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GENERALIZATION OF THE *k*-LEONARDO SEQUENCE AND THEIR HYPERBOLIC QUATERNIONS

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Summary. In this study, we examine the *k*-Leonardo, *k*-Leonardo-Lucas, and Modified *k*-Leonardo sequences, and some terms of these sequences are given. Then, we obtain the generating functions, summation formulas, etc. Also, we obtain the Binet formulas in three different ways. The first is in the known classical way, the second is with the help of the sequence's generating functions, and the third is with the help of the matrices. In addition, we examine the relations between the terms of the *k*-Leonardo, *k*-Leonardo-Lucas, Modified *k*-Leonardo, Leonardo, Leonardo-Lucas, Modified Leonardo, Francois, Fibonacci, and Lucas sequences. Moreover, we associate the terms of these sequences with matrices. Furthermore, we present on the application of these sequences to hyperbolic quaternions. For these quaternions, we give many properties such as Binet formulas. Finally, the terms of the *k*-Leonardo, *k*-Leonardo-Lucas, and Modified *k*-Leonardo sequences are associated with their hyperbolic quaternion values.

1 INTRODUCTION

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields such as Cryptology [1], Phylotaxis [2], Biomathematics [3], Chemistry [4], Engineering [5], etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the *k*-Jacobsthal, *k*-Jacobsthal-Lucas, Leonardo, *k*-Fibonacci, -Chebyshev, Copper Fibonacci sequences, etc, (see for details in [6-11]).

For $n \in \mathbb{N}$, Fibonacci numbers F and Lucas numbers are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ with } F_0 = 0 \text{ and } F_1 = 1,$$

$$L_{n+2} = L_{n+1} + L_n, \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

For and Binet formulas are given by the following relations,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

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where $\alpha = \frac{+\sqrt{5}}{}$ and $\beta = \frac{-\sqrt{5}}{}$ are the roots of the characteristic equation $\quad - \quad - 1 = 0$. Here the number α is the known golden ratio.

Leonardo numbers, which are related to Fibonacci and Lucas numbers, were first introduced by Catarino and Borges. The application of Leonardo numbers in many areas has attracted the attention of scientists. Some of these applications are coding theory [12], quaternions [13], Physics [14].

In [15] and [17], for $n \in \mathbb{N}$, the Leonardo numbers l_n, Leonardo-Lucas numbers H_n, Modified Leonardo numbers G_n, and Francois numbers f_n are defined by the recurrence relations, respectively,

$$= l_{n+1} + l_n + 1, \text{ with } l_0 = 1 \text{ and } l_1 = 1,$$
$$= H_{n+1} + H_n - 1, \text{ with } \quad = 3 \text{ and } \quad = 2,$$
$$= G_{n+1} + G_n + 1, \text{ with } \quad = 0 \text{ and } \quad = 1,$$
$$= f_{n+1} + f_n + 1, \text{ with } \quad = 2 \text{ and } \quad = 1.$$

For l_n, , , and f_n, Binet formulas are given by the following relations, respectively,

$$= \frac{2(\alpha \qquad ^1)}{} - 1, \quad = \quad + \quad + 1,$$

$$= \text{————} - 1, \text{ and } \quad = \quad + \quad + \text{————} - 1.$$

Hence, $\alpha = \frac{+\sqrt{5}}{}$ and $\beta = \frac{-\sqrt{5}}{}$ are the roots of the characteristic equation $\quad - \quad - 1 = 0$. In [16] and [17], Soykan found many properties about these sequences. Also, Modified Leonardo numbers were obtained with the help of the following equation:

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{matrix} -G & -G \\ -G & -G \\ -G & -G \end{matrix}\Bigg).$$

With the help of the recurrence relation of the Fibonacci sequence, k-sequences were introduced, and these sequences have an important place in number theory [19]. In [20], Falcon and Plaza introduced the k-Fibonacci sequence and obtained many properties related to this sequence. In addition, Falcon defined the k-Lucas sequences [21]. Moreover, Falcon applied the Hankel transform to the k-Fibonacci sequence and obtained the terms of Fibonacci sequences differently [22].

The quaternions were first described by Hamilton in 1843. In addition, quaternions are used to control rotational movements, especially in Kinematics [23], 3D games [24], mechanics [25], Eulerian angles [26], Chemistry [27]. In [28], Horadam defined Complex Fibonacci and Fibonacci quaternions, and various features were found.

The algebra of hyperbolic quaternions is an algebra that is not related to the elements of the form over the real numbers.

$$= xi_1 + yi_2 + zi_3 + \quad , x, y, z, t \in \mathbb{R}$$

He gave the properties of the q components defined in Table 1.

In [29], Macfarlane did a lot of research on hyperbolic quaternions and their properties. An expression of the general form of hyperbolic quaternions is

$$\hbar = \hbar_1 i_1 + \hbar_2 i_2 + \hbar_3 i_3 + \hbar_4 i_4 = (\hbar_1, \hbar_2, \hbar_3, \hbar_4).$$

Here, $\hbar_1, \hbar_2, \hbar_3, \hbar_4$ are the terms of the sequence and i_1, i_2, i_3, i_4 are hyperbolic quaternions.

$\cdot$	$\boldsymbol{i_1}$	$\boldsymbol{i_2}$	$\boldsymbol{i_3}$	$\boldsymbol{i_4}$
$\boldsymbol{i_1}$	i_1	i_2	i_3	i_4
$\boldsymbol{i_2}$	i_2	i_1	i_4	$-i_3$
$\boldsymbol{i_3}$	i_3	$-i_4$	i_1	i_2
$\boldsymbol{i_4}$	i_4	i_3	$-i_2$	i_1

Table 1. Hyperbolic Quaternions Units

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the k-Fibonacci and Leonardo sequences. We call these sequences the k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo sequences and denote them as $\mathcal{L}_{k,n}$, $\mathcal{H}_{k,n}$ and $\mathcal{G}_{k,n}$, respectively.

2 GENERALIZATION OF THE k-LEONARDO SEQUENCES

For $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, and Modified k-Leonardo $\mathcal{G}_{k,n}$ sequences are defined by, respectively,

$$\mathcal{L}_{k,n+2} = \mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} + k, \text{ with } \mathcal{L}_{k,0} = 1 \text{ and } \mathcal{L}_{k,1} = 1,$$

$$\mathcal{H}_{k,n+2} = \mathcal{H}_{k,n+1} + \mathcal{H}_{k,n} - k, \text{ with } \mathcal{H}_{k,0} = 3 \text{ and } \mathcal{H}_{k,1} = 2,$$

$$\mathcal{G}_{k,n+2} = \mathcal{G}_{k,n+1} + \mathcal{G}_{k,n} + k, \text{ with } \mathcal{G}_{k,0} = 0 \text{ and } \mathcal{G}_{k,1} = 1,$$

(see [15]).

Also, the third-order recurrence relations of the $\mathcal{L}_{k,n}$, $\mathcal{H}_{k,n}$, and $\mathcal{G}_{k,n}$ sequences are as follows, respectively,

$$\mathcal{L}_{k,n+3} = 2\mathcal{L}_{k,n+2} - \mathcal{L}_{k,n}, \text{ with } \mathcal{L}_{k,0} = 1, \mathcal{L}_{k,1} = 1, \text{ and } \mathcal{L}_{k,2} = k + 2,$$

$$\mathcal{H}_{k,n+3} = 2\mathcal{H}_{k,n+2} - \mathcal{H}_{k,n}, \text{ with } \mathcal{H}_{k,0} = 3, \mathcal{H}_{k,1} = 2, \text{ and } \mathcal{H}_{k,2} = 5 - k,$$

$$\mathcal{G}_{k,n+3} = 2\mathcal{G}_{k,n+2} - \mathcal{G}_{k,n}, \text{ with } \mathcal{G}_{k,0} = 0, \mathcal{G}_{k,1} = 1, \text{ and } \mathcal{G}_{k,2} = k + 1,$$

(see [15]).

Then, let's give some information about the equations of these sequences.

