{rr}}{r} = 2\frac{\theta_{\theta\theta}}{r}. \tag{3.20}$$

Eliminating $\theta_{\theta\theta}$ using this equation in (3.18) gives

$$\theta''_{rr} + \frac{4}{r}\,\theta'_{rr} = 2\kappa_1 \frac{\eta^2}{r^2}. \tag{3.21}$$

Now, this equation and (3.17) involve a single variable and can be solved by elementary methods. Moreover, the remaining variable $\theta_{\theta\theta}$ can be obtained using the solution of (3.21) via equation (3.20). The solutions we have found have the form

$$\theta_{00} = a_0 - \frac{a_1}{r} - 2\kappa_1\eta^2 \ln\frac{r}{r_0}, \tag{3.22}$$

$$\theta_{rr} = b_0 - \frac{b_1}{3r^3} + \frac{2}{3}\kappa_1\eta^2 \ln\frac{r}{r_0}, \tag{3.23}$$

$$\theta_{\theta\theta} = b_0 + \frac{b_1}{6r^3} + \frac{\kappa_1\,\eta^2}{3G_4(0)} + \frac{2}{3}\kappa_1\eta^2 \ln\frac{r}{r_0}. \tag{3.24}$$

Here a_0, a_1, b_0, b_1 and r_0 are arbitrary integration constants. Finally, Eq. (3.19) should be satisfied, and indeed it is for the $\theta_{\theta\theta}$ term above, which provides a consistency check for the solutions obtained.

To present $h_{\mu\nu}$ from $\theta_{\mu\nu}$ using the inverse of (3.9), we need the solution of the scalar field equation (3.6) as well. To fulfill this, we first consider the massless case, $m = 0$, in this part, namely the case where the scalar field does not attain a mass. This requires $K_{,\phi\phi} = 0$, since K involves the potential of the scalar field. First, we consider the case where the potential is massless or vanishing. Then, we will consider the massive case in the next subsection. Therefore, from (3.6) we have to solve

$$\varphi'' + \frac{2}{r}\varphi' = -2\kappa'\frac{\eta^2}{r^2}, \tag{3.25}$$

where κ' is defined in (3.8). The solution of this equation has the general form

$$\varphi(r) = \frac{2\kappa' M}{r} - 2\kappa'\eta^2 \ln\frac{r}{r_0}, \tag{3.26}$$

where a constant term is ignored since it can always be integrated into ϕ_0. Using the above results, the metric perturbation tensor have the form

$$\begin{aligned}
h_{00} &= \frac{a_0}{2} - \frac{a_1}{2r} + \frac{3b_0}{2} + \frac{\kappa_1\eta^2}{3} + \tilde{\varphi}(r), && (3.27)\\
h_{rr} &= \frac{a_0 - b_0}{2} - \frac{b_1}{3r^3} - \frac{a_1}{2r} - \frac{\kappa_1\eta^2}{3} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r), && (3.28)\\
h_{\theta\theta} &= \frac{a_0 - b_0}{2} + \frac{b_1}{6r^3} - \frac{a_1}{2r} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r), && (3.29)
\end{aligned}$$

where we have used the abbreviation

$$\tilde{\varphi}(r) = \frac{G_{4,\phi}(0)}{G_4(0)}\,\varphi(r). \tag{3.30}$$

There are four arbitrary integration constants a_0, a_1, b_0 and b_1 in the obtained solutions (3.27-3.29) which need to be determined using the previously known limiting solutions such as (3.14). In the following, we impose conditions to fix the values of these constants, ensuring that this solution reduces to the weak–field BV global monopole solution (3.14). Since there is no $1/r^3$ term in the BV solution (3.14) in g_{rr} and $g_{\theta\theta}$ metric terms in GR, we should set $b_1 = 0$. Also comparing h_{00} with g_{00} term in BV solution (3.14) or similar solution in BD theory [33], we can deduce that we should have $a_1 = -4M/G_4(0)$ and $a_0 = -3b_0 + 4\kappa_1\eta^2/3$. Then the line element becomes

$$\begin{aligned}
ds^2 = &-\left[1 - \kappa_1\eta^2 - \frac{2M}{G_4(0)\,r} - \tilde{\varphi}(r)\right] dt^2 \\
&+ \left[1 - 2b_0 + \frac{2M}{G_4(0)\,r} + \frac{\kappa_1\eta^2}{3} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r)\right] dr^2 \\
&+ \left[1 + \frac{2M}{G_4(0)\,r} - 2b_0 + \frac{2\kappa_1\eta^2}{3} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r)\right] d\Omega^2.
\end{aligned} \tag{3.31}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Note that since $\varphi(r)$ is already at the linear order in M and η^2, and since here we only consider linearized solutions, we can write the

solution in a conformal form as

$$\begin{aligned} ds^2 = \ & [1-\tilde{\varphi}(r)]\left[-\left(1-\kappa_1\eta^2-\frac{2M}{r}\right)dt^2\right. \\ & +\left(1-2b_0+\frac{2M}{G_4(0)\,r}+\frac{\kappa_1\eta^2}{3}-\frac{4}{3}\kappa_1\eta^2\ln\frac{r}{r_0}\right)dr^2 \\ & \left.+\left(1+\frac{2M}{G_4(0)\,r}-2b_0+\frac{2\kappa_1\eta^2}{3}-\frac{4}{3}\kappa_1\eta^2\ln\frac{r}{r_0}\right)r^2d\Omega^2\right]. \end{aligned} \tag{3.32}$$

which will help to compare previous solutions in the literature, such as the BD solution [33]. This solution is still not in the conventional global monopole form, when the scalar field vanishes, such as the Schwarzschild type solution (3.14) presented in [3]. Hence, let us proceed to obtain a solution of the Schwarzschild form (3.14) of global monopole solution first, which means no monopole term in the angular part, requiring $b_0 = \kappa_1\eta^2/3$ in (3.33), together with the following coordinate transformation

$$r = R\left(1+\frac{2}{3}\kappa_1\eta^2\ln\frac{r}{r_0}\right), \tag{3.33}$$

which gives

$$\begin{aligned} ds^2 = \ & [1-\tilde{\varphi}(R)]\left[-\left(1-\kappa_1\eta^2-\frac{2M}{G_4(0)\,R}\right)dt^2+\left(1+\frac{2M}{G_4(0)\,R}+\kappa_1\eta^2\right)dR^2\right. \\ & \left.+R^2\left(1+\frac{2M}{G_4(0)\,R}\right)d\Omega^2\right], \end{aligned} \tag{3.34}$$

at the order of $\mathcal{O}(M)$ and $\mathcal{O}(\eta^2)$. We are still a coordinate transformation away from the BV solution in its usual form. Namely the transformation

$$R = -\frac{M}{G_4(0)}+\sqrt{\frac{M^2}{G_4(0)^2}+r'^2} \tag{3.35}$$

brings the weak field solution to the desired form

$$\begin{aligned} ds^2 = & \big[1-\tilde{\varphi}(r)\big]\left[-\left(1-\kappa_1\eta^2-\frac{2M}{G_4(0)\,r}\right)dt^2\right. \\ & \left.+\left(1+\kappa_1\eta^2+\frac{2M}{G_4(0)\,r}\right)dr^2+r^2d\Omega^2\right]. \end{aligned} \tag{3.36}$$

When the scalar field vanishes, and $G_4(0) = 1$, the obtained solution is precisely the linearized form of the BV solution [3] given in eq (3.14), corresponding to a global monopole swallowed by a black hole in the order $\mathcal{O}(M)$ and $\mathcal{O}(\eta^2)$. We

could also recover another well known form of the solution by applying the following transformation

$$r = \sqrt{1-\kappa_1\eta^2}\tilde{r}, \quad t = \frac{\tilde{t}}{\sqrt{1-\kappa_1\eta^2}}, \tag{3.37}$$

resulting

$$ds^2 = \left[1-\tilde{\varphi}(\tilde{r})\right]\left[-\left(1-\frac{2M}{G_4(0)\,\tilde{r}}\right)d\tilde{t}^2 + \left(1+\frac{2M}{G_4(0)\,\tilde{r}}\right)d\tilde{r}^2 + \tilde{r}^2(1-\kappa_1\eta^2)d\Omega^2\right], \tag{3.38}$$

in which, in the vanishing M limit, gives rise to a solution conformal to the renowned global monopole metric (3.15)

$$ds^2 = \left[1+2\kappa'\eta^2\frac{G_{4,\phi}(0)}{G_4(0)}\ln\frac{\tilde{r}}{r_0}\right]\left[-d\tilde{t}^2 + d\tilde{r}^2 + (1-\kappa_1\eta^2)\,\tilde{r}^2 d\Omega^2\right], \tag{3.39}$$

describing a spacetime with solid angle deficit in the case of vanishing scalar field.

It is clear that this solution is conformal to the GR solution (3.15) and reduces to it, and also the BD [33] and scalar-tensor [48] solutions in the appropriate limits. Hence, we have found the linearized global monopole solution in the massless Horndeski theory. Note that if M given in Eq (3.36) represents a compact object in which a global monopole is swallowed by, then the Newtonian limit requires $G_{4,\phi}\kappa' + 1 = G_4(0)$, which reduces to the well known BD result $\phi_0 = (2\omega+4)/(2\omega+3)$ [46] in which $G_4(\phi) = \phi,\ G_3(\phi) = G_5(\phi) = 0,\ K = 2\omega X/\phi$.

Since, as shown in Fig. 3.1 the conformal factor in Eq. (3.39) increases indefinitely with increasing r, the solution cannot be applicable for asymptotically large values of r. In other words the linear approximation is not valid for large values of r and an exact treatment might be necessary. The solutions (3.36, 3.39) we have presented for massless Horndeski theory also valid for massless subtheories of it such as original BD theory, scalar–tensor theories without potential, etc.

3.2.2. Massive case

If we consider a massive scalar field, namely $m \neq 0$, since tensorial and scalar equations decoupled by defining auxiliary tensor $\theta_{\mu\nu}$ in (3.9), the only change in

the field equations will be in the scalar field equation (3.6), which now becomes a massive scalar field equation

$$\varphi'' + \frac{2}{r}\varphi' - m^2\varphi = -2\kappa'\frac{\eta^2}{r^2}, \tag{3.40}$$

where its well behaving solution reads

$$\varphi = \frac{2\kappa' M\, e^{-mr}}{r} + \kappa'\eta^2\left[\frac{e^{-mr}\,\mathrm{Ei}(m\,r)}{m\,r} - \frac{e^{m\,r}\,\mathrm{Ei}(-m\,r)}{m\,r}\right]. \tag{3.41}$$

Here $\mathrm{Ei}(x)$ is the exponential integral function [68]. Note that the general forms of the solution obtained in the previous section are still valid for massive Horndeski theory, since the mass of the scalar field does not affect the tensor $\theta_{\mu\nu}$. Hence, the global monopole solution corresponding to linearized massive Horndeski theory can be given by either (3.36), (3.38) or (3.39) where the scalar field φ should be given by (3.41). Inspecting the mass term of the compact object, M, reveals that now we have an effective gravitational coupling term $G(r) = \left(G_{4,\phi}(0)\kappa' e^{-m\,r} + 1\right)/G_4(0)$, which reduces to corresponding sub cases for example the massive BD case [57, 82]. In the limit of vanishing monopole, $\eta = 0$, the solution (3.41) reduces to that of a point particle in massive Horndeski theory [76].

We can again ignore the mass of the monopole term, M, for the time being, and set $M = 0$ in φ. The metric perturbation tensor $h_{\mu\nu}$ have exactly the same form as in (3.27,3.28,3.29) with the scalar field term reads as

$$\tilde{\varphi} = \frac{G_{4,\phi}(0)}{G_4(0)}\,\varphi = \frac{G_{4,\phi}(0)}{G_4(0)}\kappa'\eta^2\left[\frac{e^{-mr}\,\mathrm{Ei}(m\,r)}{m\,r} - \frac{e^{m\,r}\,\mathrm{Ei}(-m\,r)}{m\,r}\right]. \tag{3.42}$$

The general form of the metric (3.36) is also valid for massive Horndeski theory, with the only difference being that the scalar field now is given by (3.42). Namely we have

$$ds^2 = (1-\tilde{\varphi})\left[-dt^2 + dr^2 + \left(1-\kappa_1\eta^2\right)r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \tag{3.43}$$

where the term $\tilde{\varphi}$ now carries the effects of the mass of the scalar field, m, given in eq (3.42). When we look at the behavior of the scalar field, we see that far from the monopole, it vanishes, and the monopole metric effectively becomes similar to GR

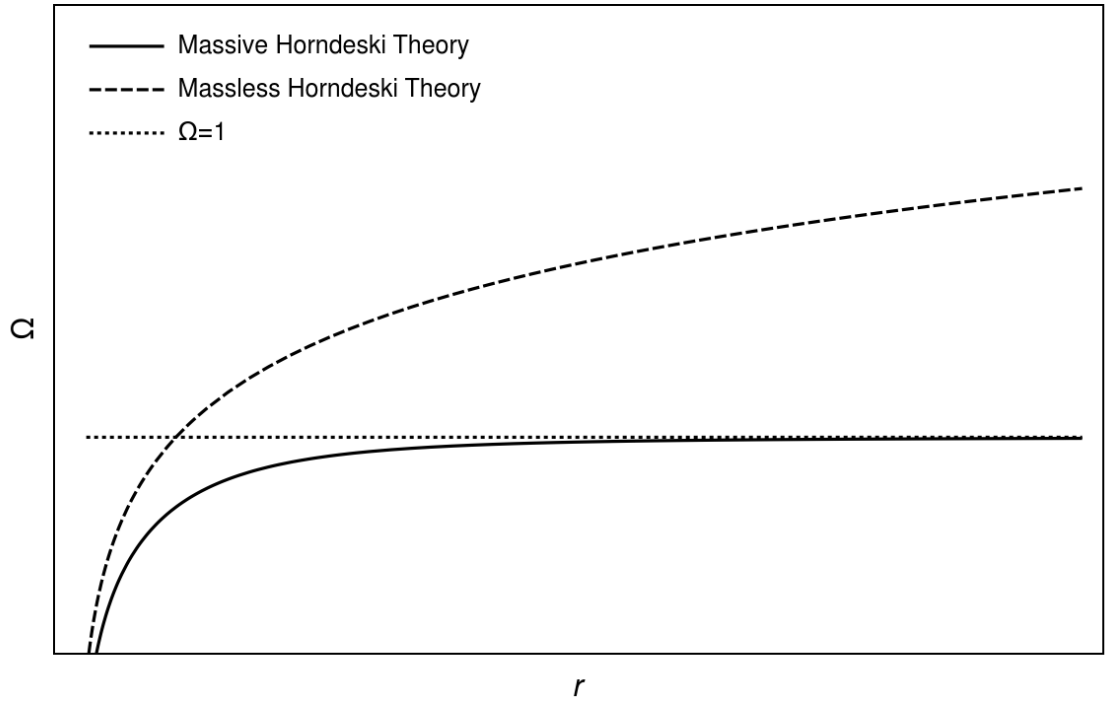


Figure 3.1 Behavior of the conformal factors ($\Omega = 1-\tilde{\varphi}$) of massless (3.39), massive (3.43) and general relativity ($\Omega = 1$) cases. It is clear that in the massive case, the conformal factor approaches unity, i.e., the GR limit for large r values, whereas in the massless case, it increases with r. Note that plots are not scaled. We choose consistent values such that these behaviors are clear to see on a single graph.

monopoles (3.15), namely

$$ds^2_{r\to\infty} \approx -dt^2 + dr^2 + \left(1-\kappa_1\eta^2\right) r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{3.44}$$

Hence, the mass of the scalar field produces a screening mechanism that diminishes the effects of the scalar field in the far region. This shows that in the presence of a massive scalar field, the behavior of the global monopoles changes dramatically from those living in massless scalar fields in Horndeski theory. Asymptotic behavior of global monopoles in massive Horndeski theory resembles that of GR. This behavior is seen in Fig. 3.1. The monopole solutions we have derived for massive Horndeski theory are also valid for subtheories of Horndeski, such as BD with a potential, scalar-tensor theories with a potential, $f(R)$ theory, etc, which were not discussed before, as far as we know. We summarize some of the differences between massless and massive cases in the following section.

