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k-Fibonacci Polynomials in The Family of Fibonacci Numbers

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Abstract

In this study, we define k-Fibonacci Polynomials by using terms of a new family of Fibonacci numbers given by Mikkawy and Sogabe. We compare the polynomials with known Fibonacci polynomial. Furthermore, we show the relationship between Pascal's triangle and the coefficient of the $k-$Fibonacci polynomials. We give some important properties of the polynomial.

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INTRODUCTION

Fibonacci numbers have many applications in mathematics [1, 2–5]. Hoggatt et al. [6] have obtained some more identities for Fibonacci and Lucas Polynomials. There exist generalizations of the Fibonacci numbers. Hoggatt and Bicknell [7] generalized the Fibonacci polynomials and introduced their relationship to diagonals of Pascal's triangle. Falcon and Plaza [8] presented the derivatives of Fibonacci polynomials as convolution of the Fibonacci polynomials and proved many relations for the derivatives of Fibonacci polynomials.

MATERIALS AND METHODS

Let *n* and $k \neq 0$ be natural numbers, then there exist unique numbers m and r such that $n = mk + r$ $(0 \leq r < k)$. The generalized k-Fibonacci numbers $F_n^{(k)}$ are defined by

$$F_n^{(k)} = \frac{1}{(\sqrt{5})^k}(\alpha^{m+2} - \beta^{m+2})^r(\alpha^{m+1} - \beta^{m+1})^{k-r}, n = mk + r$$

where, $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$.

It is well known that the relation of generalized k −Fibonacci and Fibonacci numbers is

$$F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r, \qquad n = mk + r\,.$$

Also, it is apparent that, when $k = 1$ in last equation, $m = n$ and $r = 0$, $F_n^{(1)} = F_n$.

The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

where $F_0(x) = 0, F_1(x) = 1$ and $F_2(x) = x$ and $n \geq 3$.

Definition: The 2-Fibonacci polynomials are defined by

$$F_{n+1}^{(2)}(x) = xF_n^{(2)}(x) + F_{n-1}^{(2)}(x)$$

where, $F_0^{(2)}(x) = 1$ and $F_1^{(2)}(x) = 1$ [9].

The 2-Fibonacci polynomials are given by

$$F_n^{(2)}(x) = F_{n-1}(x) + F_n(x), \ n \geq 1.$$

Definition: The 3-Fibonacci polynomials are defined by

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$F_{n+1}^{(3)}(x) = xF_n^{(3)}(x) + F_{n-1}^{(3)}(x), \quad n \geq 2$
where, $F_1^{(3)}(x) = 1$, $F_2^{(3)}(x) = 1$ [10].

The 3-Fibonacci polynomials are given by
$F_n^{(3)}(x) = F_{n-2}(x) + F_{n-1}(x), \ n > 1.$

MAIN RESULTS

Definition: The k-Fibonacci polynomials are defined by (Table 1)
$F_{n+1}^{(k)}(x) = xF_n^{(k)}(x) + F_{n-1}^{(k)}(x), n \geq k-1,$
where $F_0^{(k)}(x) = F_1^{(k)}(x) = F_{k-1}^{(k)}(x) = 1.$

It is note that we have an important relation between k-Fibonacci polynomials and usual Fibonacci polynomials as following theorem:

Theorem: The k-Fibonacci polynomials are given by
$F_n^{(k)}(x) = F_{n-k+2}(x) + F_{n-k+1}(x), \ n \geq k.$

Proof: We can prove the theorem by the induction method on n. For $n = k$, we have
$$F_k^{(k)}(x) = x + 1 = F_2(x) + F_1(x).$$

Now, assume that the theorem holds for $n = m$, that is
$$F_m^{(k)}(x) = F_{m-k+2}(x) + F_{m-k+1}(x).$$
Then, for $n = m+1$, we have
$$\begin{aligned} F_{m+1}^{(k)}(x) &= xF_m^{(k)}(x) + F_{m-1}^{(k)}(x) \\ &= x\big(F_{m-k+2}(x) + F_{m-k+1}(x)\big) \\ &\quad + (F_{m-k+1}(x) + F_{m-k}(x)) \\ &= xF_{m-k+2}(x) + F_{m-k+1}(x) + xF_{m-k+1}(x) \\ &\quad + F_{m-k}(x) \\ &= F_{m-k+3}(x) + F_{m-k+2}(x). \end{aligned}$$

We can verify that
$$F_n^{(k)}(x) = \frac{\alpha^{n-k+1}(x)(\alpha(x)+1) - \beta^{n-k+1}(x)(\beta(x)+1)}{\alpha(x) - \beta(x)}$$

where, $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$.