The characteristic equation of the k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo sequences is

$$t^3 - 2t^2 + 1 = (t-1)(t^2 - t - 1) = 0.$$

The roots of this equation are;

$$\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ \text{and}\ \delta = 1.$$

The relationship between these roots is given below;

$$\alpha + \beta + \delta = 2,\ \alpha\beta + \alpha\delta + \beta\delta = 0,\ \text{and}\ \alpha\beta\delta = -1.$$

The $\mathcal{L}_{k,n}$, $\mathcal{H}_{k,n}$, and $\mathcal{G}_{k,n}$ values for the first six n natural numbers are given below;

- $\mathcal{L}_{k,0} = 1$ $\quad \mathcal{H}_{k,0} = 3$ $\quad \mathcal{G}_{k,0} = 0,$
- $\mathcal{L}_{k,1} = 1$ $\quad \mathcal{H}_{k,1} = 2$ $\quad \mathcal{G}_{k,1} = 1,$
- $\mathcal{L}_{k,2} = k + 2$ $\quad \mathcal{H}_{k,2} = 5 - k$ $\quad \mathcal{G}_{k,2} = k + 1,$
- $\mathcal{L}_{k,3} = 2k + 3$ $\quad \mathcal{H}_{k,3} = 7 - 2k$ $\quad \mathcal{G}_{k,3} = 2k + 2,$
- $\mathcal{L}_{k,4} = 4k + 5$ $\quad \mathcal{H}_{k,4} = 12 - 4k$ $\quad \mathcal{G}_{k,4} = 4k + 3,$
- $\mathcal{L}_{k,5} = 7k + 8$ $\quad \mathcal{H}_{k,5} = 19 - 7k$ $\quad \mathcal{G}_{k,5} = 7k + 5.$

(see [15]).

In the following theorem, the Binet formulas of the k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, and Modified k-Leonardo $\mathcal{G}_{k,n}$ sequences are expressed.

Theorem 2.1. Let $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. We obtain

i. $\mathcal{L}_{k,n} = \frac{1+k}{(\alpha-\beta)(\alpha-1)}\,\alpha^n + \frac{1+k}{(\beta-\alpha)(\beta-1)}\beta^n - k$, **ii.** $\mathcal{H}_{k,n} = \frac{4-k-\alpha}{(\alpha-\beta)(\alpha-1)}\,\alpha^n + \frac{4-k-\beta}{(\beta-\alpha)(\beta-1)}\,\beta^n + k$,

iii. $\mathcal{G}_{k,n} = \frac{-1+k+\alpha}{(\alpha-\beta)(\alpha-1)}\,\alpha^n + \frac{-1+k+\beta}{(\beta-\alpha)(\beta-1)}\,\beta^n - k.$

Proof. i. The Binet form of a sequence is as follows

$$\mathcal{L}_{k,n} = x\alpha^n + y\beta^n + z\delta^n.$$

For these n values, we obtain;

$$\mathcal{L}_{k,0} = x + y + z,$$
$$\mathcal{L}_{k,1} = x\alpha + y\beta + z\delta,$$
$$\mathcal{L}_{k,n} = x\alpha^2 + y\beta^2 + z\delta^2.$$

We find

$$x = \frac{1+k}{(\alpha-\beta)(\alpha-1)},\ y = \frac{1+k}{(\beta-\alpha)(\beta-1)},\ \text{and}\ z = -k.$$

Other proofs are shown in a similar way. □

In the following theorems, we give the generating functions of the k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, and Modified k-Leonardo $\mathcal{G}_{k,n}$ sequences. In addition, we obtain Binet formulas of $\mathcal{L}_{k,n}$, $\mathcal{H}_{k,n}$, and $\mathcal{G}_{k,n}$ sequences with the help of generating functions.

Theorem 2.2. The generating functions for k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, and Modified k-Leonardo $\mathcal{G}_{k,n}$ sequences are given as follows, respectively,

i. $l(t)=\sum_{n=0}^{\infty}\mathcal{L}_{k,n}t^n=\frac{kt^2-t+1}{t^3-2t+1}$, **ii.** $h(t)=\sum_{n=0}^{\infty}\mathcal{H}_{k,n}t^n=\frac{(1-k)t^2-4t+3}{t^3-2t+1}$,

iii. $g(t)=\sum_{n=0}^{\infty}\mathcal{G}_{k,n}t^n=\frac{(k-1)t^2+t}{t^3-2t+1}$.

Proof. i. For the k-Leonardo sequence, we have

$$\begin{aligned} l(t)=\sum_{n=0}^{\infty}\mathcal{L}_{k,n}t^n &= 1+t+(k+2)t^2+\sum_{n=3}^{\infty}\mathcal{L}_{k,n}t^n \\ &= 1+t+(k+2)t^2+2\sum_{n=3}^{\infty}\mathcal{L}_{k,n-1}t^n-\sum_{n=3}^{\infty}\mathcal{L}_{k,n-3}t^n \\ &= 1+t+(k+2)t^2+2t\sum_{n=2}^{\infty}\mathcal{L}_{k,n}t^n-t^3\sum_{n=0}^{\infty}\mathcal{L}_{k,n}t^n. \\ &= \frac{kt^2-t+1}{t^3-2t+1}. \end{aligned}$$

Other proofs are shown in a similar way. □

Theorem 2.3. For $\mathcal{L}_{k,n}$, $\mathcal{H}_{k,n}$, and $\mathcal{G}_{k,n}$ the Binet formulas can be obtained with the help of the generating functions.

Proof. With the help of the roots of the characteristic equation of these sequences, the roots of the $t^3-2t+1=0$ equation become $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\delta}$. For $\mathcal{L}_{k,n}$, we have

$$\begin{aligned} \frac{kt^2-t+1}{t^3-2t+1} &= \frac{1+k}{(\alpha-\beta)(\alpha-\delta)}\frac{1}{1-\alpha t}+\frac{1+k}{(\beta-\alpha)(\beta-\delta)}\frac{1}{1-\beta t}-k\frac{1}{1-\delta t} \\ &= \frac{1+k}{(\alpha-\beta)(\alpha-\delta)}\sum_{n=0}^{\infty}\alpha^n t^n+\frac{1+k}{(\beta-\alpha)(\beta-\delta)}\sum_{n=0}^{\infty}\beta^n t^n-k\sum_{n=0}^{\infty}\delta^n t^n \\ &= \sum_{n=0}^{\infty}\Big(\frac{1+k}{(\alpha-\beta)(\alpha-\delta)}\,\alpha^n+\frac{1+k}{(\beta-\alpha)(\beta-\delta)}\beta^n-k\Big)t^n \\ &= \sum_{n=0}^{\infty}\mathcal{L}_{k,n}t^n. \end{aligned}$$

Similarly, the Binet formulas of the sequences $\mathcal{H}_{k,n}$ and $\mathcal{G}_{k,n}$ are found. □

Next, we give special sum formulas of the k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, and Modified k-Leonardo $\mathcal{G}_{k,n}$ sequences.

Theorem 2.4. Let $k\in\mathbb{R}^+$ and $n\in\mathbb{N}$. We obtain

i. $\sum_{s=0}^{n}\mathcal{L}_{k,s}=2\mathcal{L}_{k,n}+\mathcal{L}_{k,n-1}-(n-1)k-1$,

ii. $\sum_{s=0}^{n}\mathcal{H}_{k,s}=2\mathcal{H}_{k,n}+\mathcal{H}_{k,n-1}+(n-1)k-2$,

iii. $\sum_{s=0}^{n}\mathcal{G}_{k,s}=2\mathcal{G}_{k,n}+\mathcal{G}_{k,n-1}-(n-1)k-1$.