3.3. Properties of the Global Monopoles in Horndeski Theory

3.3.1. Deficit angle, Newtonian force on nonrelativistic test particles and light deflection

Let us now discuss some of the physical properties of the solutions obtained. The area of a sphere of radius r around a monopole in massless Horndeski theory, from (3.39), is given by

$$A = 4\pi r^2 \left(1 - \kappa_1 \eta^2 - \tilde{\varphi}\right), \tag{3.45}$$

which resembles the GR [9] and the BD [33] results in the appropriate limits. The area of a sphere of radius r and the deficit angle is now clearly a function of the scalar field in accordance with the corresponding BD solution instead of its constant value in GR.

Let us discuss the area formula for massive Horndeski theory. Since the area of a sphere surrounding the monopole is given as in (3.45), the only change is the scalar function $\varphi(r)$. In this case, it is given by (3.42). Due to this, if the sphere radius is small, the expression for the areas deviates from both the GR and massless Horndeski monopoles. However, if the radius of the sphere increases, for large r, the area formula, as well as solid angle deficit, becomes similar to the GR one with only difference being the coupling constant κ_1.

The solution of massless Horndeski theory can be put into Galilean coordinates by the transformation

$$T = (1 - \kappa_1 \eta^2) t, \tag{3.46}$$

$$R = r\left(1 - \frac{\kappa_1 \eta^2}{2} + \frac{\kappa_1 \eta^2}{2} \ln \frac{r}{r_0}\right), \tag{3.47}$$

giving

$$\begin{aligned} ds^2 &= [1-\tilde{\varphi}(R)]\left[-\left(1+\eta'^2\right)dT^2+\left(1-\eta'^2\ln\frac{R}{r_0}\right)\left(dX^2+dY^2+dZ^2\right)\right] \\ &= -\left[1+\kappa_1\eta^2+\tilde{\varphi}(R)\right]dT^2+\left[1-\kappa_1\eta^2\ln\frac{R}{r_0}-\tilde{\varphi}(R)\right]\left(dX^2+dY^2+dZ^2\right), \end{aligned}$$

with $R^2=X^2+Y^2+Z^2$. One advantage of these Galilean coordinates is to obtain the motion of a non-relativistic test particle using the equation

$$\ddot{X}^i=-\frac{1}{2}\frac{\partial h_{00}}{\partial X^i}=\frac{1}{2}\frac{\partial\tilde{\varphi}(R)}{\partial X^i}=-2\kappa'\eta^2\frac{G_{4,\phi}(0)}{G_4(0)}\frac{X^i}{R}. \tag{3.48}$$

These results indicate that, unlike the GR case in which the global monopole exerts no force on the test particles, nonrelativistic test particles around the global monopole in massless Horndeski theory experience an attractive force from it. In the BD limit, this expression reduces to the one given in [33].

Another change due to a massive scalar field is the force acting on non-relativistic test particles around the monopole in massive Horndeski theory. In the near region of the monopole, the particles feel a similar force. However, when the distance between the monopole and the test particles increases, the force reduces dramatically and vanishes practically at large distances. Hence, unlike the massless case, apart from the near region, the force acting on test particles becomes zero as in the GR case for massive Horndeski theory.

Note that since Horndeski monopole solution (3.39) is conformally related to the corresponding GR solution [3], the behavior of light rays has the same behavior as in GR, with only differences being related to the gravitational coupling constant. Therefore, if one considers the case where a light source S and an observer O are perfectly aligned with the global monopole, the image becomes a ring with an angular diameter

$$\delta\Omega=\kappa_1\eta^2 l/(d+l), \tag{3.49}$$

where d is the distance between the monopole and the observer, and l is from the monopole to the source. Determining $\delta\Omega$ can lead to the determination of the coupling constants in κ_1. Note that the null geodesics and hence the angular

diameter formula (3.49) do not change for both classes of monopoles since the effect of the massive scalar field manifests itself as a conformal factor, which does not affect the null geodesics.

3.4. Geodesic Motion

3.4.1. Radial geodesics and effective potential

Let us investigate particle and light motion around a global monopole in Horndeski theory. Using the line element (3.39,3.43) for *a bare monopole* without mass, let us consider the Lagrangian

$$2\mathcal{L} = -A\,\dot{t}^2 + B\,\dot{r}^2 + C\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = \epsilon, \tag{3.50}$$

where λ is an affine parameter, the overdot represents derivative with respect to λ, $\epsilon = -1$ for massive particles and $\epsilon = 0$ for photons, $A = B = 1 - \tilde{\varphi}$ is the conformal factor in (3.39) for massless case or in (3.43) massive case and $C = A(1 - \kappa_1\eta^2)r^2$. Since the problem is spherically symmetric, the motion takes place in a plane, and we can align coordinates such that the motion takes place in the equatorial plane with $\theta = \pi/2$. Then, the first integrals of the motion become

$$\dot{t} \;\; = -E/A, \tag{3.51}$$

$$\dot{\phi} \;\; = L/C, \tag{3.52}$$

where the integration constant E is the specific energy and L is the specific angular momentum of the particle. In this part, we consider a bare monopole by ignoring its mass. From the Lagrangian itself, we find

$$\dot{r}^2 \;\; = \;\; \frac{1}{A^2}\left[E^2 - \frac{L^2}{(1-\kappa_1\eta^2)r^2} + \epsilon A\right]. \tag{3.53}$$

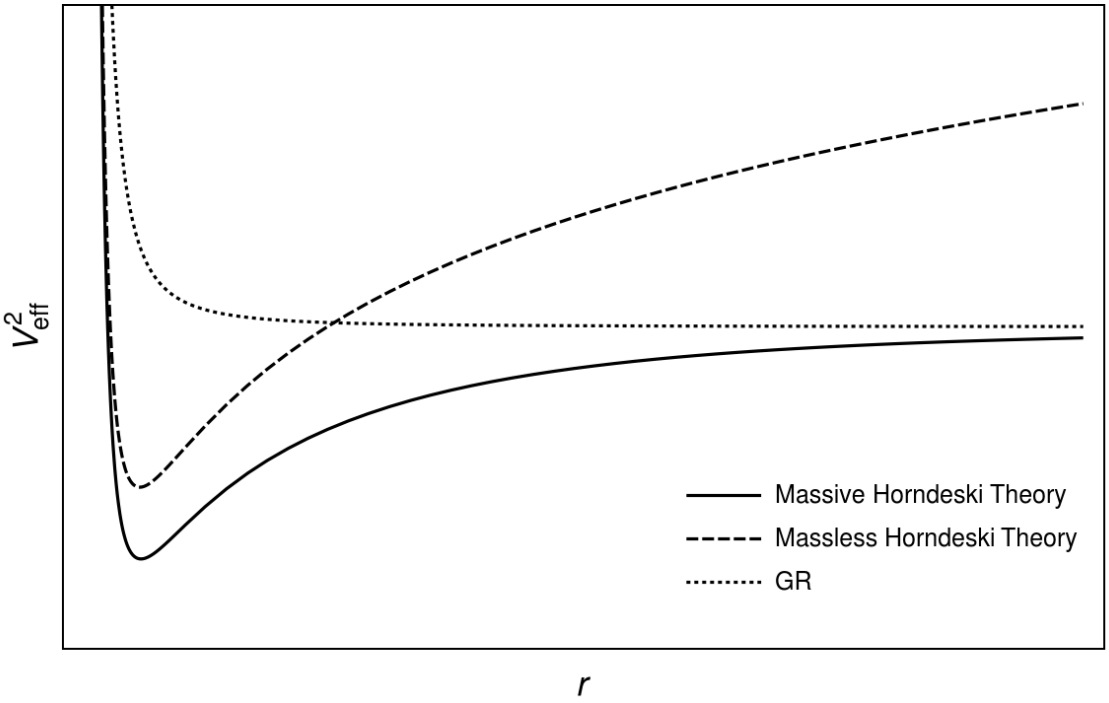


Figure 3.2 Behavior of the effective potential (3.54) for massless (3.39), massive (3.43) cases for *massive* test particles. It is clear that for the global monopoles in the massive or massless Horndeski theory, the massive particles are bounded for small r. We used physical values such that all constants equal to unity to see the behavior of the graph. Hence, the graphs are not scaled.

Defining a new affine parameter as $d\gamma = A^{-1}\, d\lambda$ yields,

$$\begin{aligned} \left(\frac{dr}{d\gamma}\right)^2 &= E^2 - V_{eff}^2, \\ V_{eff}^2 &= \frac{L^2}{(1-\kappa_1\eta^2)r^2} - (1-\tilde{\varphi}), \\ &\approx (1+\kappa_1\eta^2)\frac{L^2}{r^2} - (1-\tilde{\varphi}), \end{aligned} \tag{3.54}$$

where V_{eff} is the effective potential and at the last step we have used the linear approximation we adopted throughout in this section.

The turning points of the motion of a massive test particle are given by

$$E^2 = V_{eff}^2 = (1+\kappa_1\eta^2)\frac{L^2}{r^2} + (1-\tilde{\varphi}). \tag{3.55}$$

The behavior of the conformal factor outside the global monopole core, $1-\tilde{\varphi}$ term, can be seen in the Fig. 3.1. We see that for both massive and massless Horndeski

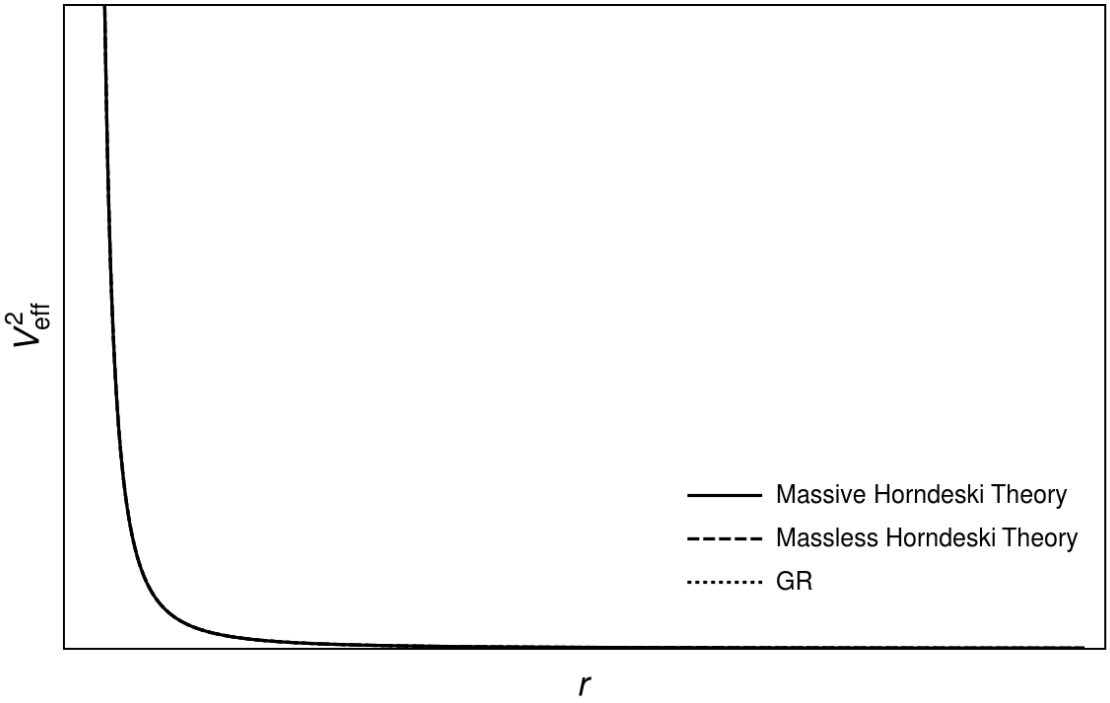


Figure 3.3 Behavior of the Effective Potential (3.54) for massless (3.39) and massive (3.43) cases for *massless* test particles. There are no bound states here as expected, and both massive and massless Horndeski theory reproduces the general relativity result. The scalar functions are used such that all constants equal to unity to see the behavior in the graph.

theory, the conformal factor is an increasing function of r. The difference is that in massless theory, massive test particles are always subject to a bounded motion around global monopoles. This cannot be said for massive theory since we only observe an upward trend until we hit the general relativity limit. These behaviors can be seen in the Fig. 3.2. Hence, the behavior of effective potential clearly distinguishes massless or massive cases. The motion of photons in the spacetime of linearized Horndeski theory has similar characteristics. For both massive and massless theories, photons reach radial infinity, and there is no bounded motion. This behavior can be seen in the Fig. 3.3.

