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On Pell and Pell-Lucas Vectors and Their Properties

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Highlights

- This paper focuses on new applications of Pell and Pell–Lucas numbers.
- The vectors obtained from the Pell and Pell–Lucas numbers have been examined.
- The study focuses on both the algebraic and sequence properties of these vectors.

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Abstract

In this paper, we define a new application of Pell and Pell-Lucas numbers, the vectors of Pell and Pell-Lucas sequences. We examine them in 2 and 3 dimensions, then we give the definition of generalized vectors. Then, we define these vectors as vector sequences. We analyze the properties of these vectors both as linear algebra and as a number sequence. We first investigate the vectorial properties of these generalized vectors. Then, we define these generalized vectors as a sequence of numbers and give the recurrence relation. We also calculate the cosines of the angles between these vectors. Finally, we derive some important identities calculated in these number sequences for these vector sequences.

1. INTRODUCTION

Fibonacci numbers were popularized in the thirteenth century by the Italian mathematician Leonardo Fibonacci through his famous work Liber Abaci. Although the sequence itself was known to earlier civilizations, Fibonacci systematically introduced it and investigated several of its properties in this work. The Fibonacci sequence is defined by a simple recurrence relation in which each term is obtained as the sum of the two preceding terms. One of the most well-known applications presented by Fibonacci is the modeling of rabbit population growth, which illustrates the natural emergence of this sequence. Owing to their simplicity and rich structure, Fibonacci numbers have continued to attract the attention of researchers, and numerous studies have been devoted to their properties and generalizations [1-14].

Motivated by Fibonacci numbers, several related integer sequences have been introduced and studied, including Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas sequences. Among these, Pell and Pell–Lucas numbers play an important role due to their close connection with quadratic Diophantine equations and linear recurrence relations [15-26].

Pell numbers P_n are defined by a second-order recurrence relation, while Pell–Lucas numbers Q_n satisfy a similar recurrence with different initial conditions. Their explicit representations are given by Binet-type formulas involving the roots of the characteristic equation $x^2 - 2x - 1 = 0$.

Pell numbers P_n are defined as follows [17].

$$P_1 = 1, P_2 = 2,\ P_n = 2P_{n-1} + P_n,\ \ n \geq 3.$$

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Pell-Lucas numbers Q_n are defined as follows [22].

$$Q_1 = 1, Q_2 = 3,\ \ Q_n = 2Q_{n-1} + Q_n,\ \ n \geq 3.$$

Binet formula of Pell, Pell- Lucas numbers are defined as follows [22].

$$P_n = \frac{a^n - b^n}{a - b},\quad Q_n = \frac{a^n + b^n}{2}$$

where a and b are roots of $x^2 - 2x - 1 = 0$.

Furthermore, these sequences can be extended to negative indices, preserving important algebraic symmetries.

Negative index Pell and Pell-Lucas numbers are defined as follows [20].

$$P_{-n} = (-1)^{n+1}P_n,\ \ Q_{-n} = (-1)^n Q_n.$$

Over the years, many fundamental identities relating Pell and Pell–Lucas numbers, such as Cassini, Catalan, Vajda, and d'Ocagne type identities, have been established. These identities form the basis for further investigations and are frequently used in the study of generalized structures associated with these sequences.

We now state a theorem that establishes the fundamental relations between Pell and Pell–Lucas numbers, which will be used in the proof of Theorem 2.1.19. This theorem summarizes the basic identities connecting these two sequences and is based on known results in the literature [18,20,22].

Theorem 1.1. It follows from the above discussion that:

1) $P_n + P_{n-1} = Q_n \ \ n \geq 2$,
2) $Q_n + Q_{n-1} = 2P_n$,
3) $P_n + Q_n = P_{n+1}$,
4) $2P_n + Q_n = Q_{n+1}$,
5) $2Q_n + 3P_n = P_{n+2}$,
6) $3Q_n + 4P_n = Q_{n+2}$,
7) $Q_{n+1} - Q_n = 2P_n$,
8) $P_{n+1} + P_{n-1} = 2Q_n$,
9) $Q_{n+1} + Q_{n-1} = 4P_n$,
10) $P_n + P_{n+1} + P_{n+3} = 3P_{n+2}$,
11) $Q_n + Q_{n+1} + Q_{n+3} = 3Q_{n+2}$,
12) $Q_{n+1} - Q_{n-1} = 2Q_n$,
13) $P_{n+1} - P_{n-1} = 2P_n$,
14) $P_{n+2} + P_{n-2} = 6P_n$,
15) $Q_{n+2} + Q_{n-2} = 6Q_n$,
16) $P_{n+2} - P_{n-2} = 4Q_n$,
17) $Q_{n+2} - Q_{n-2} = 8P_n$,
18) $2P_n + Q_{n+2} = 3Q_{n+1}$,
19) $P_{n+1} + Q_{n-1} = 3P_n$,
20) $P_{2n} = 2P_n Q_n$.

Theorem 1.2. For $m, n \geq 0$, the following equalities are satisfied [18].

1) $P_m P_{n+1} - P_{m+1}P_n = (-1)^n P_{m-n}$,

$$2)\ Q_m Q_{n+1} - Q_{m+1} Q_n = 2(-1)^{n-1} P_{m-n}.$$

Theorem 1.3. (Sum formula) [26]

$$1)\ \sum_{i=1}^{n} P_{2i} = \frac{P_{2n+1}-1}{2},$$

$$2)\ \sum_{i=1}^{n} Q_{2i} = \frac{Q_{2n+1}-1}{2}.$$

The vector representations of Fibonacci and Lucas numbers have previously been introduced and investigated from a geometric point of view, and various relationships between these vectors were established [27]. Building on this approach, Horadam vectors were later defined in two-dimensional spaces, and their fundamental algebraic properties were examined in detail, highlighting the usefulness of vector-based methods in the study of recurrence sequences [28].

Motivated by these developments, the present study focuses on the vector forms of Pell and Pell–Lucas numbers. First, these numbers are represented as vectors in two- and three-dimensional spaces, and illustrative examples are provided. Then, the constructions are generalized to vectors of arbitrary dimension. The proposed vectors are examined both geometrically and as vector sequences, and their linear algebraic and sequence-theoretic properties are investigated in detail. This unified approach reveals new connections between number theory and linear algebra and extends the scope of Pell-type sequences to higher-dimensional settings.

2. MATERIAL METHOD

In this chapter, the main materials consist of Pell and Pell–Lucas number sequences and their known algebraic properties. The methods employed are based on classical tools from number theory and linear algebra. In particular, vector representations of recurrence-based sequences are constructed, and their properties are investigated using inner products, vector products, norms, and matrix representations.

The analysis begins with the definition of Pell and Pell–Lucas vectors in low-dimensional spaces, followed by their generalization to arbitrary dimensions. Recurrence relations, Binet-type formulas, and generating functions are derived using algebraic techniques associated with second-order linear recurrences. Matrix methods are employed to describe the evolution of the vectors and to establish structural relations between different vector forms.

Furthermore, classical identities of integer sequences are extended to the vector setting by applying vector operations and algebraic manipulations. All results are obtained through analytical reasoning, and no numerical approximation or computational algorithm is required.

2.1. Pell and Pell–Lucas Vectors

In this section, Pell and Pell–Lucas vectors are introduced and their fundamental linear algebraic and vectorial properties are discussed in detail. In particular, various operations such as inner products, norms, and vector products are considered, and the cosines of the angles between these vectors are explicitly computed, providing a clear geometric interpretation. Moreover, these vectors are also treated as number sequences, and their recurrence relations and sequence-related properties are examined. This dual perspective allows the connection between linear algebra and number theory to be explored within a unified framework.

Definition 2.1.1. For $n, m \geq 0$, Pell and Pell-Lucas vectors are defined respectively,

$$i)\ \vec{P}_n^m = \begin{bmatrix} P_n \\ P_{n+1} \\ \vdots \\ P_{n+m-1} \end{bmatrix}_{m\times 1},$$

$$ii)\ \vec{Q}_n^m = \begin{bmatrix} Q_n \\ Q_{n+1} \\ \vdots \\ Q_{n+m-1} \end{bmatrix}_{m\times 1}$$

where P_n is nth Pell numbers and $Q_n$is nth Pell-Lucas numbers.