Proof. ii. From the definition of the k-Leonardo-Lucas sequence, we have

$$\begin{aligned} \mathcal{H}_{k,2} &= 2\mathcal{H}_{k,1}+\mathcal{H}_{k,0}-k, \\ \mathcal{H}_{k,3} &= 2\mathcal{H}_{k,2}+\mathcal{H}_{k,1}-k, \\ &\vdots \\ \mathcal{H}_{k,n} &= 2\mathcal{H}_{k,n-1}+\mathcal{H}_{k,n-2}-k. \end{aligned}$$

So, we have

$$-5+\sum_{s=0}^{n}\mathcal{H}_{k,s}=2\sum_{s=1}^{n-1}\mathcal{H}_{k,s}-\sum_{s=0}^{n-2}\mathcal{H}_{k,s}-(n-1)k,$$

$$-5+\sum_{s=0}^{n}\mathcal{H}_{k,s}$$
$$=2\left(-\mathcal{H}_{k,n}-\mathcal{H}_{k,0}+\sum_{s=0}^{n}\mathcal{H}_{k,s}\right)-\left(-\mathcal{H}_{k,n}-\mathcal{H}_{k,n-1}+\sum_{s=0}^{n}\mathcal{H}_{k,s}\right)-(n-1)k.$$

Thus, we obtain

$$\sum_{s=0}^{n}\mathcal{H}_{k,s}=2\mathcal{H}_{k,n}+\mathcal{H}_{k,n-1}+(n-1)k-2.$$

Other proofs are shown in a similar way. □

Theorem 2.5. Let $k\in\mathbb{R}^{+}$ and $n\in\mathbb{N}$. We obtain

i. $\sum_{s=0}^{n}\mathcal{L}_{k,2s}=\mathcal{L}_{k,2n+1}-nk,$ **ii.** $\sum_{s=0}^{n}\mathcal{L}_{k,2s+1}=\mathcal{L}_{k,2n+1}+\mathcal{L}_{k,2n}-nk-1,$

iii. $\sum_{s=0}^{n}\mathcal{H}_{k,2s}=\mathcal{H}_{k,2n+1}+nk+1,$ **iv.** $\sum_{s=0}^{n}\mathcal{H}_{k,2s+1}=\mathcal{H}_{k,2n+1}+\mathcal{H}_{k,2n}+nk+3,$

v. $\sum_{s=0}^{n}\mathcal{G}_{k,2s}=\mathcal{G}_{k,2n+1}-nk-1,$ **vi.** $\sum_{s=0}^{n}\mathcal{G}_{k,2s+1}=\mathcal{G}_{k,2n+1}+\mathcal{G}_{k,2n}-nk.$

Proof. The proofs are shown similarly to Theorem 2.4. □

3 RELATIONS AMONG SPECIAL SEQUENCES

In this chapter, we examine the relationships among the k-Leonardo, k-Leonardo-Lucas, Modified k-Leonardo, Leonardo, Leonardo-Lucas, Modified Leonardo, Francois, Fibonacci, and Lucas sequences. Finally, we associate the terms of these sequences with matrices.

In the following theorems, we examine the relations between the k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, Modified k-Leonardo $\mathcal{G}_{k,n}$, Leonardo l_n, Leonardo-Lucas H_n, Modified Leonardo G_n, and Francois f_n sequences.

Theorem 3.1. Let $k\in\mathbb{R}^{+}$, and $n\in\mathbb{N}$. The following equations are true:

$$\left.\begin{aligned}
&\textbf{i}.\ \mathscr{L}_{k,n}=\frac{-4k+15}{k^2-7k+11}\mathscr{H}_{k,n+2}+\frac{5k-14}{k^2-7k+11}\mathscr{H}_{k,n+1}-\frac{k^2-6k+12}{k^2-7k+11}\mathscr{H}_{k,n},\\
&\textbf{ii}.\ \mathscr{L}_{k,n}=\frac{k}{k^2-7k+11}\mathcal{G}_{k,n+2}-\frac{2k+1}{k^2-7k+11}\mathcal{G}_{k,n+1}+\frac{2k^2}{k^2-7k+11}\mathcal{G}_{k,n},\\
&\textbf{iii}.\ \mathscr{L}_{k,n}=\frac{-k+1}{2}l_{n+2}+\frac{k-1}{2}l_{n+1}+kl_n,\\
&\textbf{iv}.\ \mathscr{L}_{k,n}=\frac{8k+3}{5}H_{n+2}-\frac{7k+2}{5}H_{n+1}-\frac{6k+1}{5}H_n,\\
&\textbf{v}.\ \mathscr{L}_{k,n}=-kG_{n+2}+(2k+1)G_{n+1}-G_n,
\end{aligned}\right\}\quad\text{(see [15])}$$

vi. $\mathcal{L}_{k,n}=\frac{-7k+4}{11}f_{n+2}+\frac{8k-3}{11}f_{n+1}+\frac{10k-1}{11}f_n.$

Proof. i. The following relation is used for proofs;

$$\mathcal{L}_{k,n}=a\ x\ \mathcal{H}_{k,n+2}+b\ x\ \mathcal{H}_{k,n+1}+c\ x\ \mathcal{H}_{k,n}.$$

For these n values, we obtain;

$$\mathcal{L}_{k,0}=a\ x\ \mathcal{H}_{k,2}+b\ x\ \mathcal{H}_{k,3}+c\ x\ \mathcal{H}_{k,2},$$
$$\mathcal{L}_{k,1}=a\ x\ \mathcal{H}_{k,3}+b\ x\ \mathcal{H}_{k,4}+c\ x\ \mathcal{H}_{k,3},$$

$$\mathcal{L}_{k,2} = a\ x\ \mathcal{H}_{k,4} + b\ x\ \mathcal{H}_{k,5} + c\ x\ \mathcal{H}_{k,4}.$$

We find

$$a = \frac{-4k+15}{k^2-7k+11},\ b = \frac{5k-14}{k^2-7k+11}, \text{ and } c = -\frac{k^2-6k+12}{k^2-7k+11}.$$

Other proofs are shown in a similar way. □

Theorem 3.2. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

$$\left.\begin{array}{l}
\textbf{i}.\ \mathcal{H}_{k,n} = \frac{4}{k+1}\mathcal{L}_{k,n+2} - \frac{5}{k+1}\mathcal{L}_{k,n+1} - \frac{k}{k+1}\mathcal{L}_{k,n}, \\
\textbf{ii}.\ \mathcal{H}_{k,n} = \frac{3k-3}{k^2-k-1}\mathcal{G}_{k,n+2} - \frac{3k}{k^2-k-1}\mathcal{G}_{k,n+1} + \frac{-k^2+k+4}{k^2-k-1}\mathcal{G}_{k,n}, \\
\textbf{iii}.\ \mathcal{H}_{k,n} = \frac{k+3}{2}l_{n+2} - \frac{k+4}{2}l_{n+1} + \frac{-2k+1}{2}l_n, \\
\textbf{iv}.\ \mathcal{H}_{k,n} = \frac{-8k+8}{5}H_{n+2} + \frac{7k-7}{5}H_{n+1} + \frac{6k-1}{5}H_n, \\
\textbf{v}.\ \mathcal{H}_{k,n} = (k-1)G_{n+2} + (-2k+5)G_{n+1} - 4G_n,
\end{array}\right\} \quad \text{(see [15])}$$

vi. $\mathcal{H}_{k,n} = \frac{7k+11}{11}f_{n+2} - \frac{8k+11}{11}f_{n+1} - \frac{10k}{11}f_n.$

Proof. iv. If Binet formulas are used for proofs, we get

$$\begin{aligned}
\frac{-8k+8}{5}H_{n+2} + \frac{7k-7}{5}H_{n+1} + \frac{6k-1}{5}H_n &= \left(\frac{-8k+8}{5}\right)(\alpha^{n+2} + \beta^{n+2} + 1) \\
&+ \left(\frac{7k-7}{5}\right)(\alpha^{n+1} + \beta^{n+1} + 1) - (\frac{6k-1}{5})(\alpha^n + \beta^n + 1) \\
&= \frac{4-k-\alpha}{(\alpha-\beta)(\alpha-1)}\ \alpha^n + \frac{4-k-\beta}{(\beta-\alpha)(\beta-1)}\ \beta^n + k = \mathcal{H}_{k,n}.
\end{aligned}$$