3.4.2. Circular orbits

The form of the effective potential for both cases suggests that around a global monopole in Horndeski theory, circular orbits having $r = r_c$ with $\dot{r} = 0$ present

for massive test particles. Indeed, let us first calculate $E^2 = V_{eff}^2$ and $V'_{eff} = 0$ simultaneously to have the energy and angular momentum for a test particle to follow a circular geodesic for massless Horndeski case as:

$$E^2 = 1 + \frac{G_{4,\phi}(0)}{G_4(0)}\kappa'\eta^2\left[1 + 2\ln\frac{r_c}{r_0}\right], \tag{3.56}$$

$$L^2 = \frac{G_{4,\phi}(0)}{G_4(0)}\kappa'\,\eta^2 r_c^2. \tag{3.57}$$

This circular geodesic is stable since $V''_{eff} = 4G_{4,\phi}(0)\kappa\eta^2/(G_4(0)\,r_c^2) > 0$. For massive Horndeski theory, the resulting expressions for circular timelike geodesics are more complicated. Hence, we refer to them implicitly as

$$E^2 = 1 + \frac{L^2(1+\kappa_1\eta^2)}{r_c^2} - \tilde{\varphi}(r_c), \tag{3.58}$$

$$L^2 = -\frac{1}{2}r_c^3\tilde{\varphi}'(r_c). \tag{3.59}$$

Similar to the massless case, the result has similar characteristics: The second derivative of the potential for r_c is positive. Hence, the circular geodesics around a bare monopole in massive Horndeski theory are also stable.

These results permit us to speculate that the form of the effective potential and the existence of stable circular orbits may imply that, unlike GR, bare global monopoles of Horndeski theory can contribute to the structure formation in the early universe, such as galaxy and black hole formation processes, among other known mechanisms [9], by accumulating material around them due to their attractive gravitational potential, which do not exist for global monopoles in GR.

3.5. Galaxy Rotation Curves

It was shown in [83–87] that global monopoles in BD theory can play a role in the flattening of galaxy rotation curves. Here we investigate whether global monopoles of massive or massless Horndeski theory can also have the same property. To model this phenomenon, we consider a spherical static galaxy with mass M. We need to calculate the rotational velocity of stars around this galaxy with a global monopole

at the center. For the Lagrangian (3.50) with first integrals of motion (3.51,3.52) for a massive particle, we find the following radial equation

$$\dot{r}^2 + U(r) = 0, \tag{3.60}$$

with

$$U(r) = \frac{1}{B}\left(\frac{L^2}{C} - \frac{E^2}{A} + 1\right). \tag{3.61}$$

We want these stars to follow a circular, stable orbit around this galaxy. The stability condition for such orbits can be expressed as:

$$\dot{r} = U(r) = 0,\ \frac{dU}{dr} = 0,\ \frac{d^2U}{dr^2} > 0. \tag{3.62}$$

The first two conditions give the energy and the angular momentum of test particles following stable circular orbits as

$$E^2 = \frac{A^2C'}{AC' - A'C}, \tag{3.63}$$

$$L^2 = \frac{C^2A'}{AC' - CA'}. \tag{3.64}$$

The second derivative of the potential evaluated at the extrema reads

$$U'' = \frac{2A'C'}{ABC} + \frac{C'A'' - A'C''}{B(AC' - CA')}. \tag{3.65}$$

Note that for $C(r) = r^2$, these expressions reduce to the ones found in [88].

Using the definition of four velocity $U^\mu = dx^\mu/d\tau = (\dot{t}, \dot{r}, \dot{\theta} = 0, \dot{\phi})$, and the proper time relation $d\tau^2 = -ds^2$ we obtain that

$$A(r)(U^0)^2(1 - v^2) = 1, \tag{3.66}$$

where U^0 is the time component of the four velocity and v is the spatial velocity of the particle given by

$$v^2 = \frac{1}{A}\left[B\left(\frac{dr}{dt}\right)^2 + C\left(\frac{d\phi}{dt}\right)^2\right] := (v^r)^2 + (v^\phi)^2, \tag{3.67}$$

with v^r and v^ϕ being the components of the velocity v observed in an orthonormal

coordinate system. The angular component v^ϕ is given by

$$v^\phi = \sqrt{\frac{C}{A}}\,\Omega \tag{3.68}$$

where Ω is the angular velocity of the particle, which can be calculated using the first integrals of the motion as

$$\Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{AL}{CE} = \frac{A'}{C'}, \tag{3.69}$$

in which we have used the circular geodesics conditions (3.63,3.64) at the last step of the calculation. Using this result, the tangential velocity of a particle in a stable circular motion is found as

$$(v^\phi)^2 = \frac{C\,A'}{A\,C'}. \tag{3.70}$$

3.5.1. Massless Horndeski Theory

Reading the metric functions $A(r)$, $B(r)$ and $C(r)$ from the line element (3.36) with the help of the scalar field (3.26) for the massless Horndeski theory, we find that

$$(v^\phi)^2 = \frac{M}{G_4(0)\,r} - \frac{r\,\tilde{\varphi}'}{2} \tag{3.71}$$

$$= \left[\frac{1+\kappa' G_{4,\phi}(0)}{G_4(0)}\right]\frac{M}{r} + \frac{\kappa' G_{4,0}(0)}{G_4(0)}\eta^2. \tag{3.72}$$

We see that near the galactic center, we have the usual $1/r$ term due to the mass of the galaxy. Whereas far from it, the effect of the monopole is to contribute a constant term which may help to straighten the galaxy rotation curves. This result is in accordance with previous ones obtained for subtheories of massless Horndeski theory such as BD theory [85–87].

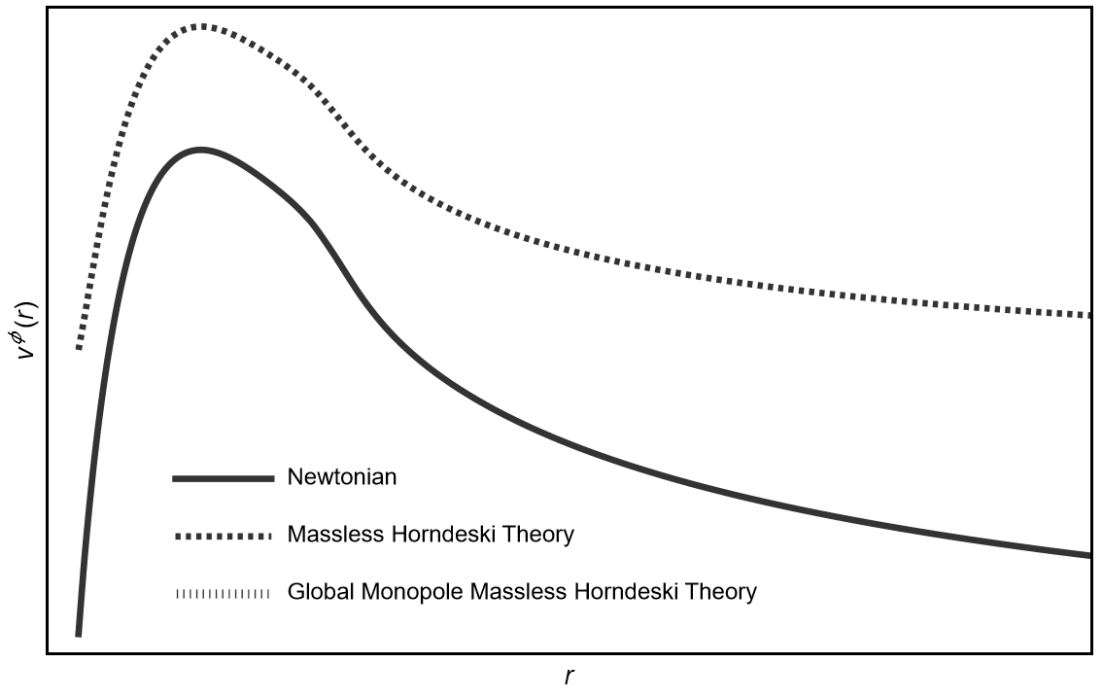


Figure 3.4 Behavior of the GRC in the massless monopole Horndeski theory for $\alpha = -0.92$ and $\eta \propto 10^{-4}$.

3.5.2. Massive Horndeski Theory

For the massive Horndeski theory, similarly using the line element (3.36) with now the scalar field (3.26) we find that

$$\begin{aligned}(v^{\phi})^2 &= \left[1+\kappa' G_{4,\phi}(0)(1+m\,r)e^{-mr}\right]\frac{M}{G_4(0)r} \\ &\quad +\frac{G_{4,\phi}(0)^2}{2G_4(0)^2}\kappa'\eta^2\left[\frac{(1+mr)\,e^{-mr}\,\mathrm{Ei}(m\,r)}{m\,r}-\frac{(1-mr)\,e^{m\,r}\,\mathrm{Ei}(-m\,r)}{m\,r}\right].\end{aligned} \tag{3.73}$$

Let us investigate this result for various cases. When we ignore the monopole term, then the expression of the rotational velocity can be put into the form:

$$(v^{\phi})^2 = \frac{G_{\infty}M}{r}\left[1+\alpha e^{-\frac{r}{r_0}}\left(1+\frac{r}{r_0}\right)\right], \tag{3.74}$$

which is exactly the expression found in [89] by Sanders for a hypothetical Yukawa type phenomenological potential where r_0 is the length scale, and α is the coupling constant of this Yukawa type potential, and G_{∞} is the gravitational coupling constant measured at the infinity. It was shown in [89] that for the range

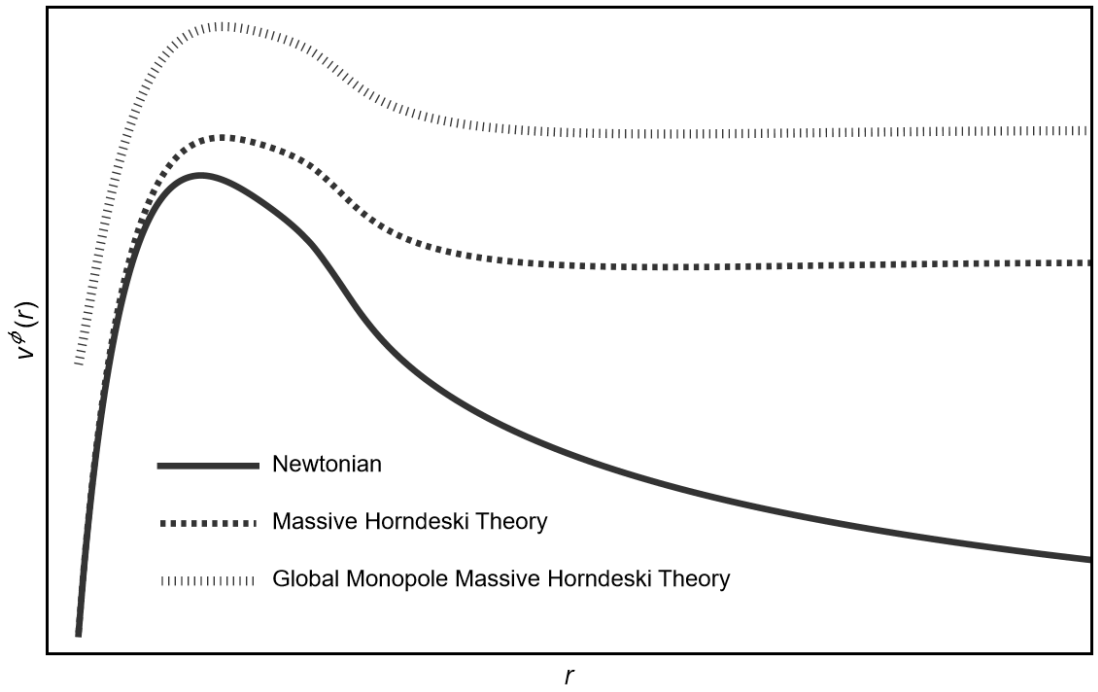


Figure 3.5 Behavior of the GRC in the massive monopole Horndeski theory for $\alpha = -0.92$, $r_0 = 40$ kpc with $\eta \propto 10^{-4}$.

$-0.95 < \alpha < -0.90$, the general properties of extended galactic rotation curves are recovered. From (3.73) we have the identifications $r_0 = 1/m$, $\alpha = \kappa' G_{4,\phi}(0)$ and $G_\infty = 1/G_4(0)$. From these, we see that in order for the Yukawa potential originated from the mass of the scalar field to explain observed galaxy rotation curves is the coupling constants to satisfy $-0.95 < \kappa' G_{4,\phi}(0) < -0.90$. For massive Brans–Dicke theory [57], this corresponds approximately to $-2.06 < \omega < -2.03$ for BD parameter, ω, which is a very unfavorable and restricted value for this parameter. Hence, only for very restrictive and negative values of α, the flat rotational curves of galaxies can be attributed to the massive scalar field in Horndeski theory without a global monopole.

Now, let us discuss whether the existence of a global monopole changes this result. If the mass of the scalar field is "heavy", i.e., $m \to \infty$, then the second term in the expression (3.73) vanishes. Therefore, for a heavy massive scalar field, the monopole term cannot explain the flattening of the rotation curves. For intermediate values of m, the second term in (3.73) is a decreasing positive function of radial

distance, hence it cannot explain the flat rotational curves. For a light scalar field, i.e., $m \to 0$, the second term in (3.73) reduces to (3.71), and it can contribute to the flattening of the rotational velocity of a star around a spherical galaxy, similar to the massless scalar field case. In summary, only if the mass of the scalar field is quite small, then we obtain the novel result that the global monopoles may possibly be responsible for the flattening of the rotational curves of the galaxies.

4. STRAIGHT COSMIC STRINGS IN HORNDESKI THEORY

In this section, we again consider the linearized field equations of Horndeski theory for straight strings. We have already been studied the linearized field equations and the details of these equations in Section 3.1, and here we only refer to the relevant equations to obtain the solutions for clarity. We begin with solutions corresponding to straight cosmic strings. This section is based on our work in [90].