Note that, $\vec{P}_n^m$ and $\vec{Q}_n^m$ are the nth Pell and Pell-Lucas vectors of length m.
If negative indices, Pell and Pell-Lucas vectors are defined respectively,

$$i)\ \vec{P}_{-n}^m = \begin{bmatrix} P_{-n} \\ P_{-n+1} \\ \vdots \\ P_{-n+m-1} \end{bmatrix}_{m\times 1},$$

$$ii)\ \vec{Q}_{-n}^m = \begin{bmatrix} Q_{-n} \\ Q_{-n+1} \\ \vdots \\ Q_{-n+m-1} \end{bmatrix}_{m\times 1}$$

where P_{-n} is $-nth$ Pell numbers and Q_{-n} is$-nth$ Pell-Lucas numbers.

Example 2.1.2.

- For $m = 2$, 2 $-$Pell and Pell-Lucas vectors are

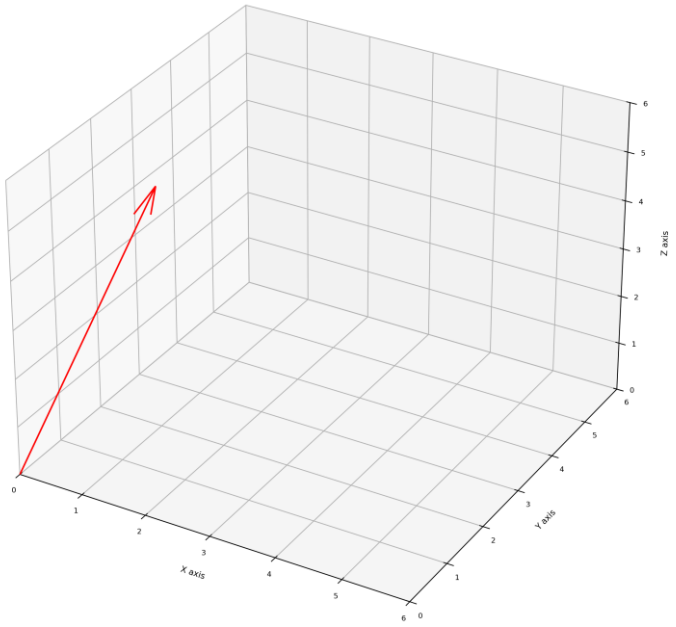
$$\vec{P}_n^2 = \begin{bmatrix} P_n \\ P_{n+1} \end{bmatrix},\quad \vec{Q}_n^2 = \begin{bmatrix} Q_n \\ Q_{n+1} \end{bmatrix}.$$

- For $m = 3$, 3 $-$Pell and Pell-Lucas vectors are

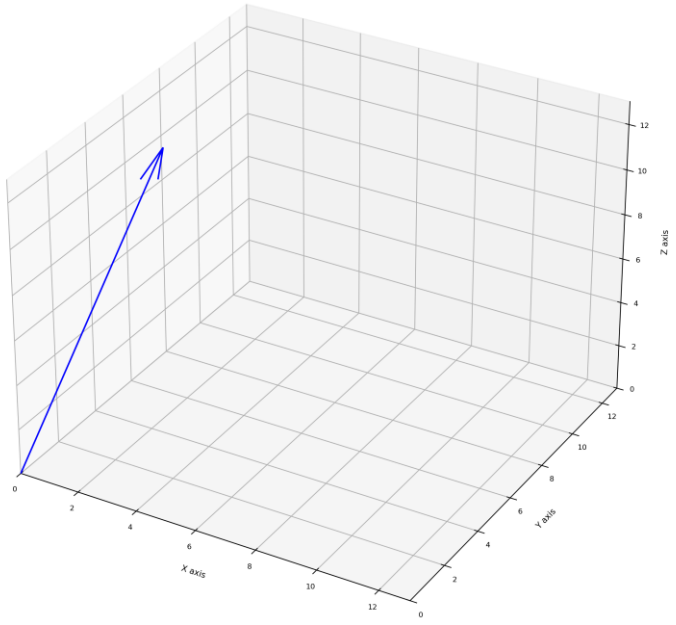
$$\vec{P}_n^3 = \begin{bmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{bmatrix},\quad \vec{Q}_n^3 = \begin{bmatrix} Q_n \\ Q_{n+1} \\ Q_{n+2} \end{bmatrix}.$$

The three-dimensional plots of the first five Pell 3-vectors and the first five Pell–Lucas 3-vectors are presented in Figures 1 and 2, respectively.