Other proofs are shown in a similar way. □

Theorem 3.3. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

$$\left.\begin{array}{l}
\textbf{i}.\ \mathcal{G}_{k,n} = -\frac{1}{k+1}\mathcal{L}_{k,n+2} + \frac{2}{k+1}\mathcal{L}_{k,n+1} + \frac{k}{k+1}\mathcal{L}_{k,n}, \\
\textbf{ii}.\ \mathcal{G}_{k,n} = \frac{-3k+12}{k^2-7k+11}\mathcal{H}_{k,n+2} + \frac{3k-9}{k^2-7k+11}\mathcal{H}_{k,n+1} + \frac{-k^2+7k-14}{k^2-7k+11}\mathcal{H}_{k,n}, \\
\textbf{iii}.\ \mathcal{G}_{k,n} = -\frac{k}{2}l_{n+2} + \frac{k+1}{2}l_{n+1} + \frac{2k-1}{2}l_n, \\
\textbf{iv}.\ \mathcal{G}_{k,n} = \frac{8k+1}{5}H_{n+2} + \frac{-7k+1}{5}H_{n+1} - \frac{6k+2}{5}H_n, \\
\textbf{v}.\ \mathcal{G}_{k,n} = (-k+1)G_{n+2} + (2k-2)G_{n+1} + G_n,
\end{array}\right\} \quad \text{(see [15])}$$

vi. $\mathcal{G}_{k,n} = \frac{-7k+1}{11}f_{n+2} + \frac{8k+2}{11}f_{n+1} + \frac{10k-3}{11}f_n.$

Proof. The proofs are shown similarly to Theorem 3.2. □

Theorem 3.4. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

i. (see [15])

- $G_n = \frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} + \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} + \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n}$,
- $G_n = -\frac{1}{k^2+k}\mathcal{L}_{k,n+2} + \frac{k+1}{k^2+k}\mathcal{L}_{k,n+1} + \frac{1}{k^2+k}\mathcal{L}_{k,n}$,
- $G_n = \frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+2} + \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n+1} - \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n}$,

ii. (see [15])

- $H_n = \frac{-3k^2+14k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} + \frac{4k^2-5k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} + \frac{-6k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n}$,
- $H_n = \frac{3k+1}{k^2+k}\mathcal{L}_{k,n+2} - \frac{4k+1}{k^2+k}\mathcal{L}_{k,n+1} - \frac{1}{k^2+k}\mathcal{L}_{k,n}$,
- $H_n = \frac{3k^2-2k-1}{k^3-k^2-k}\mathcal{G}_{k,n+2} + \frac{-4k^2+1}{k^3-k^2-k}\mathcal{G}_{k,n+1} + \frac{3k+1}{k^3-k^2-k}\mathcal{G}_{k,n}$.

iii. (see [15])

- $l_n = \frac{k-1}{k^2+k}\mathcal{L}_{k,n+2} + \frac{-k+1}{k^2+k}\mathcal{L}_{k,n+1} + \frac{1}{k}\mathcal{L}_{k,n}$,
- $l_n = \frac{-k^2+k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} + \frac{k^2+k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} + \frac{-k^2+5k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n}$,
- $l_n = \frac{k^2-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+2} - \frac{k^2+k+1}{k^3-k^2-k}\mathcal{G}_{k,n+1} + \frac{k^2+k-1}{k^3-k^2-k}\mathcal{G}_{k,n}$.

iv.

- $f_n = \frac{2k-1}{k^2+k}\mathcal{L}_{k,n+2} + \frac{-3k+1}{k^2+k}\mathcal{L}_{k,n+1} + \frac{2k+1}{k^2+k}\mathcal{L}_{k,n}$
- $f_n = \frac{-2k^2+4k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} + \frac{3k^2-4k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} + \frac{-2k^2+7k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n}$
- $f_n = \frac{2k^2-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+2} - \frac{3k^2+2k+1}{k^3-k^2-k}\mathcal{G}_{k,n+1} + \frac{2k^2+2k-1}{k^3-k^2-k}\mathcal{G}_{k,n}$.

Proof. The proofs are shown similarly to Theorem 3.1. □

In the following theorem, we examine the relationships among the k-Leonardo, k-Leonardo-Lucas, Modified k-Leonardo sequences, and special numbers sequences.

Theorem 3.5. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The relationships of the k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, and Modified k-Leonardo $\mathcal{G}_{k,n}$, sequences with the Fibonacci sequence F_n and Lucas sequence L_n are as follows:

i.

- $\mathcal{L}_{k,n} = F_{n+1} + kF_{n+1} - k$, (see [15])
- $\mathcal{L}_{k,n} = \frac{k+1}{5}L_{n+1} + \frac{2k+2}{5}L_n - k$, (see [15])
- $F_n = \frac{1}{k+1}(\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n})$,
- $L_n = \frac{1}{k+1}(-\mathcal{L}_{k,n+1} + 3\mathcal{L}_{k,n}) + \frac{2k}{k+1}$.

ii.

- $\mathcal{H}_{k,n} = (3-k)F_{n+1} - F_n + k,$ (see [15])
- $\mathcal{H}_{k,n} = \frac{-k+1}{5}L_{n+1} + \frac{2k+7}{5}L_n + k,$ (see [15])
- $F_n = \frac{-k+3}{k^2-7k+11}\mathcal{H}_{k,n+1} + \frac{k-2}{k^2-7k+11}\mathcal{H}_{k,n+1} - \frac{k}{k^2-7k+11},$
- $L_n = \frac{k-1}{k^2-7k+11}\mathcal{H}_{k,n+1} + \frac{-3k+8}{k^2-7k+11}\mathcal{H}_{k,n+1} - \frac{2k^2-7k}{k^2-7k+11}.$

iii.

- $\mathcal{G}_{k,n} = kF_{n+1} + F_n - k,$ (see [15])
- $\mathcal{G}_{k,n} = \frac{k+2}{5}L_{n+1} + \frac{2k-1}{5}L_n - k,$ (see [15])
- $F_n = \frac{k}{k^2-k-1}\mathcal{G}_{k,n+1} - \frac{k+1}{k^2-k-1}\mathcal{G}_{k,n} - \frac{k}{k^2-k-1},$
- $L_n = -\frac{k+2}{k^2-k-1}\mathcal{G}_{k,n+1} + \frac{3k+1}{k^2-k-1}\mathcal{G}_{k,n} + \frac{2k^2-k}{k^2-k-1}.$

Proof. i. If Binet formulas are used for proofs, we get

$$F_{n+1} + kF_{n+1} - k = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + k\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - k$$

$$= \frac{\alpha^n(\alpha k + \alpha) + \beta^n(k\beta + \beta)}{\alpha - \beta} - k$$

$$= \frac{1+k}{(\alpha-\beta)(\alpha-1)}\ \alpha^n + \frac{1+k}{(\beta-\alpha)(\beta-1)}\beta^n - k = \mathcal{L}_{k,n}.$$

Other proofs are shown in a similar way. □

In the following theorems, we associate the terms of the k-Leonardo $\mathcal{L}_{k,n}$, k-Leonardo-Lucas $\mathcal{H}_{k,n}$, and Modified k-Leonardo $\mathcal{G}_{k,n}$ sequences with matrices. In addition, we obtain the Binet formulas of these sequences with the help of the matrices.

Theorem 3.6. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

i. For the k-Leonardo sequence,

1. $\begin{bmatrix} \mathcal{L}_{k,n+2} \\ \mathcal{L}_{k,n+1} \\ \mathcal{L}_{k,n} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+2 \\ 1 \\ 1 \end{bmatrix}$, **2.** $\begin{bmatrix} \mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} \\ \mathcal{L}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ k+2 \end{bmatrix}$.

ii. For the k-Leonardo-Lucas sequence,

1. $\begin{bmatrix} \mathcal{H}_{k,n+2} \\ \mathcal{H}_{k,n+1} \\ \mathcal{H}_{k,n} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 5-k \\ 2 \\ 3 \end{bmatrix}$, **2.** $\begin{bmatrix} \mathcal{H}_{k,n} \\ \mathcal{H}_{k,n+1} \\ \mathcal{H}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}^n \begin{bmatrix} 3 \\ 2 \\ 5-k \end{bmatrix}$.

iii. For the Modified k-Leonardo sequence,

1. $\begin{bmatrix} \mathcal{G}_{k,n+2} \\ \mathcal{G}_{k,n+1} \\ \mathcal{G}_{k,n} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+1 \\ 1 \\ 0 \end{bmatrix}$, **2.** $\begin{bmatrix} \mathcal{G}_{k,n} \\ \mathcal{G}_{k,n+1} \\ \mathcal{G}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ k+1 \end{bmatrix}$.