The k-Fibonacci polynomials are obtained by a matrix $\mathbb{Q}_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$. Since
$$F_n^{(k)}(x) = F_{n-k+2}(x) + F_{n-k+1}(x),$$

$$\begin{aligned} \mathbb{Q}_2^{n-k} + \mathbb{Q}_2^{n-k+1} &= \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^{n-k+1} \\ &\quad + \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^{n-k} \\ &= \begin{pmatrix} F_{n+1}^{(k)}(x) & F_n^{(k)}(x) \\ F_n^{(k)}(x) & F_{n-1}^{(k)}(x) \end{pmatrix}. \end{aligned}$$

We can write the following interesting determinant identity.
$$det\begin{pmatrix} F_{n+1}^{(k)}(x) & F_n^{(k)}(x) \\ F_n^{(k)}(x) & F_{n-1}^{(k)}(x) \end{pmatrix} = \begin{cases} (-1)^{n+1}x, n \ even \\ (-1)^n x, n \ odd \end{cases}.$$

Table 1: *The First few k-Fibonacci Polynomials and the Array of their Coefficients.*

***k*-Fibonacci polynomials**	**Coefficient array**
$F_1^{(k)}(x) = 1$	1
$F_2^{(k)}(x) = 1$	1
⋮	
$F_k^{(k)}(x) = 1 + x$	1 1
$F_{k+1}^{(k)}(x) = 1 + x + x^2$	1 1 1
$F_{k+2}^{(k)}(x) = 1 + 2x + x^2 + x^3$	1 2 1 1
$F_{k+3}^{(k)}(x) = 1 + 2x + 3x^2 + x^3 + x^4$	1 2 3 1 1
$F_{k+4}^{(k)}(x) = 1 + 3x + 3x^2 + 4x^3 + x^4 + x^5$	1 3 3 4 1 1
$F_{k+5}^{(k)}(x) = 1 + 3x + 6x^2 + 4x^3 + 5x^4 + x^5 + x^6$	1 3 6 4 5 1 1
$F_{k+6}^{(k)}(x) = 1 + 4x + 6x^2 + 10x^3 + 5x^4 + 6x^5 + x^6 + x^7$	1 4 6 10 5 6 1
$F_{k+7}^{(k)}(x) = 1 + 4x + 10x^2 + 10x^3 + 15x^4 + 6x^5 + 7x^6 + x^7 + x^8$	1 4 10 10 15 6 7 1 1
$F_{k+8}^{(k)}(x) = 1 + 5x + 10x^2 + 20x^3 + 15x^4 + 21x^5 + 7x^6 + 8x^7 + x^8 + x^9$	1 5 10 20 15 21 7 8 1 1

Theorem: For $n > 0$, we have

$$F_n^{(k)}(x) = 1 + x \sum_{i=1}^{n-k+1} F_i(x).$$

Proof: Using the recurrence relation $F_{i+1}(x) = xF_i(x) + F_{i-1}(x)$,

$$\sum_{i=1}^{n} F_{i+1}(x) = x \sum_{i=1}^{n} F_i(x) + \sum_{i=1}^{n} F_{i-1}(x).$$

That is

$$F_n(x) + F_{n+1}(x) = x \sum_{i=1}^{n} F_i(x) + F_0(x) + F_1(x)$$

Since $F_0(x) = 0$, it follows that

$$F_n(x) + F_{n+1}(x) = 1 + x \sum_{i=1}^{n} F_i(x).$$

Then

$$F_n^{(k)}(x) = F_{n-k+1}(x) + F_{n-k+2}(x) = 1 + x \sum_{i=1}^{n-k+1} F_i(x).$$

Theorem: More generally, the sum of the elements along the diagonal beginning at row n is $F_n^{(k)}(x)$; that is,

$$F_n^{(k)}(x) = \sum_{j=0}^{[\frac{n-k+1}{2}]} \binom{n-k-j}{j} x^{n-k-2j} + \binom{n-k-j+1}{j} x^{n-k-2j+1}.$$

Proof: We have

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} F_n(x) y^n.$$

But

$$\frac{1}{1-2tz+z^2} = \sum_{n=0}^{\infty} \left[\sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \binom{n-j}{j} (2t)^{n-2j} \right] z^n$$

$$U_n(t) = \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \binom{n-j}{j} (2t)^{n-2j}$$

Let $z = iy$ and $t = x/2i$, where $i^2 = -1$, then $\frac{1}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n U_n(x/2i) y^n$.