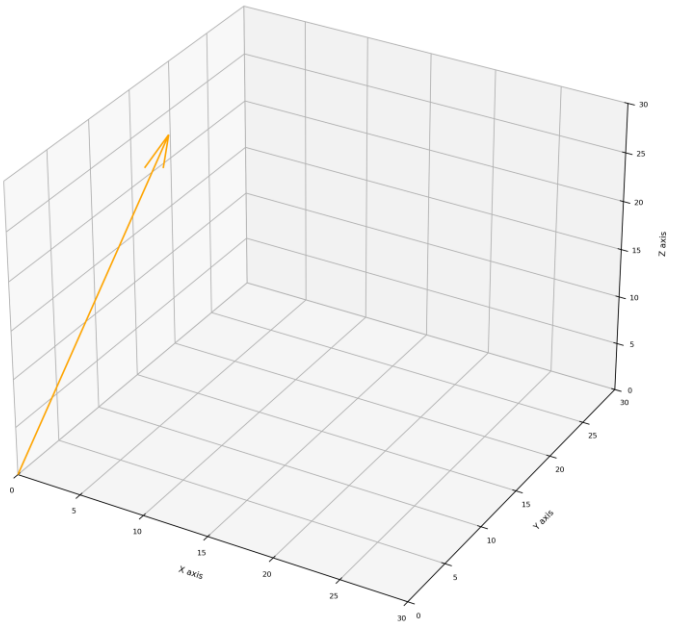
$a)\ \vec{P}_1^3$



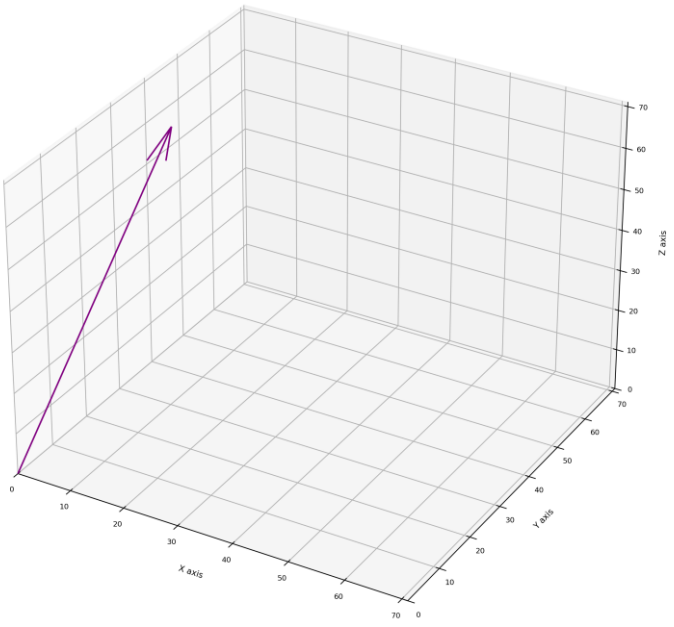
$b)\ \vec{P}_2^3$



$c)\ \vec{P}_3^3$



$d)\ \vec{P}_4^3$



$e)\ \vec{P}_5^3$

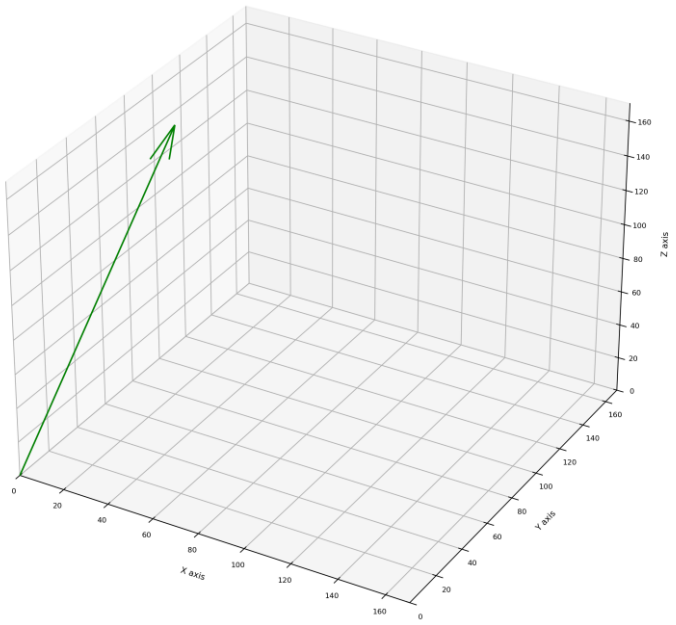
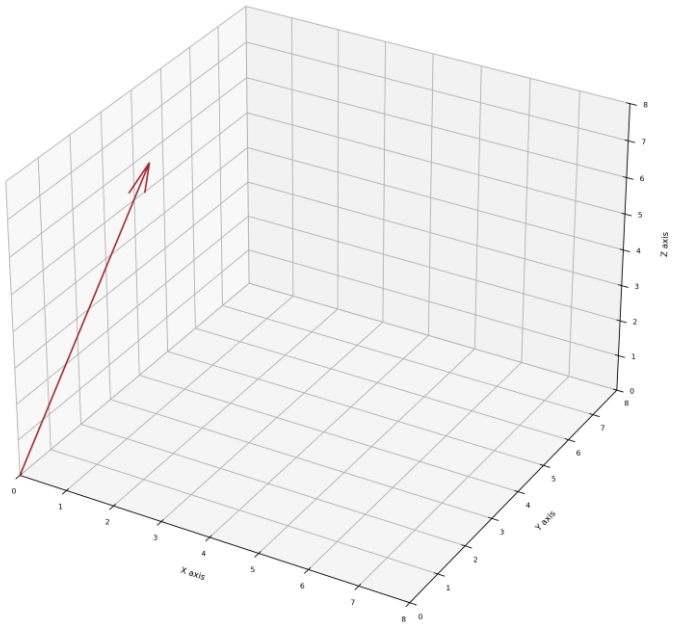
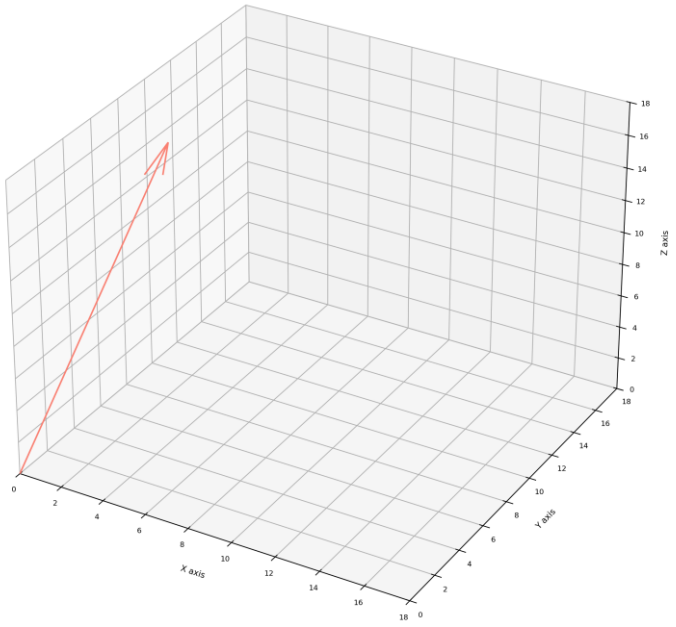


Figure 1. *$a), b), c), d), e)$ are the graphs of the Pell 3-vectors*

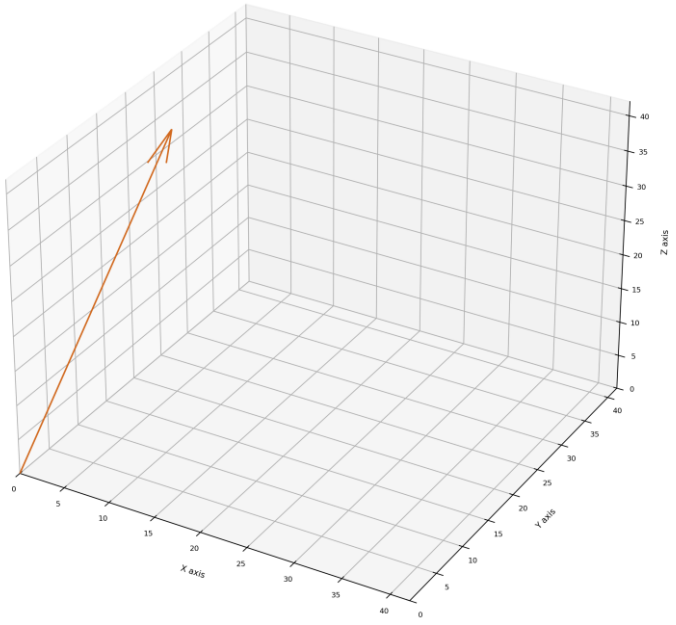
$f)\ \vec{Q}_1^3$



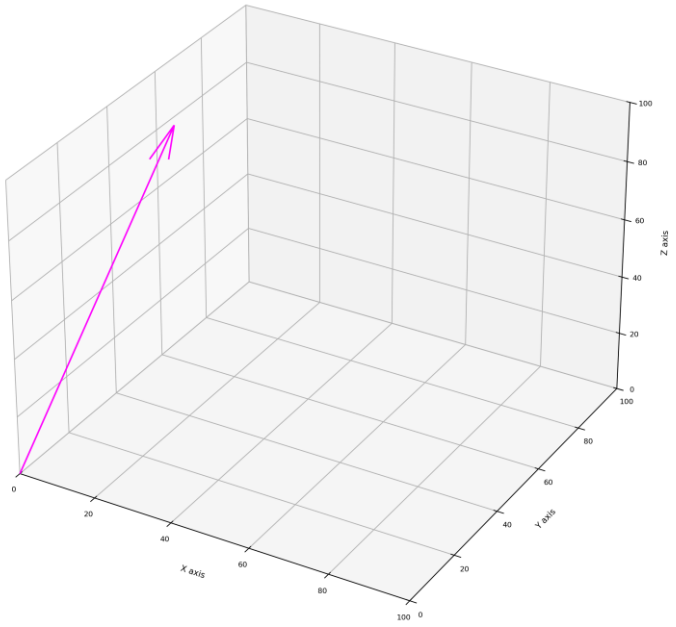
$g)\ \vec{Q}_2^3$



$h)\ \vec{Q}_3^3$



$i)\ \vec{Q}_4^3$



$k)\ \vec{Q}_5^3$

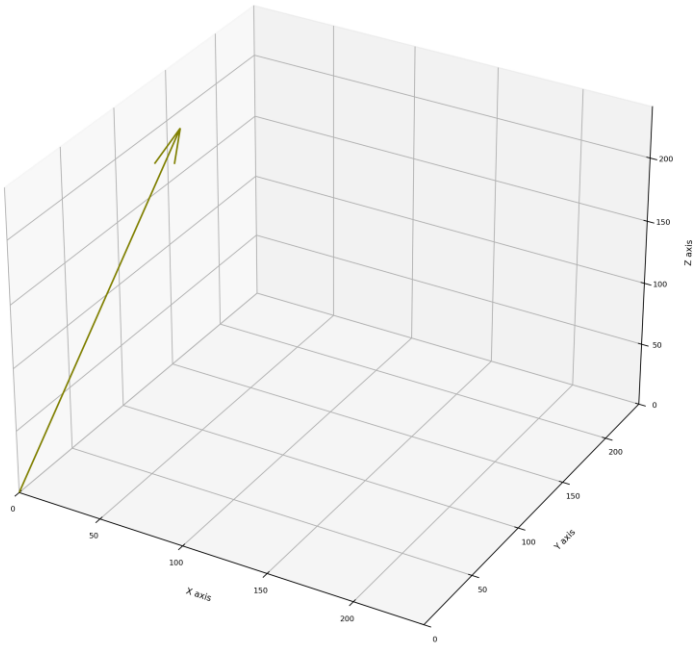


Figure 2. *$f), g), h), i), k)$ are the graphs of the Pell-Lucas 3-Vectors*

The inner (dot) product and the vector product are two fundamental operations in vector algebra, each capturing different geometric aspects of vectors. The inner product is a scalar-valued operation that measures the extent to which two vectors are aligned. It is closely related to the angle between vectors and is defined in terms of the product of their magnitudes and the cosine of the angle separating them. As a result, the inner product plays a key role in defining norms, orthogonality, and projections in vector spaces.