Proof. The proofs are shown by the induction method using definitions of sequences and properties of multiplication in matrices. □

Theorem 3.7. (Simpson Formulas, see [15]**)** Let $k \in \mathbb{R}^+$ and $t \in \mathbb{N}$. The following equations are satisfied.

i. $det\begin{bmatrix} \mathcal{L}_{k,t+2} & \mathcal{L}_{k,t+1} & \mathcal{L}_{k,t} \\ \mathcal{L}_{k,t+1} & \mathcal{L}_{k,t} & \mathcal{L}_{k,t-1} \\ \mathcal{L}_{k,t} & \mathcal{L}_{k,t-1} & \mathcal{L}_{k,t-2} \end{bmatrix} = (-1)^{t+1}k(k+1)^2,$

ii. $det\begin{bmatrix} \mathcal{H}_{k,t+2} & \mathcal{H}_{k,t+1} & \mathcal{H}_{k,t} \\ \mathcal{H}_{k,t+1} & \mathcal{H}_{k,t} & \mathcal{H}_{k,t-1} \\ \mathcal{H}_{k,t} & \mathcal{H}_{k,t-1} & \mathcal{H}_{k,t-2} \end{bmatrix} = (-1)^{t}k(11-7k+k^2),$

iii. $det\begin{bmatrix} \mathcal{G}_{k,t+2} & \mathcal{G}_{k,t+1} & \mathcal{G}_{k,t} \\ \mathcal{G}_{k,t+1} & \mathcal{G}_{k,t} & \mathcal{G}_{k,t-1} \\ \mathcal{G}_{k,t} & \mathcal{G}_{k,t-1} & \mathcal{G}_{k,t-2} \end{bmatrix} = (-1)^{t}k(1+k-k^2).$

Proof. The proofs are shown by the induction method using definition and determinant properties. □

Theorem 3.8 (see [15])**.** Let $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. The following equations are true:

i. For the k-Leonardo sequence;

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

where

$$\begin{aligned}
x_{11} &= G_{n+1} = -\tfrac{1}{k^2+k}\mathcal{L}_{k,n+3} + \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n+2} + \tfrac{1}{k^2+k}\mathcal{L}_{k,n+1}, \\
x_{12} &= -G_{n-1} = \tfrac{1}{k^2+k}\mathcal{L}_{k,n+1} - \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n} - \tfrac{1}{k^2+k}\mathcal{L}_{k,n-1}, \\
x_{13} &= -G_{n} = \tfrac{1}{k^2+k}\mathcal{L}_{k,n+2} - \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n+1} - \tfrac{1}{k^2+k}\mathcal{L}_{k,n}, \\
x_{21} &= G_{n} = -\tfrac{1}{k^2+k}\mathcal{L}_{k,n+2} + \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n+1} + \tfrac{1}{k^2+k}\mathcal{L}_{k,n}, \\
x_{22} &= -G_{n-2} = \tfrac{1}{k^2+k}\mathcal{L}_{k,n} - \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n-1} - \tfrac{1}{k^2+k}\mathcal{L}_{k,n-2}, \\
x_{23} &= -G_{n-1} = \tfrac{1}{k^2+k}\mathcal{L}_{k,n+1} - \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n} - \tfrac{1}{k^2+k}\mathcal{L}_{k,n-1}, \\
x_{31} &= G_{n-1} = -\tfrac{1}{k^2+k}\mathcal{L}_{k,n+1} + \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n} + \tfrac{1}{k^2+k}\mathcal{L}_{k,n-1}, \\
x_{32} &= -G_{n-3} = \tfrac{1}{k^2+k}\mathcal{L}_{k,n-1} - \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n-2} - \tfrac{1}{k^2+k}\mathcal{L}_{k,n-3}, \\
x_{33} &= -G_{n-2} = \tfrac{1}{k^2+k}\mathcal{L}_{k,n+2} - \tfrac{k+1}{k^2+k}\mathcal{L}_{k,n+1} - \tfrac{1}{k^2+k}\mathcal{L}_{k,n}.
\end{aligned}$$

ii. For the k-Leonardo-Lucas sequence;

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

where

$$y_{11} = G_{n+1} = \frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+3} + \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} + \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n+1},$$

$$y_{12} = -G_{n-1} = -\frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} - \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n} - \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n-1},$$

$$y_{13} = -G_n = -\frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} - \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} - \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n},$$

$$y_{21} = G_n = \frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} + \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} + \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n},$$

$$y_{22} = -G_{n-2} = -\frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n} - \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n-1} - \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n-2},$$

$$y_{23} = -G_{n-1} = -\frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} - \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n} - \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n-1},$$

$$y_{31} = G_{n-1} = \frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} + \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n} + \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n-1},$$

$$y_{32} = -G_{n-3} = -\frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n-1} - \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n-2} - \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n-3},$$

$$y_{33} = -G_{n-2} = -\frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n} - \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n-1} - \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n-2}.$$

iii. For the Modified k-Leonardo sequence;

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}$$

where

$$z_{11} = G_{n+1} = \frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+3} + \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n+2} - \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n+1},$$

$$z_{12} = -G_{n-1} = -\frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+1} - \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n} + \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n-1},$$

$$z_{13} = -G_n = -\frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+2} - \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n+1} + \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n},$$

$$z_{21} = G_n = \frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+2} + \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n+1} - \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n},$$

$$z_{22} = -G_{n-2} = -\frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n} - \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n-1} + \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n-2},$$

$$z_{23} = -G_{n-1} = -\frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+1} - \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n} + \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n-1},$$

$$z_{31} = G_{n-1} = \frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+1} + \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n} - \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n-1},$$

$$z_{32} = -G_{n-3} = -\frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n-1} - \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n-2} + \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n-3},$$

$$z_{33} = -G_{n-2} = -\frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n} - \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n-1} + \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n-2}.$$

Proof. For the Modified Leonardo sequence, the following relation is true:

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

The proofs are obtained with the help of the following relations:

- $G_n = \frac{-2k+11}{k^3-7k^2+11k}\mathcal{H}_{k,n+2} + \frac{-k^2+6k-11}{k^3-7k^2+11k}\mathcal{H}_{k,n+1} + \frac{3k-11}{k^3-k^2-k}\mathcal{H}_{k,n}$, (Theorem 3.4. i.)
- $G_n = -\frac{1}{k^2+k}\mathcal{L}_{k,n+2} + \frac{k+1}{k^2+k}\mathcal{L}_{k,n+1} + \frac{1}{k^2+k}\mathcal{L}_{k,n}$, (Theorem 3.4. i.)
- $G_n = \frac{-k+1}{k^3-k^2-k}\mathcal{G}_{k,n+2} + \frac{k^2-1}{k^3-k^2-k}\mathcal{G}_{k,n+1} - \frac{1}{k^3-k^2-k}\mathcal{G}_{k,n}$. (Theorem 3.4. i.) □

Theorem 3.8. For $\mathcal{L}_{k,n}$, $\mathcal{H}_{k,n}$ and $\mathcal{G}_{k,n}$ sequences, the Binet formulas can be obtained with the help of the matrices.