4.1. Solutions for Straight Cosmic Strings

The energy–momentum tensor for a straight cosmic string lying along the z–axis, in the zero–thickness approximation, is given by [2]

$$T_{\mu\nu} = \mu\ \delta(x)\delta(y)\mathrm{diag}(1,0,0,-1), \tag{4.1}$$

in Cartesian coordinates with $x^\mu = (t, x, y, z)$, where μ is the tension of the string. For this energy–momentum tensor, the nontrivial, linearized Horndeski field equations (3.11) read

$$\Box\theta_{00} = -\Box\theta_{zz} = \frac{-2\kappa}{G_4(0)}\mu\,\delta(x)\delta(y), \tag{4.2}$$

with solutions

$$\theta_{00} = -\theta_{zz} = \frac{\kappa\,\mu}{\pi G_4(0)}\ln\frac{r}{r_0}, \quad \theta_{xx} = \theta_{yy} = 0. \tag{4.3}$$

Here $r = \sqrt{x^2+y^2}$ and r_0 is an integration constant. The scalar field equation, (3.6), now becomes

$$(\Box_\eta - m^2)\varphi = -2\kappa'\,\mu\,\delta(x)\delta(y). \tag{4.4}$$

The solutions of this equation will be different for massless and massive scalar field cases. Hence, we obtain them separately in the following.

4.1.1. Massless scalar field case

When the function K in the action (3.1) is at most linear in the scalar field ϕ, the mass of the scalar field m defined in (3.7) vanishes. Therefore, the scalar field equation (4.4) has the following solution for massless case, i.e. $m = 0$,

$$\varphi = -\frac{\kappa' \mu}{\pi} \ln \frac{r}{r_0}. \tag{4.5}$$

Using this solution, (3.9) and (4.3), the metric perturbation tensor has nontrivial components given by

$$h_{00} = -h_{zz} = -\frac{G_{4,\phi}(0)}{\pi G_4(0)} \kappa' \mu \ln \frac{r}{r_0}, \tag{4.6}$$

$$h_{xx} = h_{yy} = \frac{G_{4,\phi}(0)\kappa' - \kappa}{\pi G_4(0)} \mu \ln \frac{r}{r_0}. \tag{4.7}$$

Hence, employing the cylindrical coordinates, the spacetime metric reads

$$ds^2 = \left[1 + \frac{G_{4,\phi}(0)}{\pi G_4(0)} \kappa' \mu \ln \frac{r}{r_0}\right] \left\{ -dt^2 + dz^2 + \left[1 - \frac{\kappa \mu}{\pi G_4(0)} \ln \frac{r}{r_0}\right] \left(dr^2 + r^2 d\theta^2\right) \right\}.$$

The transformation $r = r_0 \, (r'/a)^b$ with $a = [1 - \kappa/(2\pi G_4(0))\mu]^{-1/2}$ and $b = [1 - \kappa/(4\pi G_4(0))\mu]^{-1}$, adapted from [91], bring the line element into a more familiar form

$$ds^2 = \left[1 + \frac{G_{4,\phi}(0)}{\pi G_4(0)} \kappa' \mu \ln \frac{r'}{r_0}\right] \left\{ -dt^2 + dz^2 + dr'^2 + \left[1 - \frac{\kappa \mu}{\pi G_4(0)}\right] r'^2 d\theta^2 \right\}. \tag{4.8}$$

Hereafter, we will set $r_0 = 1$ without loss of generality. Under the GR limit $\kappa' = 0, G_4(0) = 1$, $G_3 = G_5 = K = 0$, this solution reduces to Vilenkin's weak field solution [2], whereas in the BD limit, i.e. $G_4 = \phi$, $G_3 = G_5 = 0$, $K = 2\omega X/\phi$, this solution reduces to the weak field solution presented in [32, 91] which is conformal to Vilenkin's solution. When the conformal factor survives, the solution is valid only near the string, since the weak field approximation breaks down towards radial infinity, as in the BD case [32, 91]. Apart from GR, however, for any subclass of Horndeski theory where $G_{4,\phi}(0) = 0$ and the conformal factor vanishes, including Einstein-scalar theory, Vilenkin's weak field solution is also a solution for any value of radial coordinate, at least in the linearized regime. Whether the mass of the scalar field remedies this behavior will be the subject of the following subsection.

4.1.2. Massive scalar field case

In this case, the general solution of the scalar field equation (4.4) outside the string core becomes

$$\varphi(r) = c_0\, I_0(mr) + c_1\, K_0(mr) - \frac{\kappa'\mu}{\pi}\, K_0(mr), \tag{4.9}$$

where I_α and K_α are modified Bessel functions of the first and second kind, and c_0 and c_1 are arbitrary integration constants, and they should be in the same order as $\kappa'\mu$. In order to fix the integration constants c_0 and c_1, we choose the following boundary conditions: i) we require the scalar field to vanish at the radial infinity, $\varphi(r \to \infty) \approx 0$, ii) near the string, the solution to resemble similar behavior with the massless case. Then, for i) we can choose $c_0 = 0$ since I_0 blows up, but K_0 vanishes at infinity. This is in contrast to the corresponding solution in the massless case. The boundary condition ii) can be satisfied if we choose, without loss of generality, $c_1 = 2\kappa'\mu/\pi$. Note that with this choice, the scalar field of the massless case and the massive case near the string have similar behavior.

We will show that this choice is justified when we discuss the conformal factors of both massless and massive cases in the following subsection. Under these assumptions, the massive scalar field becomes

$$\varphi(r) = \frac{\kappa'\mu}{\pi}\, K_0(mr). \tag{4.10}$$

For the massive scalar field case, we obtain the following line element

$$ds^2 = \left[1 - \frac{G_{4,\phi}(0)}{\pi\, G_4(0)}\kappa'\mu\, K_0(mr)\right]\left[-dt^2 + dz^2 + \left[1 - \frac{\kappa\mu}{\pi\, G_4(0)}\ln r\right]\left(dr^2 + r^2 d\theta^2\right)\right].$$

We can also express this line element in the transformed coordinates as given in [2], using again [91] as in previous subsection, it becomes linear in μ as

$$ds^2 = \left[1 - \frac{G_{4,\phi}(0)}{\pi G_4(0)}\kappa'\mu\, K_0(mr')\right]\left[-dt^2 + dz^2 + dr'^2 + \left(1 - \frac{\kappa\,\mu}{\pi G_4(0)}\right) r'^2 d\theta^2\right]. \tag{4.11}$$

For the large values of r', the function K_0 diverges towards zero, hence the line

element becomes

$$ds^2_{r\to\infty} \approx -dt^2 + dz^2 + dr'^2 + \left[1 - \frac{\kappa\,\mu}{\pi G_4(0)}\right] r'^2 d\theta^2, \tag{4.12}$$

which clearly approaches Vilenkin's solution in GR. Therefore, we see that the general behavior of cosmic strings in massive Horndeski theory differs from the massless case. In some sense, the mass of the scalar field screens the effects of the scalar field, leading to the GR result far from the string. It is well known that massive theories show this behavior [58–60], where the scalar field freezes beyond a point. For distant regions from the string, we have a locally flat spacetime with an angle deficit, whereas the behavior of particles near the string will be altered by the conical space behavior of GR cosmic strings. However, the behavior of photons will be the same as in GR since the line elements are conformal to the GR solution.

4.2. Properties of Horndeski Cosmic Strings

4.2.1. Conformal Factor

The solutions of Horndeski cosmic strings are conformal to those of GR. The conformal factor determines the character and differences of these objects in the corresponding theories. In the case of the massless Horndeski theory, the conformal factor is an increasing function of r; as a result, the solution is valid only near the string where the linear approximation is valid. However, for the far region, one needs to look for exact solutions to describe the behavior of these strings. By contrast, in the massive Horndeski theory, the conformal factor asymptotically approaches unity, i.e., the GR solution, which makes the solution valid everywhere. These distinct behaviors can be clearly seen in Fig. 4.1.

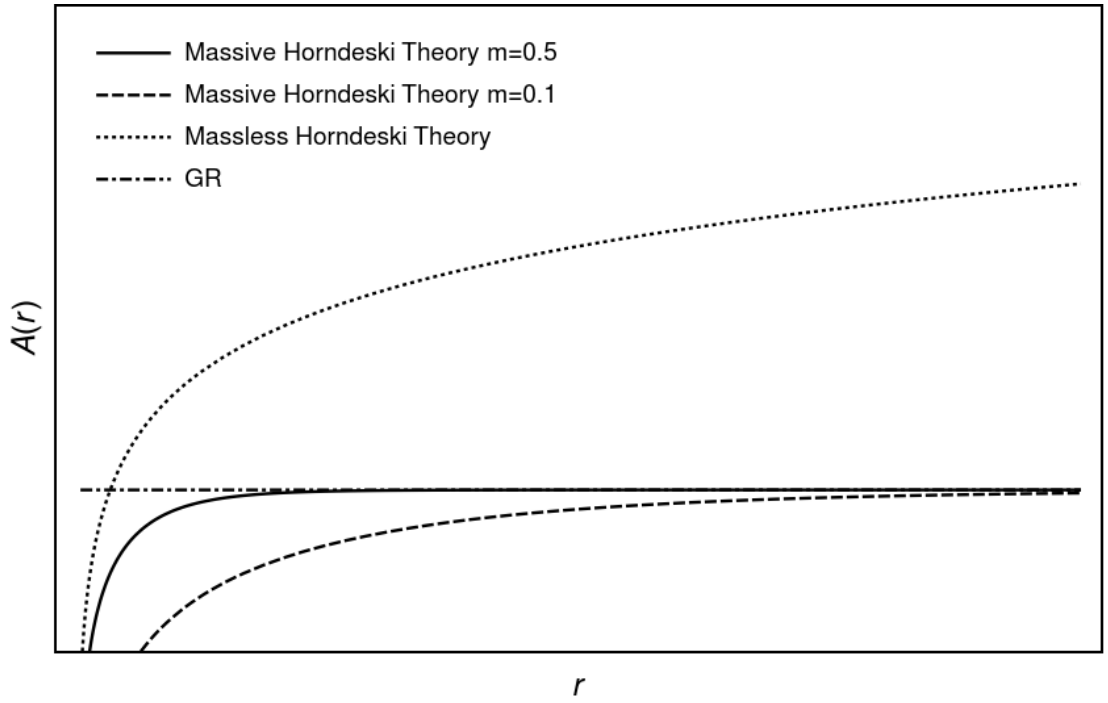


Figure 4.1 Behavior of the conformal factors of massless, massive with $m = 0.1$, $m = 0.5$ and general relativity cases. It is clear that in the massive case, the conformal factor approaches unity, i.e., the GR limit for large r values, whereas in the massless case it increases with r. The general behavior of massive and massless cases agrees with small values of r as the mass of the scalar field decreases. Note that plots are not to scale. We choose constant values such that these behaviors are clear to see on a single graph.

4.2.2. Angular Deficit

It is well known that, for straight cosmic strings in GR, the spacetime geometry is locally pseudo-Euclidean but globally conical due to the angular deficit parameter in the line element. This can be seen if we define a new angular coordinate $\theta' = (1 - \kappa\mu/2\pi)\,\theta$, then the line element of a straight cosmic string in GR becomes $ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\theta'^2$, but the range of the angular coordinate is now $\theta' \in [0, 2\pi(1-\kappa\mu/2\pi)]$, which clearly defines a conical spacetime. Since the spacetime is locally Euclidean, a cosmic string does not apply any gravitational attraction to test particles or photons around it, and their trajectories are straight lines. Hence, the differences in the motion of test particles in this spacetime compared to flat Minkowski spacetime have a topological origin. One important outcome of this topological effect is the deflection of light due to a cosmic string. In a light scattering

experiment where a photon comes from a very far radial distance and reaches very close to the string and again reaches a very far distance to the string, the change in the angular coordinate is $\Delta\theta' = (1-\kappa\mu/2\pi)\Delta\theta = \pi$ which yields $\Delta\theta = (1+\kappa\mu/2\pi)\pi$. Here, we have used the smallness of the angular deficit parameter to derive this result. Therefore, the light deflection angle due to a straight cosmic string in GR is given by

$$\delta\theta_{GR} = \kappa\mu/2 = 4\pi G\mu, \tag{4.13}$$

which is independent of the closest approach distance to the string.

For the massless Horndeski theory, the line element (4.8) has an overall conformal factor, which means the spacetime is no longer locally flat but conformally locally flat. This implies that the massive test particles will feel an attractive force towards the string, and their trajectories are no longer straight lines. The angular deficit parameter is also no longer constant but increases with increasing radial distance to the string. For the photons, since the conformal factor does not affect null geodesics, they are still straight rays, and the deflection angle has the same form as

$$\delta\theta_{Horndeski} = \frac{\kappa\,\mu}{2\,G_4(0)}. \tag{4.14}$$

For the massive Horndeski theory, we again have a conformal factor in the line element (4.11), and the angular deficit parameter is again a function of radial coordinate. Similar to the massless theory case, the trajectories of massive test particles are clearly affected by this factor. Photon trajectories are not affected by this conformal factor, and the deflection angle for the massive Horndeski theory is the same as the massless one given in (4.14). One important difference between massless and massive Horndeski theory is that, for the massive theory, the line element becomes asymptotically locally flat as given in (4.12). Hence, for the particles very far from the string, unlike in the massless case, the trajectories are straight lines, and particles feel no force due to the string. We discuss particle motion for both cases below.