In contrast, the vector product is a vector-valued operation defined specifically in three-dimensional space. The resulting vector is perpendicular to the plane spanned by the original vectors, and its magnitude is proportional to the area of the parallelogram formed by them. The direction of the vector product is determined by the right-hand rule, emphasizing its geometric nature. While the inner product captures information about parallelism and angular relationships, the vector product encodes information about orientation and perpendicularity.

Despite these differences, the inner and vector products are closely related through the angle between vectors. Both operations depend on the magnitudes of the vectors and the sine or cosine of the angle between them, respectively. In particular, the inner product involves the cosine of the angle, whereas the magnitude of the vector product involves the sine of the angle. Together, these two products provide complementary geometric descriptions of vector relationships and are often used simultaneously in the analysis of vector sequences and geometric structures.

Definition 2.1.3. (Inner Product)

$$\langle \vec{P}_n^m, \vec{Q}_n^m \rangle = \vec{P}_n^m . \vec{Q}_n^m = \left[\vec{P}_n^m\right]^T \vec{Q}_n^m = [P_n \quad P_{n+1} \quad \dots \quad P_{n+m-1}] \begin{bmatrix} Q_n \\ Q_{n+1} \\ \vdots \\ Q_{n+m-1} \end{bmatrix}.$$

This is the basic definition of the inner product, but inner product can be done in different ways. For example, it can be solved with the help of Binet's formula, the definition of which we will give in the following sections.

Theorem 2.1.4. $\vec{P}_n^2$ and $\vec{Q}_n^2$ satisfy the inner products given below.

$$i)\ \langle \vec{P}_n^2, \vec{P}_m^2 \rangle = P_{n+m+1},$$

$$ii)\ \langle P_n^2, \vec{Q}_m^2 \rangle = 2P_{n+m+1},$$

$$iii)\ \langle \vec{Q}_n^2, \vec{Q}_m^2 \rangle = Q_{n+m+1}.$$

Proof. We only give the proof for $i)$. The others is done similarly.

$$i)\ \langle \vec{P}_n^2, \vec{P}_m^2 \rangle = \begin{bmatrix} P_n \\ P_{n+1} \end{bmatrix}^T \begin{bmatrix} P_m \\ P_{m+1} \end{bmatrix}$$

$$= P_{n+m+1}.$$

Definition 2.1.5. (Vector product)

$$\vec{P}_n^3 \wedge \vec{P}_m^3 = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_2 \\ P_n & P_{n+1} & P_{n+2} \\ P_m & P_{m+1} & P_{m+2} \end{vmatrix}.$$

Theorem 2.1.6. $\vec{P}_n^3$ and $\vec{Q}_m^3$ satisfy the vectoral products given below.

$$i)\ \vec{P}_n^3 \wedge \vec{P}_k^3 = (-1)^k P_{n-k} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix},$$

$$ii)\ \vec{Q}_n^3 \wedge \vec{Q}_k^3 = 2(-1)^k P_{n-k}\begin{bmatrix}1\\2\\-1\end{bmatrix}.$$

Proof.

$$i)\ \vec{P}_n^3 \wedge \vec{P}_k^3 = \begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ P_n & P_{n+1} & P_{n+2}\\ P_k & P_{k+1} & P_{k+2}\end{vmatrix}$$

$$= \vec{e}_1\left((-1)^{k+1}P_{n-k}\right) - \vec{e}_2(P_n(2P_{k+1}+P_k) - P_k(2P_{n+1}+P_n))$$

$$+\vec{e}_3((-1)^k P_{n-k})$$

$$= \vec{e}_1\left((-1)^{k+1}P_{n-k}\right) - 2\vec{e}_2(P_nP_{k+1} - P_kP_{n+1}) + \vec{e}_3((-1)^k P_{n-k})$$

$$= \vec{e}_1\left((-1)^{k+1}P_{n-k}\right) - 2\vec{e}_2\left((-1)^k P_{n-k}\right) + \vec{e}_3\left((-1)^k P_{n-k}\right)$$

$$= (-1)^k P_{n-k}(-\vec{e}_1 - 2\vec{e}_2 + \vec{e}_3)$$

$$= (-1)^k P_{n-k}\begin{bmatrix}-1\\-2\\1\end{bmatrix}.$$

The others are done similarly.

Corollary 2.1.7. This means that the vector perpendicular to $\vec{P}_n^3$ and $\vec{P}_k^3$ is

$$(-1)^k P_{n-k}\begin{bmatrix}-1\\-2\\1\end{bmatrix}.$$

Similarly, the vector perpendicular to $\vec{Q}_n^3$ and $\vec{Q}_k^3$ is

$$2(-1)^k P_{n-k}\begin{bmatrix}1\\2\\-1\end{bmatrix}.$$

Theorem 2.1.8. For any vector $\vec{P}_n^3 = \begin{bmatrix}P_n\\P_{n+1}\\P_{n+2}\end{bmatrix}$, the vector product of this vector with the next vector is

$$\vec{P}_n^3 \wedge \vec{P}_{n+1}^3 = (-1)^n(\vec{e}_1 + 2\vec{e}_2 - \vec{e}_3)$$

of the form.

Similarly, for any vector $\vec{Q}_n^3 = \begin{bmatrix}Q_n\\Q_{n+1}\\Q_{n+2}\end{bmatrix}$, the vector product of this vector with the next vector is

$$\vec{Q}_n^3 \wedge \vec{Q}_{n+1}^3 = 2(-1)^n(-\vec{e}_1 - 2\vec{e}_2 + \vec{e}_3)$$

of the form.

Proof. If we use Theorem 2.6 we get the following. For $k = n + 1$, we have

$$\begin{aligned}\vec{P}_n^3 \wedge \vec{P}_{n+1}^3 &= (-1)^{n+1} P_{n-(n+1)}(-\vec{e}_1 - 2\vec{e}_2 + \vec{e}_3) \\ &= (-1)^{n+1} P_{-1}(-\vec{e}_1 - 2\vec{e}_2 + \vec{e}_3)\end{aligned}$$

If $P_{-1} = 1$, then

$$\vec{P}_n^3 \wedge \vec{P}_{n+1}^3 = (-1)^n - (-\vec{e}_1 - 2\vec{e}_2 + \vec{e}_3) = (-1)^n(\vec{e}_1 + 2\vec{e}_2 - \vec{e}_3).$$

Corollary 2.1.9. The all $\vec{P}_n^3$ vectors are perpendicular to vector $(1,2,-1) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and the all $\vec{Q}_n^3$ vectors are perpendicular to vector $(-1\,,-2,1) = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

In what follows, we present the matrix representations associated with two-dimensional Pell and Pell–Lucas vectors. These matrix formulations are constructed to describe the transformation and structural properties of the vectors and provide a convenient algebraic framework for analyzing their behavior in the two-dimensional case.