Proof. The following relation is used for proof (see for details Corollary 3.1 in [30]).

$$t_n = \frac{1}{\det(\Lambda)}\sum_{j=1}^{i} r_{m+1-j}\det(\Lambda_j).$$

Thus,

$$\mathcal{L}_{k,n} = \frac{1}{\det(\Lambda)}\sum_{j=1}^{i} \mathcal{L}_{k,m+1-j}\det(\Lambda_j).$$

Let $m = i = 3$,

$$\Lambda = \begin{bmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ 1 & 1 & 1 \end{bmatrix}, \Lambda_1 = \begin{bmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ 1 & 1 & 1 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ and } \Lambda_3 = \begin{bmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ 1 & 1 & 1 \end{bmatrix}.$$

So, we obtain

$$\mathcal{L}_{k,n} = \frac{1}{\det(\Lambda)}\sum_{j=1}^{3} \mathcal{L}_{k,4-j}\det(\Lambda_j)$$

$$= \frac{1}{\det(\Lambda)}\left(\mathcal{L}_{k,3}\det(\Lambda_1) + \mathcal{L}_{k,2}\det(\Lambda_2) + \mathcal{L}_{k,1}\det(\Lambda_3)\right)$$

$$= \frac{1+k}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{1+k}{(\beta-\alpha)(\beta-1)}\beta^n - k.$$

Similarly, the Binet formula of the $\mathcal{H}_{k,n}$, and $\mathcal{G}_{k,n}$ sequences are found. □

4 APPLICATIONS

In this chapter, we present on the application of generalized k-Blaise sequences to hyperbolic quaternions. Then we define hyperbolic k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo quaternions. For these quaternions, we give many properties such as

Binet formulas. Finally, the terms of the k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo sequences are associated with their hyperbolic quaternion values.

Definition 4.1. For $n \in \mathbb{N}$, the hyperbolic k-Leonardo $\breve{H}\mathcal{L}_n$, hyperbolic k-Leonardo-Lucas $\breve{H}\mathcal{H}_n$, and hyperbolic Modified k-Leonardo $\breve{H}\mathcal{G}_n$ quaternions are defined by, respectively,

$$\breve{H}\mathcal{L}_{k,n} = \mathcal{L}_{k,n}i_1 + \mathcal{L}_{k,n+1}i_2 + \mathcal{L}_{k,n+2}i_3 + \mathcal{L}_{k,n+3}i_4 = (\mathcal{L}_{k,n}, \mathcal{L}_{k,n+1}, \mathcal{L}_{k,n+2}, \mathcal{L}_{k,n+3}),$$

$$\breve{H}\mathcal{H}_{k,n} = \mathcal{H}_{k,n}i_1 + \mathcal{H}_{k,n+1}i_2 + \mathcal{H}_{k,n+2}i_3 + \mathcal{H}_{k,n+3}i_4 = (\mathcal{H}_{k,n}, \mathcal{H}_{k,n+1}, \mathcal{H}_{k,n+2}, \mathcal{H}_{k,n+3}),$$

$$\breve{H}\mathcal{G}_{k,n} = \mathcal{G}_{k,n}i_1 + \mathcal{G}_{k,n+1}i_2 + \mathcal{G}_{k,n+2}i_3 + \mathcal{G}_{k,n+3}i_4 = (\mathcal{G}_{k,n}, \mathcal{G}_{k,n+1}, \mathcal{G}_{k,n+2}, \mathcal{G}_{k,n+3}).$$

where $\mathcal{L}_{k,n}$ is n^{th} k-Leonardo sequence, $\mathcal{H}_{k,n}$ is n^{th} k-Leonardo-Lucas sequence, $\mathcal{G}_{k,n}$ is Modified k-Leonardo sequence and i_1, i_2, i_3 and i_4 are the hyperbolic quaternion units in table 1.

Let now give the first two terms of the $\breve{H}\mathcal{L}_{k,n}$, $\breve{H}\mathcal{H}_{k,n}$, and $\breve{H}\mathcal{G}_{k,n}$, respectively,

- $\breve{H}\mathcal{L}_{k,0} = i_1 + i_2 + (k+2)i_3 + (2k+3)i_4$,
- $\breve{H}\mathcal{L}_{k,1} = i_1 + (k+2)i_2 + (2k+3)i_3 + (4k+5)i_4$,
- $\breve{H}\mathcal{H}_{k,0} = 3i_1 + 2i_2 + (5-k)i_3 + (7-2k)i_4$,
- $\breve{H}\mathcal{H}_{k,1} = 2i_1 + (5-k)i_2 + (7-2k)i_3 + (12-4k)i_4$,
- $\breve{H}\mathcal{G}_{k,0} = i_2 + (k+1)i_3 + (2k+2)i_4$,
- $\breve{H}\mathcal{G}_{k,1} = i_1 + (k+1)i_2 + (2k+2)i_3 + (4k+3)i_4$.

Definition 4.2. For $n \in \mathbb{N}$, the conjugate of hyperbolic k-Leonardo $\breve{H}\mathcal{L}_n^*$, hyperbolic k-Leonardo-Lucas $\breve{H}\mathcal{H}_n^*$, and Modified k-Leonardo $\breve{H}\mathcal{G}_n^*$ quaternions are defined by, respectively,

$$\breve{H}\mathcal{L}_n^* = \mathcal{L}_{k,n}i_1 - \mathcal{L}_{k,n+1}i_2 - \mathcal{L}_{k,n+2}i_3 - \mathcal{L}_{k,n+3}i_4 = (\mathcal{L}_{k,n}, -\mathcal{L}_{k,n+1}, -\mathcal{L}_{k,n+2}, -\mathcal{L}_{k,n+3}),$$

$$\breve{H}\mathcal{H}_n^* = \mathcal{H}_{k,n}i_1 - \mathcal{H}_{k,n+1}i_2 - \mathcal{H}_{k,n+2}i_3 - \mathcal{H}_{k,n+3}i_4 = (\mathcal{H}_{k,n}, -\mathcal{H}_{k,n+1}, -\mathcal{H}_{k,n+2}, -\mathcal{H}_{k,n+3}),$$

and

$$\breve{H}\mathcal{G}_n^* = \mathcal{G}_{k,n}i_1 - \mathcal{G}_{k,n+1}i_2 - \mathcal{G}_{k,n+2}i_3 - \mathcal{G}_{k,n+3}i_4 = (\mathcal{G}_{k,n}, -\mathcal{G}_{k,n+1}, -\mathcal{G}_{k,n+2}, -\mathcal{G}_{k,n+3}).$$

Definition 4.3. For $n \in \mathbb{N}$, the norms of the hyperbolic k-Leonardo $\breve{H}\mathcal{L}_{k,n}$, hyperbolic k-Leonardo-Lucas $\breve{H}\mathcal{H}_{k,n}$, and Modified k-Leonardo $\breve{H}\mathcal{G}_{k,n}$ quaternions are defined by, respectively,

$$\left\|\breve{H}\mathcal{L}_{k,n}\right\| = \sqrt{\mathcal{L}_{k,n}^2 + \mathcal{L}_{k,n+1}^2 + \mathcal{L}_{k,n+2}^2 + \mathcal{L}_{k,n+3}^2},$$

$$\left\|\breve{H}\mathcal{H}_{k,n}\right\| = \sqrt{\mathcal{H}_{k,n}^2 + \mathcal{H}_{k,n+1}^2 + \mathcal{H}_{k,n+2}^2 + \mathcal{H}_{k,n+3}^2},$$

and

$$\left\|\breve{H}\mathcal{G}_{k,n}\right\| = \sqrt{\mathcal{G}_{k,n}^2 + \mathcal{G}_{k,n+1}^2 + \mathcal{G}_{k,n+2}^2 + \mathcal{G}_{k,n+3}^2}.$$

In the following theorems, we examine the relations between the $\breve{H}\mathcal{L}_{k,n}$, $\breve{H}\mathcal{L}_{k,n}^*$, $\breve{H}\mathcal{H}_{k,n}$, $\breve{H}\mathcal{H}_{k,n}^*$, $\breve{H}\mathcal{G}_{k,n}$, and $\breve{H}\mathcal{G}_{k,n}^*$ quaternions.