4.2.3. Geodesics and Effective Potential

Let us discuss the motion of test particles and photons around the straight cosmic strings in Horndeski theory. Since the theory we employ is minimally coupled to matter, test particles follow geodesics. The line element that we have derived has the general form

$$ds^2 = A(r)\left[-dt^2 + dr^2 + dz^2 + (1-\alpha)\, r^2 d\theta^2\right], \tag{4.15}$$

where $A(r)$ is the conformal factor in the solutions (4.8) or (4.11) and

$$\alpha = \frac{\kappa\mu}{\pi\, G_4(0)} \tag{4.16}$$

is the angular deficit parameter. We can use the Lagrangian $2L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \epsilon$, where overdot represents derivative with respect to an affine parameter, λ, and $\epsilon = -1, 0$ for particles or photons, respectively. We find the first integrals of the motion as:

$$\dot{t} = \frac{E}{A}, \quad \dot{\theta} = \frac{L}{A(1-\alpha)r^2}, \tag{4.17}$$

where E and L are the specific energy and angular momentum of the particle. Considering the motion on the plane perpendicular to the string with $z =$ constant, we find the following expression

$$\dot{r}^2 = \frac{1}{A^2}\left[E^2 - \frac{L^2}{(1-\alpha)r^2} + \epsilon A\right]. \tag{4.18}$$

Defining a new affine parameter $d\lambda = A\, d\gamma$, we obtain an effective potential as

$$\left(\frac{dr}{d\gamma}\right)^2 = E^2 - V(r), \tag{4.19}$$

with

$$V(r) = \frac{L^2}{(1-\alpha)r^2} - \epsilon A(r). \tag{4.20}$$

Investigating the effective potential, V, we see that for massive particles, i.e., $\epsilon = -1$, the cosmic strings of massless Horndeski theory, there is a potential well and test

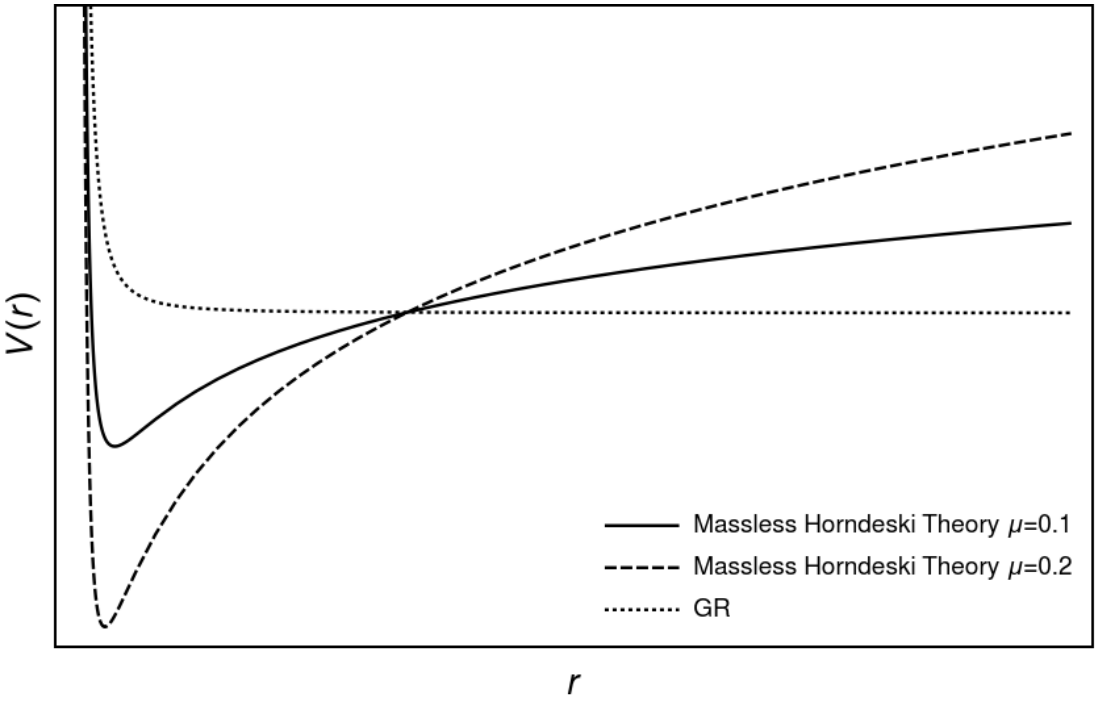


Figure 4.2 Behavior of the effective potential of massless Horndeski theory with $\mu = 0.1$ and $\mu = 0.2$. Observe that, when we double the string tension μ, the trapping energy of a particle decreases, and the depth of the trapping region increases. Meaning that a particle that is trapped in a gravitational region with a higher string tension in massless Horndeski theory has a low chance of escaping from that field. The graph is plotted for $L = 0.01$. Other constants are chosen such that the behavior is reflected on a single graph.

particles are making bounded motion among turning points given by the equation

$$E^2 = \frac{L^2}{(1-\alpha)r^2} + A, \tag{4.21}$$

as shown in Figs. (4.3) and (4.4). For the massive Horndeski theory, in the near region the string is gravitationally active and applies attractive forces to the test particles around it, in contrast to the asymptotic region where the conformal factor vanishes and the potential approaches a finite value. Hence, we have a semi-infinite potential well, depending on the specific energy and angular momentum, and the test particles may or may not escape to radial infinity. Therefore, there could be bounded motion for particles depending on their specific energy and angular momentum. The particles far from the string (at the asymptotic region $r \to \infty$) effectively feel no radial force, since the potential vanishes there, and the case will be similar to local cosmic strings in GR. This can again be seen in Figs. (4.3) and (4.4). These figures

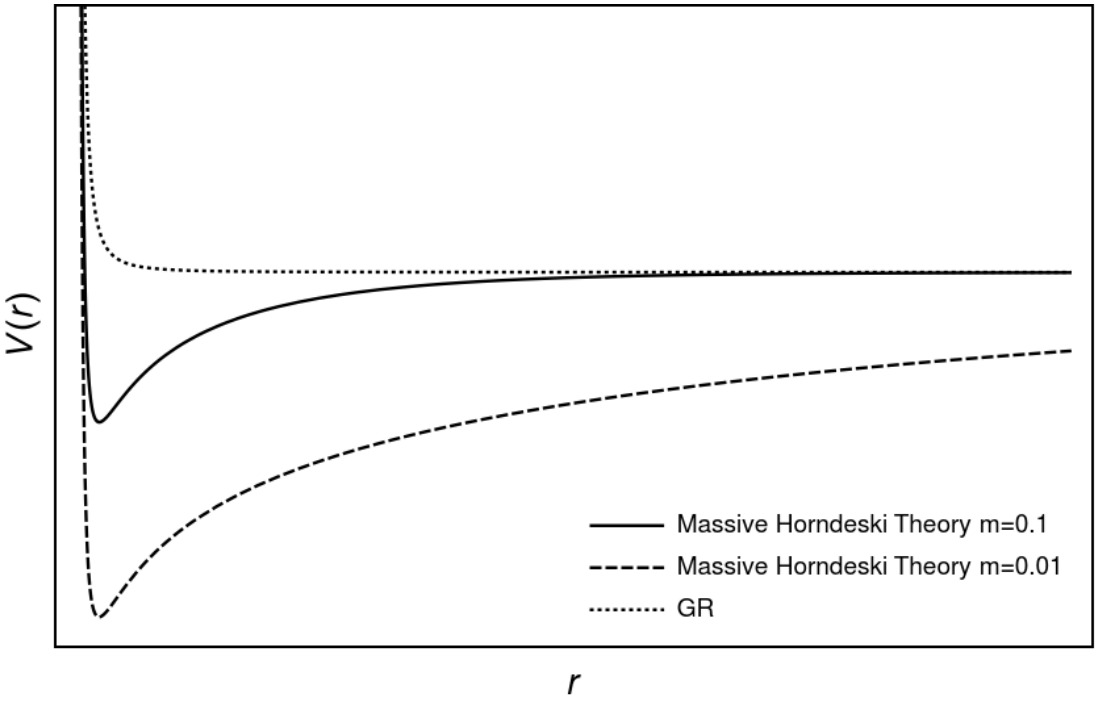


Figure 4.3 Behavior of the effective potential of massive Horndeski theory with $m = 0.1, m = 0.01$ and $\mu = 0.1$. It is clear that there exists a global minimum and then increases with r. Hence, we have circular stable geodesics, and test particles are trapped in this potential. We see that as the mass of the scalar field decreases, the trapping energy of a particle also decreases.

also show that test particles cannot reach the string for both cases, except for the $L = 0$ case.

4.2.4. Circular Geodesics

The shape of the effective potential shows that we can have stable circular geodesics at $r = r_c$ for massive test particles obeying $V(r) = E^2$, $V'(r_c) = 0$, and $V''(r_c) > 0$. These conditions yield the energy and angular momentum of the test particle following the circular geodesic as

$$E^2 = \frac{L^2}{(1-\alpha)r^2} + A(r_c), \tag{4.22}$$

$$L^2 = \frac{1}{2}(1-\alpha)r^3 A'(r_c), \tag{4.23}$$

$$V''(r_c) = A''(r_c) + \frac{3}{r_c}A'(r_c) > 0, \tag{4.24}$$

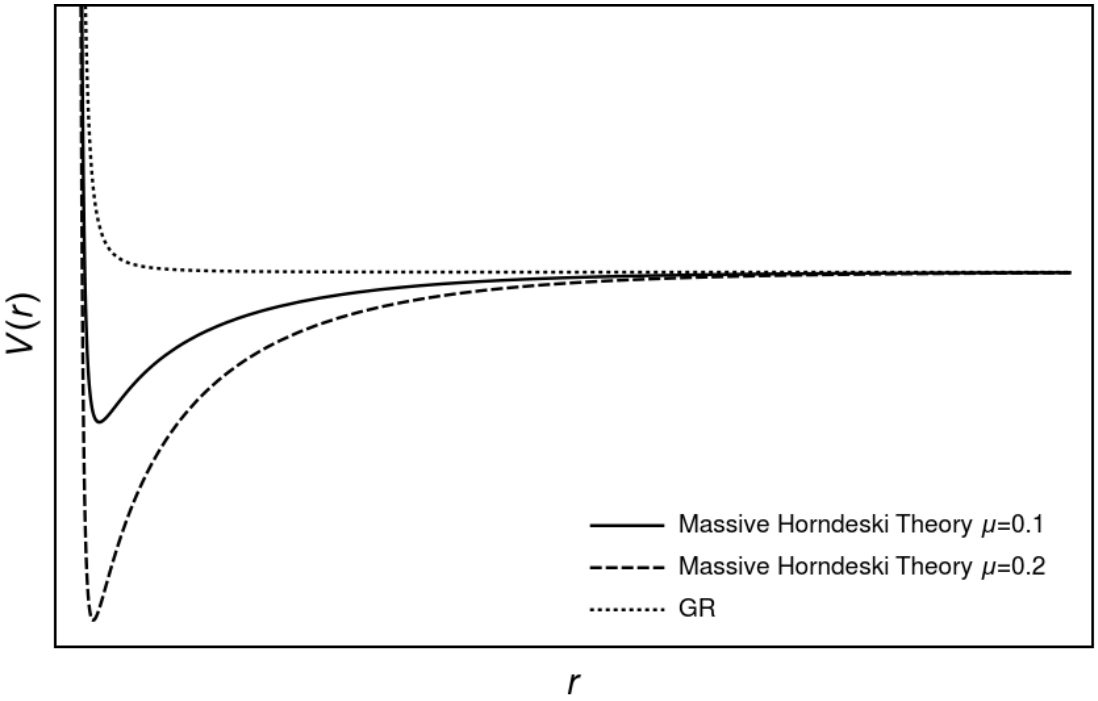


Figure 4.4 Behavior of the effective potential of massive Horndeski theory with $\mu = 0.1, \mu = 0.2$ and $m = 0.2$. Observe that, when we double the string tension μ, the trapping energy of a particle decreases, and the depth of the trapping region increases. Meaning a particle that is trapped in a gravitational region with a higher string tension in massive Horndeski theory has a low chance of escaping from that field.

where in the last equation we have used the value of L^2 given in (4.23).

For the massless Horndeski theory, the specific energy and angular momentum of the particle explicitly becomes

$$E^2 = 1 + \frac{L^2}{(1-\alpha)r_c^2} + \frac{G_{4,\phi}(0)\kappa'\,\mu}{\pi G_4(0)}\,\ln r_c, \tag{4.25}$$

$$L^2 = \frac{1}{2}(1-\alpha)\frac{G_{4,\phi}(0)\kappa'\,\mu}{\pi G_4(0)}\,r_c^2, \tag{4.26}$$

and the second derivative of the potential is positive, namely

$$V''(r_c) = \frac{2}{r_c^2}\frac{G_{4,\phi}(0)\kappa'\,\mu}{\pi G_4(0)} > 0. \tag{4.27}$$

This proves the existence of time-like stable circular geodesics around a straight cosmic string in massless Horndeski theory.

In the massive Horndeski theory, we have similar results. Namely, investigation

of the equations (4.22,4.23,4.24) shows that there is a stable circular geodesic around a straight cosmic string in the massive Horndeski theory as well. These results show that for both the massless and the massive Horndeski theory, unlike GR, straight cosmic strings are gravitationally active and accumulate matter around them, which may help the formation of large-scale structures of matter around them, in addition to other mechanisms [9] known for straight cosmic strings in GR.

5. WIGGLY COSMIC STRINGS IN HORNDESKI THEORY

In this section, we shift our attention to wiggly cosmic strings and their gravitational effects in linearized Horndeski theory. Again, we use the field equations presented in detail in Section 3.1 and refer to them as needed. This section is based on our work in [90].

5.1. Solutions for Wiggly Cosmic Strings

The dynamics and properties of wiggly strings are discussed in detail in the excellent monograph of Vilenkin and Shellard [9], who note that the kinks and wiggles will have small length scales compared to the length scale of a cosmic string network and the string will appear smooth [23]. The dominant observational consequence is that the effective mass per unit length $\tilde{\mu}$ and the effective tension $\tilde{T}$ of the string will obey the equation of state $\tilde{\mu}\tilde{T} = \mu^2$ [2]. For an observer far from the typical length scale of a string, the energy–momentum tensor will effectively have the following form [2, 9]:

$$T_{\mu\nu} = \mathrm{diag}(\tilde{\mu}, 0, 0, -\tilde{T})\,\delta(x)\,\delta(y). \tag{5.1}$$

For this energy–momentum tensor, the nontrivial linearized field tensor equations (3.11) will have the following form,

$$\Box\theta_{00} = \frac{-2\kappa\,\tilde{\mu}}{G_4(0)}\,\delta(x)\delta(y), \quad \Box\theta_{zz} = \frac{2\kappa\,\tilde{T}}{G_4(0)}\,\delta(x)\delta(y), \tag{5.2}$$

with corresponding solutions in cylindrical coordinates

$$\theta_{00} = -\frac{\kappa\,\tilde{\mu}}{\pi G_4(0)}\,\ln r, \quad \theta_{zz} = \frac{\kappa\,\tilde{T}}{\pi G_4(0)}\ln r, \quad \theta_{xx} = \theta_{yy} = 0. \tag{5.3}$$

The scalar field equation (3.6) takes the following form for wiggly strings,

$$(\Box_\eta - m^2)\varphi = -\kappa'\,(\tilde{\mu} + \tilde{T})\,\delta(x)\delta(y). \tag{5.4}$$

This equation can give different solutions for massless and massive cases, as in the straight strings. Hence, we consider these cases separately in the following subsections.