Theorem 2.1.10. For

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

matrix, $\vec{P}_n^2$ and $\vec{Q}_n^2$ are provide the following equations.

$$i)\ \vec{P}_{n+1}^2 = T\vec{P}_n^2,$$

$$ii)\ \vec{Q}_{n+1}^2 = T\vec{Q}_n^2.$$

Proof. We only give the proof for i). The other part is done similarly.

$$i)\quad T\vec{P}_n^2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} P_n \\ P_{n+1} \end{bmatrix} = \begin{bmatrix} P_{n+1} \\ P_{n+2} \end{bmatrix} = \vec{P}_{n+1}^2.$$

Likewise, let us give the matrix application for 3D Pell and Pell-Lucas vectors**.**

Theorem 2.1.11. For

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

matrix, $\vec{P}_n^3$ and $\vec{Q}_n^3$ provide the following equations.

$$i)\ \vec{P}_{n+1}^3 = T\vec{P}_n^3,$$

$$ii)\ \vec{Q}_{n+1}^3 = T\vec{Q}_n^3,$$

Proof. We only give the proof for i). The other part is done similarly.

$$i) \quad T\vec{P}_n^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{bmatrix} = \vec{P}_{n+1}^3.$$

After presenting the matrix representations for two- and three-dimensional vectors, we proceed to introduce the corresponding formulation for generalized Pell and Pell–Lucas vectors. The following theorem establishes this generalized matrix representation and summarizes its main properties.

Theorem 2.1.12. For

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix}_{m \times m}$$

matrix, $\vec{P}_n^m$ and $\vec{Q}_n^m$ provide the following equations.

$$i) \ \vec{P}_{n+1}^m = T\vec{P}_n^m,$$

$$ii) \ \vec{Q}_{n+1}^m = T\vec{Q}_n^m.$$

Proof.

$i)$

$$T\vec{P}_n^m = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix}_{m \times m} \begin{bmatrix} P_n \\ P_{n+1} \\ \vdots \\ P_{n+m-1} \end{bmatrix}_{m \times 1}$$

The resulting matrix product is derived directly from the recurrence relation satisfied by the Pell numbers.

Theorem 2.1.12. Pell and Pell-Lucas vectors satisfy the following vector recurrence relations,

$$i) \ \vec{P}_n^m = 2\vec{P}_{n-1}^m + \vec{P}_{n-2}^m \text{ for } n \geq 2 \, ,$$

$$ii) \ \vec{Q}_n^m = 2\vec{Q}_{n-1}^m + \vec{Q}_{n-2}^m \text{ for } n \geq 2 \, .$$

Proof. We only give the proof for $i)$. The others are done similarly.

$i)$

$$\vec{P}_n^m = \begin{bmatrix} P_n \\ P_{n+1} \\ \vdots \\ P_{n+m-1} \end{bmatrix} = \begin{bmatrix} 2P_{n-1} + P_{n-2} \\ 2P_n + P_{n-1} \\ \vdots \\ 2P_{n+m-2} + P_{n+m-2} \end{bmatrix}$$

$$= \begin{bmatrix} 2P_{n-1} \\ 2P_n \\ \vdots \\ 2P_{n+m-2} \end{bmatrix} + \begin{bmatrix} P_{n-2} \\ P_{n-1} \\ \vdots \\ P_{n+m-2} \end{bmatrix}$$

$$= 2\vec{P}_{n-1}^m + \vec{P}_{n-2}^m.$$

Binet-type formulas provide explicit representations for terms of recurrence-based sequences and play a crucial role in revealing their structural properties. Such closed-form expressions allow recurrence relations to be analyzed through algebraic and analytic methods and offer valuable insights into the asymptotic behavior of the sequences. In the context of Pell and Pell–Lucas numbers, Binet formulas establish a direct connection between the sequences and the roots of their characteristic equations. In the following theorem, we extend this classical approach to the vector setting and present Binet-type formulas for Pell and Pell–Lucas vectors.

Theorem 2.1.13. (Binet's Formula) For $n \geq 0$, we have

$$i)\ \vec{P}_n^m = \frac{1}{a-b}\left[a^n\vec{a} - b^n\vec{b}\right],$$

$$ii)\ \vec{Q}_n^m = \frac{1}{2}\left[a^n\vec{a} + b^n\vec{b}\right]$$

where

$$\vec{a} = \begin{bmatrix} 1 \\ a \\ a^2 \\ \vdots \\ a^{m-1} \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ b \\ b^2 \\ \vdots \\ b^{m-1} \end{bmatrix}.$$

Proof.

$i)\ \vec{P}_n^m = \begin{bmatrix} P_n \\ P_{n+1} \\ \vdots \\ P_{n+m-1} \end{bmatrix}$ from $P_n = \frac{a^n-b^n}{a-b}$, we have

$$\vec{P}_n^m = \begin{bmatrix} \frac{a^n - b^n}{a-b} \\ \frac{a^{n+1} - b^{n+1}}{a-b} \\ \vdots \\ \frac{a^{n+m-1} - b^{n+m-1}}{a-b} \end{bmatrix} = \frac{1}{a-b}\begin{bmatrix} a^n - b^n \\ a^{n+1} - b^{n+1} \\ \vdots \\ a^{n+m-1} - b^{n+m-1} \end{bmatrix}$$

$$= \frac{1}{a-b}\left[a^n\vec{a} - b^n\vec{b}\right].$$

Theorem 2.1.14. (Binet's Formula) For negative indices,

$$i)\ \vec{P}_{-n}^m = \frac{(-1)^n}{a-b}\left[b^n\vec{a} - a^n\vec{b}\right],$$

$$ii)\ \vec{Q}_{-n}^m = \frac{(-1)^n}{2}\left[b^n\vec{a} + a^n\vec{b}\right]$$

where

$$\vec{a} = \begin{bmatrix} 1 \\ a \\ a^2 \\ \vdots \\ a^{m-1} \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ b \\ b^2 \\ \vdots \\ b^{m-1} \end{bmatrix}.$$

Proof. For the proof, we use the Binet formula of Pell vectors. Thus, if $-n$ is used instead of n, we have

$$i)\ \vec{P}_{-n}^{m} = \frac{1}{a-b}\left[a^{-n}\vec{a} - b^{-n}\vec{b}\right] \text{ since } ab = -1,$$

$$= \frac{1}{a-b}\left[(-b)^{n}\vec{a} - (-a)^{n}\vec{b}\right]$$

$$= \frac{(-1)^{n}}{a-b}\left[b^{n}\vec{a} - a^{n}\vec{b}\right].$$

The proof of the remaining case follows by analogous arguments.

Theorem 2.1.15. We have

$$i)\ \vec{a}.\vec{a} = \begin{cases} a^{m-1}\sqrt{2}P_m, & m\ even \\ a^{m-1}Q_m, & m\ odd \end{cases},$$

$$ii)\ \vec{a}.\vec{b} = \begin{cases} 0, & m\ even \\ 1, & m\ odd \end{cases},$$

$$iii)\ \vec{b}.\vec{b} = \begin{cases} -b^{m-1}\sqrt{2}P_m, & m\ even \\ b^{m-1}Q_m, & m\ odd \end{cases}.$$

Proof.

$$i)\ \vec{a}.\vec{a} = [1\ a\ \dots\ a^{m-1}]\begin{bmatrix} 1 \\ a \\ \vdots \\ a^{m-1} \end{bmatrix}$$

$$= 1 + a^2 + a^4 + \cdots + a^{2m-2}$$

$$= \sum_{i_0}^{m-1} a^{2i} = \sum_{i_0}^{m-1} (aa)^{i}$$

from $ab = -1$, we have

$$\vec{a}.\vec{a} = \sum_{i=0}^{m-1}\left(-\frac{a}{b}\right)^{i}$$

$$= \frac{\left(-\frac{a}{b}\right)^{m} - 1}{\left(-\frac{a}{b} - 1\right)}$$

$$= \left(\frac{1}{b}\right)^{m-1}\left(\frac{(-1)^{m-1}a^{m} - b^{m}}{-a-b}\right)$$

$$\vec{a}.\vec{a} = (-a)^{m-1}\left(\frac{(-1)^{m}a^{m} - b^{m}}{-a-b}\right).$$

If m is even, then

$$\vec{a}.\vec{a} = a^{m-1}\left(\frac{a^{m} - b^{m}}{a+b}\right) = a^{m-1}a^{m-1}\sqrt{2}P_m.$$

If m is odd, then

$$\vec{a}.\vec{a} = a^{m-1}\left(\frac{-a^{m} - b^{m}}{-a-b}\right) = a^{m-1}\left(\frac{a^{m} + b^{m}}{a+b}\right) = a^{m-1}Q_m.$$

Hence, the proof is complete. The remaining parts can be obtained by analogous arguments. As a consequence of Theorem 2.7, we obtain the following result.