Theorem 4.1. Let $n \in \mathbb{N}$. The following equations are true:

i. $\breve{H}\mathcal{L}_{k,n+3} = 2\breve{H}\mathcal{L}_{k,n+2} - \breve{H}\mathcal{L}_{k,n}$, **ii.** $\breve{H}\mathcal{L}_{k,n+3}^* = 2\breve{H}\mathcal{L}_{k,n+2}^* - \breve{H}\mathcal{L}_{k,n}^*$,

iii. $\breve{H}\mathcal{H}_{k,n+3} = 2\breve{H}\mathcal{H}_{k,n+2} - \breve{H}\mathcal{H}_{k,n}$, **iv.** $\breve{H}\mathcal{H}_{k,n+3}^* = 2\breve{H}\mathcal{H}_{k,n+2}^* - \breve{H}\mathcal{H}_{k,n}^*$,

v. $\breve{H}\mathcal{G}_{k,n+3} = 2\breve{H}\mathcal{G}_{k,n+2} - \breve{H}\mathcal{G}_{k,n}$, **vi.** $\breve{H}\mathcal{G}_{k,n+3}^* = 2\breve{H}\mathcal{G}_{k,n+2}^* - \breve{H}\mathcal{G}_{k,n}^*$.

Proof. i. If the definition is used, we have

$$\begin{aligned}2\breve{H}\mathcal{L}_{k,n+2} - \breve{H}\mathcal{L}_{k,n} &= 2\left(\mathcal{L}_{k,n+2}i_1 + \mathcal{L}_{k,n+3}i_2 + \mathcal{L}_{k,n+4}i_3 + \mathcal{L}_{k,n+5}i_4\right)\\ &\quad -\left(\mathcal{L}_{k,n}i_1 + \mathcal{L}_{k,n+1}i_2 + \mathcal{L}_{k,n+2}i_3 + \mathcal{L}_{k,n+3}i_4\right)\\ &= \left(2\mathcal{L}_{k,n+2} - \mathcal{L}_{k,n}\right)i_1 + \left(2\mathcal{L}_{k,n+3} - \mathcal{L}_{k,n+1}\right)i_2 + \left(2\mathcal{L}_{k,n+4} - \mathcal{L}_{k,n+2}\right)i_3\\ &\quad +\left(2\mathcal{L}_{k,n+5} - \mathcal{L}_{k,n+3}\right)i_4.\end{aligned}$$

Since, $\mathcal{L}_{k,n+3} = 2\mathcal{L}_{k,n+2} - \mathcal{L}_{k,n}$. Thus, we obtain

$$\breve{H}\mathcal{L}_{k,n+3} = 2\breve{H}\mathcal{L}_{k,n+2} - \breve{H}\mathcal{L}_{k,n}.$$

Other proofs are shown in a similar way. □

Theorem 4.2. Let $n \in \mathbb{N}$. We obtain

i. $\breve{H}\mathcal{L}_{k,n} = \frac{-4k+15}{k^2-7k+11}\breve{H}\mathcal{H}_{k,n+2} + \frac{5k-14}{k^2-7k+11}\breve{H}\mathcal{H}_{k,n+1} - \frac{k^2-6k+12}{k^2-7k+11}\breve{H}\mathcal{H}_{k,n}$,

ii. $\breve{H}\mathcal{L}_{k,n} = \frac{k}{k^2-7k+11}\breve{H}\mathcal{G}_{k,n+2} - \frac{2k+1}{k^2-7k+11}\breve{H}\mathcal{G}_{k,n+1} + \frac{2k^2}{k^2-7k+11}\breve{H}\mathcal{G}_{k,n}$,

iii. $\breve{H}\mathcal{H}_{k,n} = \frac{4}{k+1}\breve{H}\mathcal{L}_{k,n+2} - \frac{5}{k+1}\breve{H}\mathcal{L}_{k,n+1} - \frac{k}{k+1}\breve{H}\mathcal{L}_{k,n}$,

iv. $\breve{H}\mathcal{H}_{k,n} = \frac{3k-3}{k^2-k-1}\breve{H}\mathcal{G}_{k,n+2} - \frac{3k}{k^2-k-1}\breve{H}\mathcal{G}_{k,n+1} + \frac{-k^2+k+4}{k^2-k-1}\breve{H}\mathcal{G}_{k,n}$,

v. $\breve{H}\mathcal{G}_{k,n} = -\frac{1}{k+1}\breve{H}\mathcal{L}_{k,n+2} + \frac{2}{k+1}\breve{H}\mathcal{L}_{k,n+1} + \frac{k}{k+1}\breve{H}\mathcal{L}_{k,n}$,

vi. $\breve{H}\mathcal{G}_{k,n} = \frac{-3k+12}{k^2-7k+11}\breve{H}\mathcal{H}_{k,n+2} + \frac{3k-9}{k^2-7k+11}\breve{H}\mathcal{H}_{k,n+1} + \frac{-k^2+7k-14}{k^2-7k+11}\breve{H}\mathcal{H}_{k,n}$.

Proof. iii. The following relation is used for proofs;

$$\breve{H}\mathcal{H}_{k,n} = ax\breve{H}\mathcal{L}_{k,n+2} + bx\breve{H}\mathcal{L}_{k,n+1} + cx\breve{H}\mathcal{L}_{k,n}.$$

For these n values, we obtain;

$$\breve{H}\mathcal{H}_{k,0} = ax\breve{H}\mathcal{L}_{k,2} + bx\breve{H}\mathcal{L}_{k,1} + cx\breve{H}\mathcal{L}_{k,0},$$

$$\breve{H}\mathcal{H}_{k,1} = ax\breve{H}\mathcal{L}_{k,3} + bx\breve{H}\mathcal{L}_{k,2} + cx\breve{H}\mathcal{L}_{k,1},$$

$$\breve{H}\mathcal{H}_{k,2} = ax\breve{H}\mathcal{L}_{k,4} + bx\breve{H}\mathcal{L}_{k,3} + cx\breve{H}\mathcal{L}_{k,2}.$$

We find

$$a = \frac{4}{k+1},\ b = -\frac{5}{k+1},\ \text{and}\ c = -\frac{k}{k+1}.$$

Thus, we obtain

$$\breve{H}\mathcal{H}_{k,n} = \frac{4}{k+1}\breve{H}\mathcal{L}_{k,n+2} - \frac{5}{k+1}\breve{H}\mathcal{L}_{k,n+1} - \frac{k}{k+1}\breve{H}\mathcal{L}_{k,n}.$$

Other proofs are shown in a similar way. □

In the following theorem, the Binet formulas of the $\breve{H}\mathcal{L}_{k,n}$, $\breve{H}\mathcal{L}^*_{k,n}$, $\breve{H}\mathcal{H}_{k,n}$, $\breve{H}\mathcal{H}^*_{k,n}$, $\breve{H}\mathcal{G}_{k,n}$, and $\breve{H}\mathcal{G}^*_{k,n}$ quaternions are expressed.

Theorem 4.3. Let $n \in \mathbb{N}$. We obtain

i. $\breve{H}\mathcal{L}_{k,n} = \overline{\alpha}\frac{1+k}{(\alpha-\beta)(\alpha-1)} + \overline{\beta}\frac{1+k}{(\beta-\alpha)(\beta-1)} - \overline{\delta}k,$

ii. $\breve{H}\mathcal{L}^*_{k,n} = \overline{\alpha}^*\frac{1+k}{(\alpha-\beta)(\alpha-1)} + \overline{\beta}^*\frac{1+k}{(\beta-\alpha)(\beta-1)} - \overline{\delta}^*k,$

iii. $\breve{H}\mathcal{H}_{k,n} = \overline{\alpha}\frac{4-k-\alpha}{(\alpha-\beta)(\alpha-1)} + \overline{\beta}\frac{4-k-\beta}{(\beta-\alpha)(\beta-1)} + \overline{\delta}k,$

iv. $\breve{H}\mathcal{H}^*_{k,n} = \overline{\alpha}^*\frac{4-k-\alpha}{(\alpha-\beta)(\alpha-1)} + \overline{\beta}^*\frac{4-k-\beta}{(\beta-\alpha)(\beta-1)} + \overline{\delta}^*k,$

v. $\breve{H}\mathcal{G}_{k,n} = \overline{\alpha}\frac{-1+k+\alpha}{(\alpha-\beta)(\alpha-1)} + \overline{\beta}\frac{-1+k+\beta}{(\beta-\alpha)(\beta-1)} - \overline{\delta}k,$

vi. $\breve{H}\mathcal{G}^*_{k,n} = \overline{\alpha}^*\frac{-1+k+\alpha}{(\alpha-\beta)(\alpha-1)} + \overline{\beta}^*\frac{-1+k+\beta}{(\beta-\alpha)(\beta-1)} - \overline{\delta}^*k,$

where

$$\overline{\alpha} = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 = (1, \alpha, \alpha^2, \alpha^3),\ \overline{\alpha}^* = (1, -\alpha, -\alpha^2, -\alpha^3),$$

$$\overline{\beta} = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 = (1, \beta, \beta^2, \beta^3),\ \overline{\beta}^* = (1, -\beta, -\beta^2, -\beta^3),$$

$$\overline{\delta} = i_1 + i_2 + i_3 + i_4 = (1,1,1,1)\ \text{and}\ \overline{\delta}^* = (1, -1, -1, -1).$$