5.1.1. Massless case

For the massless scalar field case, $m = 0$, the scalar field equation (5.4) has the solution

$$\varphi = -\frac{\kappa'(\tilde{\mu}+\tilde{T})}{2\pi}\,\ln r. \tag{5.5}$$

Using this solution and also (3.9) and (5.2), the metric perturbation terms become

$$h_{00} = \left[\frac{-\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} - \frac{G_{4,\phi}\kappa'(\tilde{\mu}+\tilde{T})}{2\pi G_4(0)}\right]\ln r, \tag{5.6}$$

$$h_{zz} = \left[\frac{-\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} + \frac{G_{4,\phi}\kappa'(\tilde{\mu}+\tilde{T})}{2\pi G_4(0)}\right]\ln r, \tag{5.7}$$

$$h_{xx} = h_{yy} = \left[\frac{-\kappa(\tilde{\mu}+\tilde{T})}{2\pi G_4(0)} + \frac{G_{4,\phi}\kappa'(\tilde{\mu}+\tilde{T})}{2\pi G_4(0)}\right]\ln r. \tag{5.8}$$

The line element around a linearized wiggly cosmic string in massless Horndeski theory in cylindrical coordinates can be written as

$$\begin{aligned} ds^2 = \left[1+\frac{G_{4,\phi}(0)\kappa'(\tilde{\mu}+\tilde{T})}{2\pi\, G_4(0)}\ln r\right]\Bigg\{ &-\left(1+\frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)}\ln r\right)dt^2 \\ &+\left(1-\frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)}\ln r\right)dz^2 + \left[1-\frac{\kappa\left(\tilde{\mu}+\tilde{T}\right)}{2\pi G_4(0)}\ln r\right]\left(dr^2+r^2d\theta^2\right)\Bigg\}, \end{aligned} \tag{5.9}$$

which is conformal to the metric given in [9]. Therefore, the transformation

$$\left[1-\frac{\kappa}{2\pi G_4(0)}\left(\tilde{\mu}+\tilde{T}\right)\ln r\right]r^2 = \left[1-\frac{\kappa\left(\tilde{\mu}+\tilde{T}\right)}{2\pi G_4(0)}\right]r'^2 \tag{5.10}$$

brings the line element in the more familiar form

$$ds^2 = \left[1 + \frac{G_{4,\phi}(0)\kappa'(\tilde{\mu}+\tilde{T})}{2\pi\, G_4(0)} \ln r'\right] \Bigg\{ - \left[1 + \frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r'\right] dt^2$$
$$+ \left[1 - \frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r'\right] dz^2 + dr'^2 + \left[1 - \frac{\kappa\left(\tilde{\mu}+\tilde{T}\right)}{2\pi G_4(0)}\right] r'^2 d\theta^2 \Bigg\}. \qquad (5.11)$$

This solution is also conformal to the corresponding GR solution given again in [9]. We see that the surfaces of constant t and z are conformally equivalent to cones with deficit angle $\kappa(\tilde{\mu}+\tilde{T})/[2\pi G_4(0)]$. Since the conformal factor is unbounded in radial infinity, this solution can only describe an isolated wiggly string up to the distance where the weak field approximation breaks down. Another important observation is that, for the appropriate choice of parameters of Horndeski theory, this solution reduces to the corresponding wiggly string solution for BD theory [43]. Hence, here, we have generalized the wiggly string solution given in [9] for GR and in [43] into Horndeski theory for the special case where the scalar field is massless. The massive scalar field case will be discussed in the next subsection.

5.1.2. Massive scalar field case

For the case where the scalar field of Horndeski theory is massive, which originates from the existence of an arbitrary potential in Horndeski action, the scalar field equation (5.4) resembles the form of the corresponding straight string equation (4.4). Therefore, we can use the same reasoning to derive the solution to this equation. This leads to the fact that we should ignore the part of the solution involving the modified Bessel function of the first kind, which blows up at the radial infinity. Therefore, the solution of (5.4) will only involve the modified Bessel function of the second kind, which can be written without loss of generality as,

$$\varphi(r) = \frac{\kappa'(\tilde{\mu}+\tilde{T}) - 4c_1'}{2\pi} K_0(mr). \qquad (5.12)$$

The solution now involves an arbitrary integration constant, c_1'. To determine c_1'' we can use the fact that when $\tilde{\mu} \to \tilde{T}$, the solution should be reduced to the

corresponding straight string solution (4.10). This boundary condition indicates that $c_1' = 0$. Hence, finally, for the massive scalar field, we obtain

$$\varphi(r) = \frac{\kappa'(\tilde{\mu}+\tilde{T})}{2\pi} K_0(mr). \tag{5.13}$$

The tensorial components $\theta_{\mu\nu}$ do not change and the non-zero metric perturbation tensor components become

$$h_{00} = -\frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r + \frac{\kappa'(\tilde{\mu}+\tilde{T})}{2\pi} \frac{G_{4,\phi}(0)}{G_4(0)} K_0(mr), \tag{5.14}$$

$$h_{zz} = -\frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r - \frac{\kappa'(\tilde{\mu}+\tilde{T})}{2\pi} \frac{G_{4,\phi}(0)}{G_4(0)} K_0(mr), \tag{5.15}$$

$$h_{xx} = h_{yy} = -\frac{\kappa(\tilde{\mu}+\tilde{T})}{2\pi G_4(0)} \ln r - \frac{\kappa'(\tilde{\mu}+\tilde{T})}{2\pi} \frac{G_{4,\phi}(0)}{G_4(0)} K_0(mr). \tag{5.16}$$

The line element around a wiggly cosmic string in the presence of a massive scalar field in Horndeski theory can be written as

$$\begin{aligned} ds^2 = &\left[1 - \frac{\kappa'(\tilde{\mu}+\tilde{T})}{2\pi} \frac{G_{4,\phi}(0)}{G_4(0)} K_0(mr)\right] \left\{ -\left[1 + \frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r\right] dt^2 \right. \\ &\left. + \left[1 - \frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r\right] dz^2 + \left[1 - \frac{\kappa\left(\tilde{\mu}+\tilde{T}\right)}{2\pi G_4(0)} \ln r\right] \left(dr^2 + r^2 d\theta^2\right) \right\}. \end{aligned} \tag{5.17}$$

This solution is compatible with the literature in the appropriate GR limit [9]. Similar to the massless scalar field case, this solution can also be put in a more convenient form by applying the transformation similar to (5.10), which gives

$$\begin{aligned} ds^2 = &\left[1 - \frac{\kappa'(\tilde{\mu}+\tilde{T})}{2\pi} \frac{G_{4,\phi}(0)}{G_4(0)} K_0(mr')\right] \left\{ -\left[1 + \frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r'\right] dt^2 \right. \\ &\left. + \left[1 - \frac{\kappa(\tilde{\mu}-\tilde{T})}{2\pi G_4(0)} \ln r'\right] dz^2 + dr'^2 + \left[1 - \frac{\kappa\left(\tilde{\mu}+\tilde{T}\right)}{2\pi G_4(0)}\right] r'^2 d\theta^2 \right\}. \end{aligned} \tag{5.18}$$

We again see that the surfaces of constant t and z are conformally equivalent to cones with deficit angle $\kappa(\tilde{\mu}+\tilde{T})/[2\pi G_4(0)]$. We study further properties of wiggly strings for both cases in the next subsection.

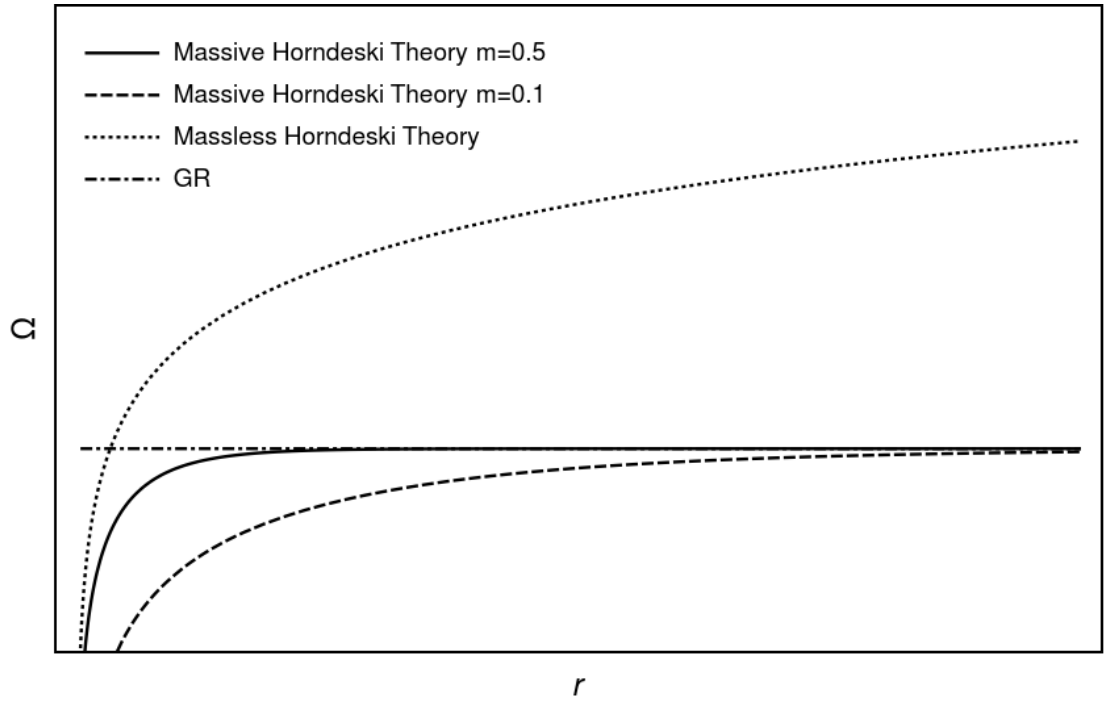


Figure 5.1 Behavior of the conformal factors for wiggly cosmic strings in linearized Horndeski theory. We observe the same graphical trend as the straight string! Here Ω represents the conformal factors of wiggly cosmic string solutions of the corresponding cases (5.9) and (5.17).

5.2. Properties of Wiggly Cosmic Strings

5.2.1. Conformal factor

Let us discuss some properties of wiggly strings in linearized Horndeski theory. The form of the solutions of Horndeski theory is conformal to wiggly strings in GR. Hence, the conformal factor determines some of their characteristic behavior. For example, for the massless Horndeski theory, the conformal factor increases unboundedly, similar to straight strings in Horndeski theory. Hence, the solution is only valid near the string, and for the far region, an exact solution may be needed to determine the properties of these strings. For the massive Horndeski theory, however, the conformal factor approaches unity far from the string, and the effect of the scalar field is screened. Therefore, the solution resembles the wiggly string solution in GR, far from the symmetry axis where the string lies. If we plot these conformal factors, we will obtain curves given in Fig. 5.1, which are similar to the graphs of straight

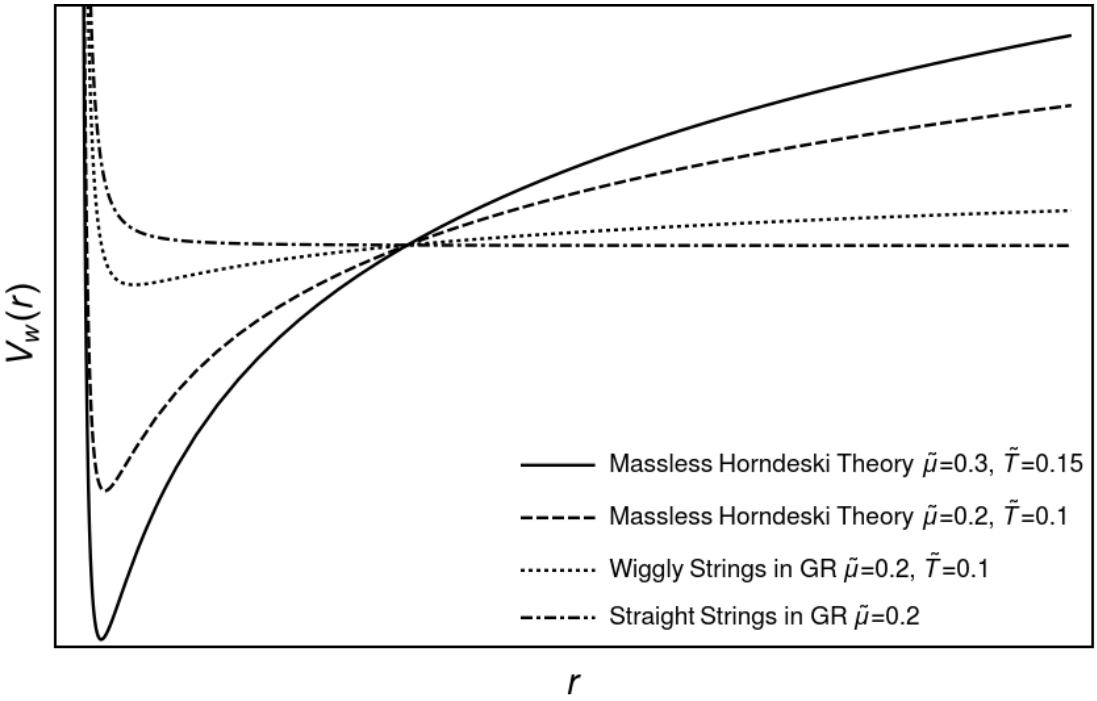


Figure 5.2 Behavior of the effective potential of massless Horndeski theory with $\tilde{\mu} = 0.2, \tilde{T} = 0.1$ and $\tilde{\mu} = 0.3, \tilde{T} = 0.15$. The graph is plotted for $L = 0.1$. Other constants are chosen such that the behavior is reflected on a single graph.

strings presented in the Fig. 4.1. Hence, the mass of the scalar field screens the effect of the scalar field in the far region, similar to the behavior of straight strings we have studied in the previous section or global monopoles in massive Horndeski theory in Section 3.