Theorem 2.1.16. These vectors satisfy the inner products given below.

$$i)\ \langle \vec{P}_n^m, \vec{P}_n^m \rangle = \begin{cases} \frac{Q_m Q_{2n+m-1} - (-1)^n}{4}, & m\ \ odd \\ \frac{P_m P_{2n+m-1}}{2}, & m\ \ even \end{cases},$$

$$ii)\ \langle \vec{Q}_n^m, \vec{Q}_n^m \rangle = \begin{cases} \frac{Q_m Q_{2n+m-1} + (-1)^n}{2}, & m\ \ odd \\ P_m P_{2n+m-1}, & m\ \ even \end{cases},$$

$$iii)\ \langle \vec{P}_n^m, \vec{Q}_k^m \rangle = \begin{cases} \left(\sum_{i=1,3,\ldots,(m-1)} P_{n+k+i}\right) + P_{n+m-1}P_{k+m-1}, & m\ \ odd \\ \sum_{i=1,3,\ldots,(m-1)} P_{n+k+i}, & m\ \ even \end{cases}.$$

Proof.

$$i)\ \langle \vec{P}_n^m, \vec{P}_n^m \rangle = \frac{1}{a-b}\left[a^n\vec{a} - b^n\vec{b}\right].\frac{1}{a-b}\left[a^n\vec{a} - b^n\vec{b}\right]$$

$$= \frac{a^{2n}\vec{a}.\vec{a} - a^n b^n \vec{a}.\vec{b} - b^n a^n \vec{b}.\vec{a} + b^{2n}\vec{b}.\vec{b}}{(a-b)^2}$$

If m is even, then

$$\langle \vec{P}_n^m, \vec{P}_n^m \rangle = \frac{a^{2n}a^{m-1}\sqrt{2}P_m - b^{2n}b^{m-1}\sqrt{2}P_m}{(a-b)^2}$$

$$= \frac{\sqrt{2}P_m P_{2n+m-1}}{(a-b)} = \frac{P_m P_{2n+m-1}}{2}.$$

If m is odd, then

$$\langle \vec{P}_n^m, \vec{P}_n^m \rangle = \frac{a^{2n}a^{m-1}Q_m - a^n b^n - b^n a^n + b^{2n}b^{m-1}Q_m}{(a-b)^2}$$

$$= \frac{2Q_m Q_{2n+m-1} - 2(-1)^n}{(a-b)^2}$$
$$= \frac{Q_m Q_{2n+m-1} - (-1)^n}{4}.$$

Thus, the proof is complete. The remaining part follows by analogous arguments.

Corollary 2.1.17. The norms of the $\vec{a}$ and $\vec{b}$ vectors calculate as follows.

$$i)\ \|\vec{a}\| = \begin{cases} \sqrt{a^{m-1}\sqrt{2}P_m}, & m\ \ even \\ \sqrt{a^{m-1}Q_m}, & m\ \ odd \end{cases},$$

$$ii)\ \|\vec{a}\|\|\vec{b}\| = \begin{cases} \sqrt{2}P_m, & m\ \ even \\ Q_m, & m\ \ odd \end{cases},$$

$$iii)\|\vec{b}\| = \begin{cases} \sqrt{-b^{m-1}\sqrt{2}P_m}, & m\ \ even \\ \sqrt{b^{m-1}Q_m}, & m\ \ odd \end{cases}.$$

Theorem 2.1.18. We have, for all integers n

$$i)\ a^n\vec{a} = \vec{P}_{n-1}^m + a\vec{P}_n^m,$$

$$ii)\ b^n\vec{b} = \vec{P}_{n-1}^m + b\vec{P}_n^m.$$

Proof.

$$i)\ \vec{P}_{n-1}^m + a\vec{P}_n^m = \begin{bmatrix} P_{n-1} \\ P_n \\ \vdots \\ P_{n+m-2} \end{bmatrix} + a\begin{bmatrix} P_n \\ P_{n+1} \\ \vdots \\ P_{n+m-1} \end{bmatrix}$$

$$= \begin{bmatrix} a^n \\ a^{n+1} \\ \vdots \\ a^{n+m-1} \end{bmatrix} = a^n\begin{bmatrix} 1 \\ a \\ \vdots \\ a^{m-1} \end{bmatrix}$$

$$= a^n\vec{a}.$$

We now present a comprehensive collection of fundamental relations between the vector forms of Pell and Pell–Lucas numbers. These relations describe the interactions between the two vector sequences under various algebraic operations and reveal their structural similarities and differences. The obtained identities extend well-known properties of the corresponding scalar sequences to the vector setting and provide a unified framework for further theoretical developments. The results given here play a central role in the subsequent analysis and form the basis for several applications discussed later.

Theorem 2.1.19. The fundamental identities of the basic Pell vectors and Pell-Lucas vectors are as follows.

1) $\vec{P}_n^m + \vec{P}_{n-1}^m = Q_n^m$,

2) $\vec{Q}_n^m + \vec{Q}_{n-1}^m = 2P_n^m$,

3) $\vec{P}_n^m + \vec{Q}_n^m = \vec{P}_{n+1}^m$,

4) $2\vec{P}_n^m + \vec{Q}_n^m = \vec{Q}_{n+1}^m$,

5) $2\vec{Q}_n^m + 3P_n^m = \vec{P}_{n+2}^m$,

6) $3\vec{Q}_n^m + 4P_n^m = \vec{Q}_{n+2}^m$,

7) $\vec{Q}_{n+1}^m - \vec{Q}_n^m = 2\vec{P}_n^m$,

8) $\vec{P}_{n+1}^m + \vec{P}_{n-1}^m = 2\vec{Q}_n^m$,

11) $\vec{Q}_n^m + \vec{Q}_{n+1}^{+m} + \vec{Q}_{n+3}^m = 3\vec{P}_{n+2}^m$,

12) $\vec{P}_{n+1}^m - \vec{P}_{n-1}^m = 2\vec{P}_n^m$,

13) $\vec{Q}_{n+1}^m - \vec{Q}_{n-1}^m = 2\vec{Q}_n^m$,

14) $\vec{P}_{n+2}^m + \vec{P}_{n-2}^m = 6\vec{P}_n^m$,

15) $\vec{Q}_{n+2}^m + \vec{Q}_{n-2}^m = 6\vec{Q}_n^m$,

16) $\vec{P}_{n+2}^m - \vec{P}_{n-2}^m = 4\vec{Q}_n^m$,

17) $\vec{Q}_{n+2}^m - \vec{Q}_{n-2}^m = 8\vec{P}_n^m$,

18) $2\vec{P}_n^m + \vec{Q}_{n+2}^m = 3\vec{Q}_{n+1}^m$,

9) $\vec{Q}_{n+1}^m + \vec{Q}_{n-1}^m = 4\vec{P}_n^m$,

10) $\vec{P}_n^m + \vec{P}_{n+1}^m + \vec{P}_{n+3}^m = 3\vec{P}_{n+2}^m$,

19) $\vec{P}_{n+1}^m + \vec{Q}_{n-1}^m = 3\vec{P}_n^m$,

20) $2\vec{P}_n^m . \vec{Q}_n^m = P_{2n+2m-1} - P_{2n+1}$.

Proof.

1) $\vec{P}_n^m + \vec{P}_{n-1}^m = \begin{bmatrix} P_n + P_{n-1} \\ P_{n+1} + P_n \\ \vdots \\ P_{n+m-1} + P_{n+m-2} \end{bmatrix}$

Since Theorem 1.1. 1), we have

$$\vec{P}_n^m + \vec{P}_{n-1}^m = \begin{bmatrix} Q_n \\ Q_{n+1} \\ \vdots \\ Q_{n+m-1} \end{bmatrix} = Q_n^m.$$

The remaining parts of the theorem follow by applying the relations stated in Theorem 1.1 in an analogous manner.