Proof. i. The Binet form of the hyperbolic k-Leonardo quaternions is

$$\breve{H}\mathcal{L}_{k,n} = x\frac{1+k}{(\alpha-\beta)(\alpha-1)} + y\frac{1+k}{(\beta-\alpha)(\beta-1)} - zk.$$

With the initial conditions, the following equations are obtained. For $n = \{0, 1, 2\}$, when the resulting equation is solved

$$x = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 = \overline{\alpha}$$

$$y = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 = \overline{\beta},$$

$$z = i_1 + i_2 + i_3 + i_4 = \overline{\delta}.$$

Other proofs are shown in a similar way. □

In the following theorems, we associate the terms of the k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo sequences with their hyperbolic quaternion values.

Theorem 4.4. We obtain

i. $\breve{H}\mathcal{L}_{k,n} + \breve{H}\mathcal{L}^*_{k,n} = 2\mathcal{L}_{k,n}i_1$, **ii.** $\breve{H}\mathcal{L}^2_{k,n} = 2\mathcal{L}_{k,n}\breve{H}\mathcal{L}_{k,n} + (\|\breve{H}\mathcal{L}_{k,n}\|^2 - 2\mathcal{L}^2_{k,n})i_1$,

iii. $\breve{H}\mathcal{H}_{k,n} + \breve{H}\mathcal{H}^*_{k,n} = 2\mathcal{H}_{k,n}i_1$, **iv.** $\breve{H}\mathcal{H}^2_{k,n} = 2\mathcal{H}_{k,n}\breve{H}\mathcal{H}_{k,n} + (\|\breve{H}\mathcal{H}_{k,n}\|^2 - 2\mathcal{H}^2_{k,n})i_1$,

v. $\breve{H}\mathcal{G}_{k,n} + \breve{H}\mathcal{G}^*_{k,n} = 2\mathcal{G}_{k,n}i_1$, **vi.** $\breve{H}\mathcal{G}^2_{k,n} = 2\mathcal{G}_{k,n}\breve{H}\mathcal{G}_{k,n} + (\|\breve{H}\mathcal{G}_{k,n}\|^2 - 2\mathcal{G}^2_{k,n})i_1$.

Proof. ii. If the definition is used, we have

$$\breve{H}\mathcal{L}^2_n = (\mathcal{L}_{k,n}i_1 + \mathcal{L}_{k,n+1}i_2 + \mathcal{L}_{k,n+2}i_3 + \mathcal{L}_{k,n+3}i_3)(\mathcal{L}_{k,n}i_1 + \mathcal{L}_{k,n+1}i_2 + \mathcal{L}_{k,n+2}i_3 + \mathcal{L}_{k,n+3}i_4)$$
$$= \left(\mathcal{L}^2_{k,n} + \mathcal{L}^2_{k,n+1} + \mathcal{L}^2_{k,n+2} + \mathcal{L}^2_{k,n+3}\right)i_1 + (2\mathcal{L}_{k,n}\mathcal{L}_{k,n+1})i_2 + (2\mathcal{L}_{k,n}\mathcal{L}_{k,n+2})i_3$$
$$+(2\mathcal{L}_{k,n}\mathcal{L}_{k,n+3})i_4$$
$$= 2\mathcal{L}_{k,n}\left(\mathcal{L}_{k,n}i_1 + \mathcal{L}_{k,n+1}i_2 + \mathcal{L}_{k,n+2}i_3 + \mathcal{L}_{k,n+3}i_4\right) + (-\mathcal{L}^2_{k,n} + \mathcal{L}^2_{k,n+1} + \mathcal{L}^2_{k,n+2} + \mathcal{L}^2_{k,n+3})i_1.$$

Thus, we obtain

$$\breve{H}\mathcal{L}^2_{k,n} = 2\mathcal{L}_{k,n}\breve{H}\mathcal{L}_{k,n} + (\|\breve{H}\mathcal{L}_{k,n}\|^2 - 2\mathcal{L}^2_{k,n})i_1.$$

Other proofs are shown in a similar way. □

Theorem 4.5. We obtain

i. $i_1\breve{H}\mathcal{L}_{k,n} - i_2\breve{H}\mathcal{L}_{k,n+1} - i_3\breve{H}\mathcal{L}_{k,n+2} - i_4\breve{H}\mathcal{L}_{k,n+3} = (\mathcal{L}_{k,n} - \mathcal{L}_{k,n+2} - \mathcal{L}_{k,n+4} + \mathcal{L}_{k,n+6})i_1$.

ii. $i_1\breve{H}\mathcal{H}_{k,n} - i_2\breve{H}\mathcal{H}_{k,n+1} - i_3\breve{H}\mathcal{H}_{k,n+2} - i_4\breve{H}\mathcal{H}_{k,n+3} = (\mathcal{H}_{k,n} - 6\mathcal{H}_{k,n+2} + 5\mathcal{H}_{k,n+3})i_1$,

iii. $i_1\breve{H}\mathcal{G}_{k,n} - i_2\breve{H}\mathcal{G}_{k,n+1} - i_3\breve{H}\mathcal{G}_{k,n+2} - i_4\breve{H}\mathcal{G}_{k,n+3} = (-4\mathcal{G}_{k,n} + 4\mathcal{G}_{k,n+2})i_1$.

Proof. i. If the definition is used, we have

$$i_1\breve{H}\mathcal{L}_{k,n} - i_2\breve{H}\mathcal{L}_{k,n+1} - i_3\breve{H}\mathcal{L}_{k,n+2} - i_4\breve{H}\mathcal{L}_{k,n+3} = \left(\mathcal{L}_{k,n} - \mathcal{L}_{k,n+2} - \mathcal{L}_{k,n+4} + \mathcal{L}_{k,n+6}\right)i_1$$
$$+\left(\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n+1} - \mathcal{L}_{k,n+5} + \mathcal{L}_{k,n+5}\right)i_2 + \left(\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n+4} - \mathcal{L}_{k,n+2} - \mathcal{L}_{k,n+4}\right)i_3$$
$$+(\mathcal{L}_{k,n+3} - \mathcal{L}_{k,n+3} + \mathcal{L}_{k,n+3} - \mathcal{L}_{k,n+3})i_4$$
$$= (\mathcal{L}_{k,n} - \mathcal{L}_{k,n+2} - \mathcal{L}_{k,n+4} + \mathcal{L}_{k,n+6})i_1.$$

Other proofs are shown in a similar way. □

5 CONCLUSIONS

In this study, we examine the k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo sequences. Then, we found the features of these sequences. In addition, we obtained the relationships between the terms of these sequences. Moreover, we examined k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo sequences with Fibonacci and Lucas numbers. Finally, we associated the terms of these sequences with matrices. We presented on the application of generalized k-Leonardo sequences to hyperbolic quaternions. Then we defined

hyperbolic k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo quaternions. For these quaternions, we given many properties such as Binet formulas. Finally, the terms of the k-Leonardo, k-Leonardo-Lucas, and Modified k-Leonardo sequences associated with their hyperbolic quaternion values. If this study is examined, such features can be found in other sequences such as Fermat, Olivier, and Mersenne sequences.

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