Multiplying the last equation by y, we get

$$\frac{y}{1-xy-y^2} = \sum_{n=0}^{\infty} i^n U_n(x/2i) y^{n+1}.$$

It follows that

$$F_{n-k+1}(x) = i^{n-k} U_{n-k}(x/2i) = i^{n-k} \sum_{j=0}^{[\frac{n-k+1}{2}]} (-1)^j \binom{n-k-j}{j} (x/i)^{n-k-2j} = \sum_{j=0}^{[\frac{n-k+1}{2}]} \binom{n-k-j}{j} x^{n-k-2j}$$

and

$$F_{n-k+2}(x) = \sum_{j=0}^{[\frac{n-k+2}{2}]} \binom{n-k-j-1}{j} (2t)^{n-k-2j+1}.$$

Since $F_n^{(k)}(x) = F_{n-k+1}(x) + F_{n-k+2}(x)$, we get that

$$F_n^{(k)}(x) = \sum_{j=0}^{[\frac{n-k+1}{2}]} \binom{n-k-j}{j} x^{n-k-2j} + \binom{n-k-j-1}{j} x^{n-k-2j+1}.$$

Theorem: The derivative of the k-Fibonacci polynomial $F_n^{(k)}(x)$ is

$$(F_n^{(k)}(x))' = \sum_{i=1}^{n-k+1} F_i(x).\left(F_{n-k-i+1}(x) + F_{n-k-i+2}(x)\right).$$

Proof. It is known that

$$F_n^{(k)}(x) = F_{n-k+1}(x) + F_{n-k+2}(x)$$

where, $F_n'(x)$ denotes the derivative of $F_n(x)$ with respect to x and $n \geq 1$.
Computing the derivative both side of

$$(F_n^{(3)}(x))' = (F_{n-k+1}(x) + F_{n-k+2}(x))'.$$

$$F'_{n-k+1}(x) = \sum_{i=1}^{n-k} F_i(x)F_{n-k-i+1}(x),$$

$$F'_{n-k+2}(x) = \sum_{i=1}^{n-k+1} F_i(x)F_{n-k-i+2}(x).$$

We get

$$(F_n^{(k)}(x))' = \sum_{i=1}^{n-k+1} F_i(x)\big(F_{n-k-i+1}(x) + F_{n-k-i+2}(x)\big)$$

since $F_0^{(k)}(x) = F_1^{(k)}(x) = \cdots = F_{k-1}^{(k)}(x) = 1.$

Example: The derivative of

$F_7^{(3)}(x) = x^5 + x^4 + 4x^3 + 3x^2 + 3x + 1$
$\sum_{i=1}^{5} F_i(x)[F_{6-i}(x) + F_{5-i}(x)] = F_1(x)[F_5(x) + F_4(x)] + F_2(x)[F_4(x) + F_3(x)] + F_3(x)[F_3(x) + F_2(x)] + F_4(x)[F_2(x) + F_1(x)] + F_5(x)[F_1(x) + F_0(x)] = F_1(x)F_5(x) + F_1(x)F_4(x) + F_2(x)F_4(x) + F_2(x)F_3(x) \quad + F_3(x)F_3(x) + F_3(x)F_2(x) + F_4(x)F_2(x) + F_4(x)F_1(x) + F_5(x)F_1(x) + F_5(x)F_0(x) = 2F_1(x)F_5(x) + 2F_1(x)F_4(x) + 2F_2(x)F_3(x) + 2F_2(x)F_4(x) + F_3^2(x) = 2.1.(x^4 + 3x^2 + 1) + 2 \cdot 1 \cdot (x^3 + 2x) + 2 \cdot x \cdot (x^2 + 1) + 2 \cdot x \cdot (x^3 + 2x) + (x^2 + 1)^2 = 2x^4 + 6x^2 + 2 + 2x^3 + 4x + 2x^3 + 2x + 2x^4 + 4x^2 + x^4 + 2x^2 + 1 = 5x^4 + 4x^3 + 12x^2 + 6x + 3.$

CONCLUSIONS

In the present paper, we define k-Fibonacci polynomials and give some identities between the k-Fibonacci polynomials and Fibonacci polynomials. Also, we present some properties of the k-Fibonacci polynomials and introduce the derivative of the k-Fibonacci polynomials. This enables us to give in a straightforward way several formulas for the sums of such polynomials. We can give the following equality: $F_n^{(2)}(x) = F_{n+1}^{(3)}(x) = F_{n+2}^{(4)}(x) = \cdots = F_{2n-2}^{(n)}$, for $n > 0$.

These identities can be used to develop new identities of polynomials.

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