Theorem 2.1.20.(Generating Functions) For x real numbers, we have

$$1)\ \sum_{n=0}^{\infty} \vec{P}_n^m\, x^n = \frac{\vec{P}_0^m + \vec{Q}_0^m x}{1-2x-x^2},$$

$$2)\ \sum_{n=0}^{\infty} \vec{Q}_n^m\, x^n = \frac{\vec{Q}_0^m + 2\vec{P}_0^m x}{1-2x-x^2}.$$

Proof.

1) Let $G = \sum_{n=0}^{\infty} \vec{P}_n^m\, x^n$.

$$G = \vec{P}_0^m + \vec{P}_1^m x + \cdots + \vec{P}_n^m x^m \ldots \qquad (*)$$

(*) If we multiply both sides of the equation by $2x$ and x^2, we have

$$2xG = \vec{P}_0^m x + \vec{P}_1^m x^2 + \cdots + \vec{P}_n^m x^{m+1} \ldots,$$

$$x^2 G = \vec{P}_0^m x^2 + \vec{P}_1^m x^3 + \cdots + \vec{P}_n^m x^{m+2} \ldots .$$

If the necessary operations are carried out in accordance with the recurrence relation of Pell vectors, the following is obtained.

$$G - 2xG - x^2 G = \vec{P}_0^m + \vec{P}_1^m x - \vec{P}_0^m x$$

Thus,

$$G(1 - 2x - x^2) = \vec{P}_0^m + \vec{Q}_0^m x$$

$$G = \frac{\vec{P}_0^m + \vec{Q}_0^m x}{1-2x-x^2}.$$

In this way, the proof is complete. The remaining part follows by analogous arguments.

2.2. The Angle Between These Vectors

In this section, we examine the inner (dot) products of the vectors defined in the previous sections and analyze the geometric relationships between them. In particular, by computing these inner products, we determine the angles between the corresponding vectors, which provides a clearer geometric interpretation of their mutual positions and structural properties.

Lemma 2.2.1. For $n \in \mathbb{Z}$, we have

$$i)\ \vec{P}_n^m.\vec{a} = \begin{cases} a^{m+n-1}P_m, & m\ \ even \\ \frac{a^{m+n-1}Q_m - b^n}{\sqrt{2}}, & m\ \ odd \end{cases},$$

$$ii)\ \vec{P}_n^m.\vec{b} = \begin{cases} b^{m+n-1}P_m, & m\ even \\ \frac{a^n - b^{m+n-1}Q_m}{\sqrt{2}}, & m\ odd \end{cases},$$

$$iii)\ \vec{Q}_n^m.\vec{a} = \begin{cases} \frac{a^{m+n-1}P_m}{\sqrt{2}}, & m\ even \\ \frac{a^{m+n-1}Q_m + b^n}{2}, & m\ odd \end{cases},$$

$$iii)\ \vec{Q}_n^m.\vec{b} = \begin{cases} \frac{-b^{m+n-1}P_m}{\sqrt{2}}, & m\ even \\ \frac{b^{m+n-1}Q_m + a^n}{2}, & m\ odd \end{cases}.$$

Proof. Using the Binet formulas of the $\vec{a}$ and $\vec{P}_n^m$, we obtain

$$i)\ \vec{P}_n^m.\vec{a} = \frac{1}{a-b}\left[a^n\vec{a} - b^n\vec{b}\right].\vec{a}$$

$$= \frac{1}{a-b}\left[a^n\vec{a}.\vec{a} - b^n\vec{b}.\vec{a}\right].$$

From Theorem 2.1.15.

$$i)\ \vec{a}.\vec{a} = \begin{cases} a^{m-1}\sqrt{2}P_m, & m\ \ even \\ a^{m-1}Q_m, & m\ \ odd \end{cases},$$

we have

$$\vec{P}_n^m.\vec{a} = \begin{cases} a^{m+n-1}P_m, & m\ \ even \\ \frac{a^{m+n-1}Q_m - b^n}{\sqrt{2}}, & m\ \ odd \end{cases}.$$

The other parts are shown similarly.

Theorem 2.2.2. For $n \in \mathbb{Z}$, the cosines of $\vec{P}_n^m, \vec{Q}_n^m, \vec{a}, \vec{b}$ between the vectors are calculated as follows.

$$i)\ cos_{(\vec{a},\vec{b})} = \begin{cases} 0, & m\ \ even \\ \frac{1}{Q_m}, & m\ \ odd \end{cases},$$

$$ii)\ cos_{(\vec{a},\vec{a})} = 1,$$

$$iii)\ cos_{(\vec{b},\vec{b})} = 1.$$

Proof.

$i)\ \|\vec{a}\|\|\vec{b}\|cos_{(\vec{a},\vec{b})} = \vec{a}.\vec{b}$

$$cos_{(\vec{a},\vec{b})} = \frac{\vec{a}.\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

from Theorem 2.1.15 and Corollary 2.1.17., we have

$$cos_{(\vec{a},\vec{b})} = \begin{cases} 0, & m \ \ even \\ \frac{1}{Q_m}, & m \ \ odd \end{cases}.$$

The remaining parts follow by analogous arguments.

Theorem 2.2.3. For $n \in \mathbb{Z}$ the cosine of the angle between the vectors $\vec{P}_n^m$, $\overrightarrow{a}$, $\vec{Q}_n^m$ and $\vec{b}$ is computed as follows, where m is an even integer.

$$i)\ cos_{(\vec{P}_n^m,\vec{a})} = (a-b)^{-\frac{1}{2}}\sqrt{\frac{2a^{2n+m-1}}{Q_{2n+m-1}}}\ ,$$

$$ii)\ cos_{(\vec{P}_n^m,\vec{b})} = 2(b-a)^{-\frac{1}{2}}\sqrt{\frac{b^{2n+m-1}}{P_{2n+m-1}}},$$

$$iii)\ cos_{(\vec{Q}_n^m,\vec{a})} = (a-b)^{-\frac{1}{2}}\sqrt{\frac{a^{2n+m-1}}{P_{2n+m-1}}},$$

$$iv)\ cos_{(\vec{Q}_n^m,\vec{b})} = -(b-a)^{-\frac{1}{2}}\sqrt{\frac{b^{2n+m-1}}{P_{2n+m-1}}}.$$

Proof. If m is even then,

$i\ )\ \|\vec{P}_n^m\|\|\vec{a}\|cos_{(\vec{P}_n^m,\vec{a})} = \vec{P}_n^m.\vec{a}$

$$= \sqrt{\frac{P_m P_{2n+m-1}}{2}}\sqrt{a^{m-1}\sqrt{2}P_m}cos_{(\vec{P}_n^m,\vec{a})}$$

$$= a^{m+n-1}P_m$$

$$= \sqrt{a^{m-1}P_{2n+m-1}\frac{(a-b)}{4}}.$$

2.3. Some Identities of the Pell and Pell-Lucas Vectors

In this section, several classical identities that play an important role in the theory of integer sequences are extended to the vector setting. In particular, these identities are transferred to vector forms and explicitly computed for Pell and Pell–Lucas vectors. The resulting vector identities generalize well-known relationships of the corresponding number sequences and demonstrate that many fundamental properties remain valid in a higher-dimensional vector framework. This extension provides a deeper insight into the algebraic structure of Pell and Pell–Lucas vectors and highlights the consistency between scalar sequences and their vector representations.

Cassini identity is one of the fundamental relations in the theory of recurrence-based integer sequences and plays a central role in revealing their intrinsic algebraic structure. This identity reflects the deep connection between consecutive terms of a sequence and provides important insights into symmetry, invariance, and determinant-like behavior associated with second-order recurrence relations. Beyond its classical scalar form, Cassini's identity serves as a powerful tool for understanding stability and consistency properties of sequences such as Fibonacci, Pell, and Pell–Lucas numbers.

When extended to the vector setting, Cassini's identity acquires a richer geometric interpretation.

Theorem 2.3.1. (Cassini Identities) We have, for $n \geq 1$

$$i)\ \vec{P}_{n-1}^{m}.\vec{P}_{n+1}^{m} - \left(\vec{P}_{n}^{m}\right)^{2} = \begin{cases} 0, & m\ even \\ (-1)^{n}, & m\ \ odd \end{cases},$$

$$ii)\ \vec{Q}_{n-1}^{m}.\vec{Q}_{n+1}^{m} - \left(\vec{Q}_{n}^{m}\right)^{2} = \begin{cases} 0, & m\ even \\ 2(-1)^{n-1}, & m\ \ odd \end{cases}.$$

Proof. Using the Binet formula, the result follows.