5.2.2. Geodesics and effective potential

We consider motion perpendicular to the symmetry axis of the string. Therefore, we take $dz = 0$ in (5.9) for massless and (5.17) massive Horndeski theory. The following transformation

$$\left[1 - \frac{\kappa\tilde{\mu}}{\pi G_4(0)} \ln r\right] r^2 = \left[1 - \frac{\kappa\tilde{\mu}}{\pi G_4(0)}\right] r''^2 \tag{5.19}$$

brings the corresponding line elements (5.9) or (5.17) of the form

$$ds^2 = A_w(r) \left[-dt^2 + dr^2 + (1-\beta)\, r^2 d\theta^2\right] \tag{5.20}$$

where we have removed double prime for clarity, β represents the angular deficit parameter in these coordinates, given by

$$\beta = \frac{\kappa\tilde{\mu}}{\pi\, G_4(0)}, \tag{5.21}$$

and A_w describes the conformal factors of the wiggly cosmic string solutions after the transformation (5.19). For the massless case, we have the following conformal factor

$$A_w(r) = \left[1 + \frac{\kappa(\tilde{\mu} - \tilde{T})}{2\pi G_4(0)} \ln r + \frac{G_{4,\phi}(0)\kappa'(\tilde{\mu} + \tilde{T})}{2\pi\, G_4(0)} \ln r\right], \tag{5.22}$$

and for massive case the conformal factor now becomes

$$A_w(r) = \left[1 + \frac{\kappa(\tilde{\mu} - \tilde{T})}{2\pi G_4(0)} \ln r - \frac{\kappa'(\tilde{\mu} + \tilde{T})}{2\pi} \frac{G_{4,\phi}(0)}{G_4(0)} K_0(mr)\right]. \tag{5.23}$$

The line element (5.20), given for $dz = 0$, closely resembles the one given in [9]. For the subspace perpendicular to the symmetry axis of the string, the line element is conformal to a straight string solution of GR, with the differences being those of the mass per unit length and coupling constant. The conformal factor clearly differs from straight strings in Horndeski theory with contributions due to wiggly strings in GR.

Since the form of the line element (5.20) resembles that of straight strings (4.15) in massless or massive Horndeski theory with the identification $A(r) \to A_w(r)$, $\mu \to \tilde{\mu}$, and $\alpha \to \beta$, we can use the general equations of motion of particles around straight strings presented in Section 4.2.3 also for wiggly strings for the motion of test particles moving in the plane perpendicular to the symmetry axis of the string defined by $\dot{z} = 0$. Note that, in general, the results may differ, since conformal factors A_w are different for wiggly strings compared to straight ones for both cases. Coincidentally, we obtain the general equation of effective potential for wiggly cosmic strings, similar to straight ones (4.20) in linear Horndeski theory, as follows:

$$V(r) = \frac{L^2}{(1-\beta)r^2} - \epsilon A_w(r). \tag{5.24}$$

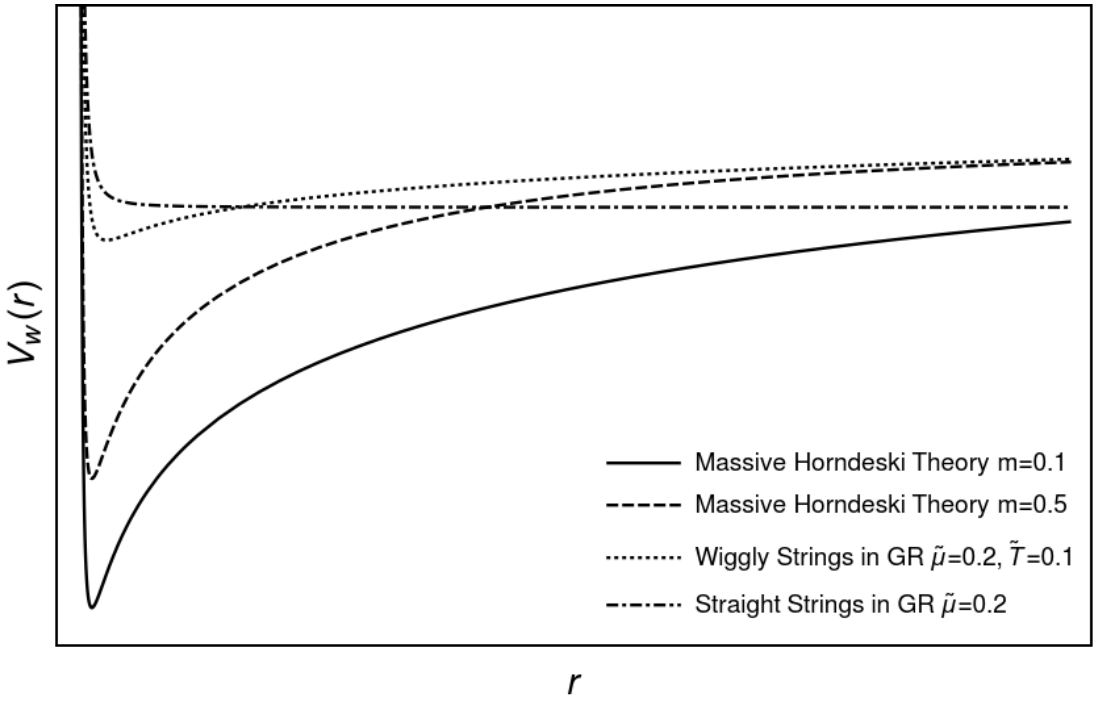


Figure 5.3 Behavior of the effective potential of massive Horndeski theory with $m = 0.1$ and $m = 0.5$. Higher scalar mass results in a deeper effective potential. For lower masses of the scalar field, effective potential approaches to the GR limit of wiggly strings more rapidly. The graph is plotted for $L = 0.1$, and for Massive Horndeski Theory $\tilde{\mu} = 0.2.$, $\tilde{T} = 0.1$.

We observe that in Fig. (5.2), the general trend of the effective potential of the wiggly strings in massless Horndeski theory does not change, as expected, compared to straight strings. Since the leading function of the effective potential in massless theory for both straight and wiggly strings is proportional to $\ln(r)$. If we consider the massive Horndeski theory, however, we see that the general behavior of the effective potential has important differences between straight and wiggly strings. As shown in Figs. (5.3) and (5.4), we see that the potential is no longer bounded at radial infinity, since now the conformal factor is dominated by the logarithmic function as opposed to straight strings in massive theory where the conformal factor is bounded from above and tends to reach the GR result, as clearly seen in Figs. (4.3) and (4.4). Hence, at infinity, we observe a similar behavior of the effective potential for massless or massive Horndeski theories for wiggly strings.

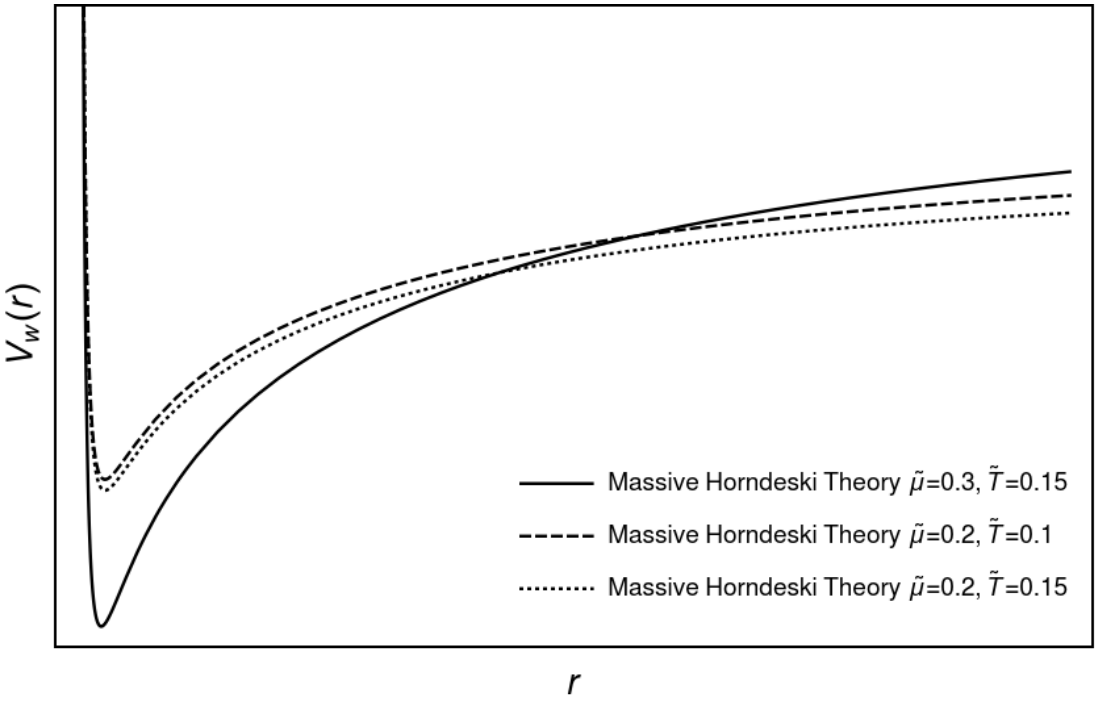


Figure 5.4 Behavior of the effective potential in massive Horndeski theory for different effective tension values $\tilde{T}$ and mass density values $\tilde{\mu}$. Increasing $\tilde{\mu}$ keeping $\tilde{T}$ constant shifts the effective potential downward, results in a deeper effective potential compared to keeping $\tilde{\mu}$ constant and increasing $\tilde{T}$. The graph is plotted for $L = 0.1$ and, $m = 0.5$.

5.2.3. Circular geodesics

The shape of the effective potential (5.24) for wiggly strings implies that, similar to straight ones, there are circular geodesics around these strings for both massive and massless Horndeski theory. The general equations for circular geodesics have the same form as straight strings, as given in (4.22,4.23,4.24), with the identifications $A \to A_w$, $\alpha \to \beta$. For the massless Horndeski theory, the specific energy and angular momentum of a test particle following a circular geodesic around a wiggly cosmic string are given by:

$$E^2 = 1 + \frac{L^2}{(1-\beta)r_c^2} + \frac{\kappa(\tilde{\mu}-\tilde{T})}{\pi\, G_4(0)} \ln r_c + \frac{G_{4,\phi}(0)\kappa'(\tilde{\mu}+\tilde{T})}{\pi\, G_4(0)} \ln r_c, \qquad (5.25)$$

$$L^2 = \frac{(1-\beta)}{4\pi\, G_4(0)} \Big[\kappa(\tilde{\mu}-\tilde{T}) + G_{4,\phi}(0)\kappa'(\tilde{\mu}+\tilde{T})\Big]\, r_c^2. \qquad (5.26)$$

These circular geodesics are stable if $V'' > 0$, which is

$$V''(r_c) = \frac{1}{\pi\, G_4(0)\, r_c^2} \left[\kappa(\tilde{\mu} - \tilde{T}) + G_{4,\phi}(0)\kappa'(\tilde{\mu} + \tilde{T})\right]. \tag{5.27}$$

Hence the circular geodesics are definitely stable for physically favorable case $\tilde{\mu} > \tilde{T}$.

An important difference between straight and wiggly strings, shown in these expressions, is that for wiggly strings, even in the GR limit ($\kappa' = 0$), circular geodesics exist. These expressions also show that the specific energy and angular momentum should be greater for a particle with the same radius to follow a circular geodesic compared to straight strings for $\tilde{\mu} > \tilde{T}$ or vice versa.

The existence of circular geodesics is another indication of the fact that the wiggly strings are gravitationally active and accumulate matter around them, similar to the straight strings in Horndeski theory. Note that the wiggly strings still have this behavior in the GR limit, as opposed to the straight ones.

6. VELOCITY KICK ON PARTICLES PASSING THROUGH A STRING IN (MASSIVE) HORNDESKI THEORY

In Sections 4.2 and 5.2, we have discussed bounded geodesics around straight and wiggly strings. Here, we will consider the unbounded motion of particles passing near such strings. Hence, we will investigate the effect of wiggly and straight strings on such trajectories. We will assume a large velocity for the particle and small impact parameter such that $v^2 >> h_{00}$ is satisfied [9]. We start with the line element of the form (5.20) in the Cartesian coordinates as,

$$ds^2 = f(x,y)\left[-dt^2 + dx^2 + dy^2\right], \tag{6.1}$$

with an angular deficit $\tilde{\Delta} = \pi\beta$ where β is given in (5.21) (for regular cosmic strings, $\tilde{\Delta} = \pi\alpha$ where α is given in (4.16)). Here, $f(x,y)$ represents the conformal factor of various cosmic string solutions we have derived in Sections (4) and (5) in the Cartesian coordinates. Note that this function takes distinct forms with respect to whether the string is wiggly or straight and/or whether the scalar field is massive or massless. The linearized equations of motion of the particle immediately follow as,

$$\partial_t^2 x = -\frac{1}{2}\left[1 - (\partial_t x)^2 - (\partial_t y)^2\right]\partial_x f(x,y), \tag{6.2}$$

$$\partial_t^2 y = -\frac{1}{2}\left[1 - (\partial_t x)^2 - (\partial_t y)^2\right]\partial_y f(x,y). \tag{6.3}$$

Following [9], we choose the x-axis as the initial trajectory with conditions $x = v\,t, y = y_0$. Then, after passing the string, the small velocity gain of the particle in its y-component can be found as

$$v_y = \frac{1}{v}\int_{-\infty}^{\infty} \partial_t^2 y\, dx = -\frac{1}{2v\gamma^2}\int_{-\infty}^{\infty} \partial_y f(x,y)\Big|_{y=y_0} dx, \tag{6.4}$$

where $\gamma^2 = 1/(1-v^2)$ and $y \in \mathbb{R}$. In the following, we will solve this integral separately for all cases we investigated in Sections (4) and (5).

6.0.1. Case I: Straight String in Massless Horndeski Theory

For straight strings in massless Horndeski theory, the conformal factor can be read from (4.8),

$$f(x,y) = 1 + \frac{G_{4,\phi}(0)}{\pi G_4(0)} \kappa' \mu \, \ln \sqrt{x^2+y^2}.$$

Therefore the trajectory integral (6.4) is,

$$\begin{aligned} v_y &= -\frac{\kappa' \mu \, G_{4,\phi}(0)}{2\pi v \gamma^2 G_4(0)} \int_{-\infty}^{\infty} \partial_y \ln \sqrt{x^2+y^2} \Big|_{y=y_0} dx, \\ &= -\mathrm{Sgn}(y_0) \, \frac{\kappa' \mu \, G_{4,\phi}(0)}{2v \, \gamma^2 \, G_4(0)}. \end{aligned} \tag{6.5}$$

The sign depends on whether y_0 is positive or negative. Consider two particles with the same initial velocity v passing through the opposite side of the string. Meaning that one particle propagates with $y_0 > 0$, while the other $y_0 < 0$. Hence, the velocity difference of these particles is given as,

$$|\Delta v_y| = \frac{\mu}{G_4(0)} \left[v\kappa + \frac{\kappa' \, G_{4,\phi}(0)}{v \, \gamma^2} \right]. \tag{6.6}$$

Here, the first term comes from the deficit angle $\tilde{\Delta} = \pi\alpha$. Note that for straight cosmic strings in GR, there is no gravitational force exerted by the source. Meaning that this change in speed in y-direction represented by the first term in (6.6) is due to the global structure of the string topology [9]. The second term is the contribution of the massless Horndeski theory, where the straight strings apply a gravitational force on the particles around them. With this term, we observe that the velocity gain is a boost relative to GR, since $v, \kappa', \gamma > 0$.

6.0.2. Case II: Straight String in Massive Horndeski Theory

In this case the conformal factor is given as (4.11),

$$f(x,y) = 1 - \frac{G_{4,\phi}(0)}{\pi G_4(0)} \kappa' \mu \, K_0(m\sqrt{x^2+y^2}).$$

The velocity integral (6.4) is,

$$v_y = \frac{G_{4,\phi}(0)}{2\pi v\gamma^2 G_4(0)}\kappa'\mu \int_{-\infty}^{\infty} \partial_y K_0(m\sqrt{x^2+y^2})\Big|_{y=y_0} dx, \tag{6.7}$$

We will solve this integral separately since it involves Bessel function, K_0. We start with,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \partial_y K_0(m\sqrt{x^2+y^2})\Big|_{y=y_0} dx = \int_{-\infty}^{\infty} \frac{-my_0}{\sqrt{x^2+y_0^2}} K_1(m\sqrt{x^2+y_0^2})\, dx. \\ &= -2my_0 \int_0^{\infty} \frac{K_1(m\sqrt{x^2+y_0^2})}{\sqrt{x^2+y_0^2}}\, dx. \end{aligned} \tag{6.8}$$

In the last line, we used the fact that the integrand is even under the transformation $x \to -x$. Changing the coordinates as $z^2 = x^2 + y_0^2$ yields,

$$I = -2my_0 \int_{|y_0|}^{\infty} \frac{K_1(mz)}{\sqrt{z^2-y_0^2}}\, dz. \tag{6.9}$$

In order to put this integral into a much more familiar form we further define the transformation $z = |y_0|\cosh\theta$, resulting

$$I = -2my_0 \int_0^{\infty} K_1(m|y_0|\cosh\theta)\, d\theta. \tag{6.10}$$

Now, from [92] we have,

$$K_\mu(x)K_\nu(x) = 2\int_0^{\infty} K_{\mu+\nu}(2x\cosh t)\cos[(\mu-\nu)t]\, dt. \tag{6.11}$$

For $\mu = \nu = 1/2$, $t \to \theta$ and $x = m|y_0|/2$ we obtain,

$$\int_0^{\infty} K_1(m|y_0|\cosh\theta)\, d\theta = \frac{1}{2}K_{1/2}^2(x) = \frac{\pi\, e^{-m|y_0|}}{m|y_0|}. \tag{6.12}$$

Exploiting this result in (6.7) gives the velocity as follows,

$$v_y = -\mathrm{Sgn}(y_0)\frac{G_{4,\phi}(0)\kappa'\mu}{2v\gamma^2 G_4(0)} e^{-m|y_0|} \tag{6.13}$$

And the velocity difference in this case is,

$$|\Delta v_y| = \frac{\mu}{G_4(0)}\left(v\kappa + \frac{\kappa'\, G_{4,\phi}(0)}{v\,\gamma^2} e^{-m|y_0|}\right). \tag{6.14}$$

Comparing this result to our previous one, we see that, in the massive Horndeski theory, the initial placement of the particle in the y-axis directly affects the speed of the particle, v_y due to the coupling of the mass of the scalar field, m and the impact parameter y_0 in $e^{-m|y_0|}$. The effect of this term diminishes as $y_0 \to \infty$, and the result approaches to GR result. This is the result of the screening effect of the massive scalar field, which we have also observed in the effective potential analysis in the earlier sections. Furthermore, we recover the velocity difference expression for straight strings (6.6), when $m \to 0$.

6.0.3. Case III: Wiggly String in Massless Horndeski Theory

In the previous two cases, we have mainly calculated all the necessary integrals. Hence, in this case and the next one, we will present our results directly. The conformal factor for massless Horndeski theory is given explicitly as (5.22), and in Cartesian coordinates it reads,

$$f(x,y) = 1 + \frac{\ln\sqrt{x^2+y^2}}{2\pi\, G_4(0)} \left[\kappa(\tilde{\mu}-\tilde{T}) + \kappa'(\tilde{\mu}+\tilde{T})\, G_{4,\phi}(0)\right].$$

Since the leading function is exactly the same as massless straight strings (6.5), the result of velocity gain is immediate,

$$v_y = -\mathrm{Sgn}(y_0)\frac{\kappa(\tilde{\mu}-\tilde{T}) + \kappa'(\tilde{\mu}+\tilde{T})G_{4,\phi}(0)}{4v\gamma^2 G_4(0)}. \tag{6.15}$$

The velocity difference is found as,

$$|\Delta v_y| = \frac{\tilde{\mu}v\kappa}{G_4(0)} + \frac{1}{2v\gamma^2 G_4(0)}\left[\kappa(\tilde{\mu}-\tilde{T}) + \kappa'(\tilde{\mu}+\tilde{T})G_{4,\phi}(0)\right]. \tag{6.16}$$

This expression differs from our straight string result (6.6), with the addition of the second term due to $\tilde{\mu} \neq \tilde{T}$ for wiggly strings, which is the consequence of the gravitational force of the wiggly string, in which this term is also present in GR [9]. The last term is due to the scalar field of Horndeski theory for a wiggly string, which is not present in the corresponding GR expression [9].

6.0.4. Case IV: Wiggly String in Massive Horndeski Theory

Using the conformal factor (5.23) in Cartesian coordinates, we found the velocity gain for massive Horndeski theory for wiggly strings as,

$$v_y = -\frac{\mathrm{Sgn}(y_0)}{4v\,\gamma^2 G_4(0)}\left[\kappa(\tilde{\mu}-\tilde{T})+\kappa'(\tilde{\mu}+\tilde{T})G_{4,\phi}(0)e^{-m|y_0|}\right]. \tag{6.17}$$

and the velocity difference between particles was,

$$|\Delta v_y| = \frac{\tilde{\mu}v\kappa}{G_4(0)}+\frac{1}{2v\,\gamma^2 G_4(0)}\left[\kappa(\tilde{\mu}-\tilde{T})+\kappa'(\tilde{\mu}+\tilde{T})G_{4,\phi}(0)e^{-m|y_0|}\right]. \tag{6.18}$$

From this general expression, one can deduce the three previous cases of v_y. For example, to obtain our massless straight string result, one puts $m \to 0, \tilde{\mu} = \tilde{T} = \mu$ and obtains Eq. (6.6). The screening effect again reveals itself in the last term, where the effect diminishes with increasing $|y_0|$ and disappears as $|y_0| \to \infty$, far from the string, reducing the expression to the GR one.

In this section, we derived the change in the velocity of particles passing through a cosmic string in various settings. We have obtained the corresponding velocity gain of the particles, which basically comes from three effects. The first one is the topological term due to the global conicity of the spacetime, which is present for all cases. The second one is due to the gravitational force of the strings, which does not exist only for straight strings in GR. The last term is the effect of the scalar field, non-minimally coupled to gravity. Hence, it exists as long as $G_{4,\phi} \neq 0$. In other words, this effect is not present for the theories with $G_{4,\phi} = 0$ such as GR, Einstein-scalar theory, Quintessence/K-Essence, etc, where either there is no scalar field or the coupling of the scalar field to the gravity is minimal. The last term also carries the effects of the mass of the scalar field as a decay factor, with the screening effect, which only exists for the massive scalar theories, such as BD with a potential, $f(R)$ theories, etc. Note that these results are valid as long as the linear approximation is valid.

In astrophysical or cosmological applications, this expression can also easily be used to calculate the relative velocity, that is, $u = \gamma|\tilde{\Delta}v_y|$, where the effects of a

string relativistically moving in a perfect fluid background can be considered. The imprints of the wakes due to a relativistic string can be investigated in a dark matter background or in the cosmic microwave background, and observational consequences can be studied, using the results obtained here for the strings of the massless or massive Horndeski theory, in future works.

7. CONCLUSION

In this thesis, we examined several attempts in the literature to construct a rotating global monopole spacetime using the JN method and its variants in Section 2. By explicitly deriving the required BL coordinate transformations, we demonstrated that the BL transformation functions have a θ dependence. However, this dependence violates the integrability conditions for BL coordinates and shows that the resulting metrics in [17, 61, 64] cannot represent the solution of the rotating global monopole spacetime. Resulting line elements rely on incorrect coordinate transformations as a result, contain off-diagonal metric components incompatible with a proper BL form for a rotating body. The main reason for this inconsistency lies in the complexification step of the JN method: the monopole parameter does not properly transform under the procedure since it has no dependence on the complexified coordinate r. Consequently, the JN algorithm is unsuitable for generating rotating generalizations of global monopole solutions. Even alternative approaches, such as the method proposed in [66], fail to produce a fully reliable rotating monopole spacetime. We have derived the field equations for the line element obtained by the JNA method [66] given by Eq. (2.47), and explicitly showed that this solution does not correspond to the rotating global monopole black hole. We also remarked that neither (2.32) nor the metric (2.47) satisfies the slow rotation limit [62], which supports the results we produced. Therefore, our findings [69], together with a recent no-go theorem [70] strongly indicate that rotating global monopole solutions may not exist within the standard framework of GR. This motivates a shift of focus in the subsequent parts of the thesis toward alternative directions, rather than pursuing a rotating global monopole metric that appears fundamentally obstructed.

In Section 3, we have presented two distinct solutions of linearized Horndeski theory corresponding to global monopoles obeying Barriola and Vilenkin's approximation for their energy–momentum tensor. The first case can be called the massless Horndeski theory, whereas the second case is the massive one, due to the existence of an arbitrary potential term in the action. The existence of the potential changes the

asymptotic behavior of the spacetime as it possesses a screening mechanism to suppress the effects of the long–range scalar field and effectively make it a short–range one. Due to this mechanism, the asymptotic behavior of the global monopole becomes the same as the monopoles in GR, as seen in Fig. 3.1.

We have also investigated several physical properties of the solutions for both theories and see the effect of the mass and hence the screening effect in the asymptotical behavior of these solutions. For example, the geodesic equations clearly show different behavior for massless or massive cases of the theory, as we have seen by investigating the radial motion of test particles. The effective potential Figs. 3.2 and 3.3 illustrate these behavior for massive or null test particles. Namely, the local properties of both cases have similar characteristics, whereas far from the monopole, the properties of the massive Horndeski theory, unlike the massless case, approach those of GR.

We have also analyzed the effects of global monopoles on the galaxy rotation curves. We have seen that, similar to the Brans–Dicke theory, for a massless scalar field, global monopoles may explain the flattening of the galaxy rotation curves. For the case of the massive scalar field, however, only if the mass of the scalar field is "light", i.e., $m \to 0$, then the global monopoles may explain this phenomenon. For intermediate and heavy scalar fields, however, the global monopoles cannot explain the flattening of the galaxy rotation curves. As far as we know, this and other novel results we have discussed throughout the section, especially about monopoles on massive scalar-tensor theories, were not discussed before in the literature.

In Section 4, we first discussed solutions of the linearized field equations of Horndeski theory corresponding to a static straight cosmic string lying along the z–axis for both massless and massive scalar field cases. We have obtained the spacetime line element and the scalar field configuration for both cases in appropriate forms, which makes the comparison easier to the known versions of corresponding solutions for GR and BD theories. The solutions of massless and massive theories have distinct features, which can be tractable in the behavior of the conformal factors of

the corresponding line elements and test particle motion around the strings. The existence of the mass of the scalar field results in a screening effect, and the scalar field effectively freezes from a distance from the string. The result of this behavior is that the cosmic strings in massive theory are asymptotically indistinguishable from corresponding ones in GR, except for the differences in the coupling constants in the angular deficit parameter. For the massless theory, however, the behavior of the strings is very different. The conformal factor asymptotically increases, undoubtedly making the solution valid only in regions not very far from the string. Moreover, the particles make a bounded motion, and stable circular geodesics exist for a particle around the string at constant z.

We have presented the second type of solutions in Section 5, which are corresponding to wiggly cosmic strings for both massless and massive theory. We have again obtained the solutions for both cases and presented them in appropriate forms for further analysis. The solutions resemble the form of the earlier solutions found for GR and BD, where the BD solution is conformal to GR. The difference between massless and massive theories is, again, revealed in the conformal factors. The behavior of the conformal factors is just the same as straight strings that we discussed in sections 4.1 and 4.2. The mass of the scalar field again shows a screening effect, and wiggly strings in massive Horndeski theory become equivalent to strings in GR when viewed at asymptotically radial distances. Similar to straight strings, the particles around these strings make bounded motion as long as the weak field approximation is valid.

In Section 6, we investigated the effects of the straight or wiggly strings in massless or massive Horndeski theory on the particle velocity passing through the string. We generalize the results of [9] by finding the modifications not only for the effect of the straight string topology and the gravitational force of the wiggly string on the particle velocity, but also for the massive and massless scalar field. We observed velocity gains in both massive and massless linear Horndeski theory for straight and wiggly string spacetimes. These velocity "kicks" can have significant effects on the background matter distributions on intergalactic dark matter and cosmic fluid dis-

tributions. A future direction stemming from this work is to investigate the motion of a cosmic string in the dark matter and cosmic fluid distributions, revealing the effects of the scalar field and its mass on these fluids by considering the effects of the modifications we have obtained in Horndeski theory on the cosmic string wakes.

8. PUBLICATIONS

- M. H. Seçuk, "Comment on "Gravitational field of a rotating global monopole"," *Phys. Rev. D*, vol. 102, p. 068501, 2020.
- M. H. Seçuk and Ö. Delice, "Global monopoles in Horndeski theory," *The European Physical Journal C*, vol. 85, no. 7, p. 806, 2025.
- M. H. Seçuk and Ö. Delice, "Straight and wiggly cosmic strings in Horndeski theory," *arXiv:2511.10578*, 2025.

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