$i)\ \vec{P}_{n-1}^{m}.\vec{P}_{n+1}^{m} - \left(\vec{P}_{n}^{m}\right)^{2}$

$= \frac{1}{(a-b)^{2}}\left(-a^{n-1}b^{n+1}\vec{a}.\vec{b} - b^{n-1}a^{n+1}\vec{b}.\vec{a} + a^{n}b^{n}\vec{a}.\vec{b} + b^{n}a^{n}\vec{b}.\vec{a}\right).$

If m is even:

$$\vec{P}_{n-1}^{m}.\vec{P}_{n+1}^{m} - \left(\vec{P}_{n}^{m}\right)^{2} = 0.$$

If is m is odd:

$$\vec{P}_{n-1}^{m}.\vec{P}_{n+1}^{m} - \left(\vec{P}_{n}^{m}\right)^{2} = \frac{\vec{a}\vec{b}\,(-a^{n+1}b^{n-1} - b^{n+1}a^{n-1} + 2a^{n}b^{n})}{(a-b)^{2}}$$

$$= \frac{(ab)^{n-1} - (a-b)^{2}}{(a-b)^{2}}$$

$$= (-1)^{n}.$$

The other part is shown similarly.

Theorem 2.3.2. (Catalan Identities) We have, for $n \geq t$

$$i)\ \vec{P}_{n-t}^{m}.\vec{P}_{n+t}^{m} - \left(\vec{P}_{n}^{m}\right)^{2} = \begin{cases} 0, & m\ \ even \\ (-1)^{n-t+1}\frac{\left(a^{t}-b^{t}\right)^{2}}{8}, & m\ \ \ odd \end{cases},$$

$$ii)\ \vec{Q}_{n-t}^{m}.\vec{Q}_{n+t}^{m} - \left(\vec{Q}_{n}^{m}\right)^{2} = \begin{cases} 0, & m\ even \\ (-1)^{n-t}\frac{\left(a^{t}-b^{t}\right)^{2}}{4}, & m\ odd \end{cases}.$$

Proof.

$i)\ \vec{P}_{n-1}^{m}.\vec{P}_{n+1}^{m} - \left(\vec{P}_{n}^{m}\right)^{2}$

$$= \frac{1}{(a-b)^2}\left(-a^{n-t}b^{n+t}\vec{a}.\vec{b} - b^{n-t}a^{n+t}\vec{b}.\vec{a} + a^n b^n \vec{a}.\vec{b} + b^n a^n \vec{b}.\vec{a}\right).$$

If m is even:

$$\vec{P}^m_{n-t}.\vec{P}^m_{n+t} - \left(\vec{P}^m_n\right)^2 = 0.$$

If is m is odd:

$$\vec{P}^m_{n-t}.\vec{P}^m_{n+t} - \left(\vec{P}^m_n\right)^2 = \frac{\vec{a}\vec{b}\,(-a^{n+t}b^{n-t} - b^{n+t}a^{n-t} + 2a^n b^n)}{(a-b)^2}$$

$$= \frac{-(ab)^{n-t}(a^t - b^t)^2}{(a-b)^2}$$

$$= (-1)^{n-t+1}\frac{\left(a^t - b^t\right)^2}{8}.$$

The remaining case can be established by applying the same arguments. It is worth noting that, when $t = 1$ the obtained result reduces to the classical Cassini identity.

Theorem 2.3.3. (Vajda Identities) We have, for $n + k \le r$

$$i)\ \vec{P}^m_{n+k}.\vec{P}^m_{r-k} - \vec{P}^m_n.\vec{P}^m_r = \begin{cases} 0, & m\ even \\ (-1)^n P_k P_{r-k-n}, & m\ \ odd \end{cases},$$

$$ii)\ \vec{Q}^m_{n+k}.\vec{Q}^m_{r-k} - \vec{Q}^m_n.\vec{Q}^m_r = \begin{cases} 0, & m\ even \\ 2(-1)^{n+1} P_k P_{r-k-n}, & m\ \ odd \end{cases}.$$

Proof.

$i)\ \vec{P}^m_{n+k}.\vec{P}^m_{r-k} - \vec{P}^m_n.\vec{P}^m_r$

$$= \frac{1}{(a-b)^2}\left(-a^{n+k}b^{r-k}\vec{a}.\vec{b} - b^{n+k}a^{r-k}\vec{b}.\vec{a} + a^n b^r \vec{a}.\vec{b} + b^n a^r \vec{b}.\vec{a}\right).$$

If m is even:

$$\vec{P}^m_{n-t}.\vec{P}^m_{n+t} - \left(\vec{P}^m_n\right)^2 = 0.$$

If is m is odd:

$$\vec{P}^m_{n+k}.\vec{P}^m_{r-k} - \vec{P}^m_n.\vec{P}^m_r = \frac{a^n b^r\left(1 - \left(\frac{a}{b}\right)^k\right) + b^n a^r\left(1 - \left(\frac{b}{a}\right)^k\right)}{(a-b)^2}$$

$$= \frac{a^n b^r\left(\frac{b^k - a^k}{b^k}\right) + b^n a^r\left(\frac{a^k - b^k}{a^k}\right)}{(a-b)^2}$$

$$= P_k \frac{a^n b^n (a^{r-k-n} - b^{r-k-n})}{(a-b)}$$

$$= (-1)^n P_k P_{r-k-n}.$$

Hence, the proof is complete. The remaining part can be established by analogous arguments.

Theorem 2.3.4. (d'Ocagne Identities) We have, for $n \leq k$

$$i)\ \vec{P}_{n+1}^{m}.\vec{P}_{k}^{m} - \vec{P}_{k+1}^{m}.\vec{P}_{n}^{m} = \begin{cases} 0, & m\ even \\ (-1)^{n}P_{k-n}, & m\ odd \end{cases},$$

$$ii)\ \vec{Q}_{n+1}^{m}.\vec{Q}_{k}^{m} - \vec{Q}_{k+1}^{m}.\vec{Q}_{n}^{m} = \begin{cases} 0, & m\ even \\ 2(-1)^{n}P_{k-n}, & m\ odd \end{cases}.$$

Proof.

$i)\ \vec{P}_{n+1}^{m}.\vec{P}_{k}^{m} - \vec{P}_{k+1}^{m}.\vec{P}_{n}^{m}$

$$= \frac{1}{(a-b)^{2}}\left(-a^{n+1}b^{k}\vec{a}.\vec{b} - b^{n+1}a^{k}\vec{b}.\vec{a} + a^{k+1}b^{n}\vec{a}.\vec{b} + b^{k+1}a^{n}\vec{b}.\vec{a}\right).$$

If m is even:

$$\vec{P}_{n-t}^{m}.\vec{P}_{n+t}^{m} - \left(\vec{P}_{n}^{m}\right)^{2} = 0.$$

If is m is odd:

$$\vec{P}_{n+k}^{m}.\vec{P}_{r-k}^{m} - \vec{P}_{n}^{m}.\vec{P}_{r}^{m} = \frac{a^{n}b^{k}\,(b-a) + b^{n}a^{k}(a-b)}{(a-b)^{2}}$$

$$= P_{k}\frac{a^{n}b^{n}(a^{k-n} - b^{k-n})}{(a-b)}$$

$$= (-1)^{n}P_{k-n}.$$

Therefore, the proof is complete. The remaining part follows by a similar argument.

Theorem 2.3.5.(Honsberger Identities) We have, for $k \geq 1$

$$i)\ \vec{P}_{k-1}^{m}.\vec{P}_{n}^{m} + \vec{P}_{k}^{m}.\vec{P}_{n+1}^{m} = \begin{cases} P_{m}Q_{k+m+n-1}, & m\ even \\ Q_{m}P_{k+m+n-1}, & m\ odd \end{cases},$$

$$ii)\ \vec{Q}_{k-1}^{m}.\vec{Q}_{n}^{m} + \vec{Q}_{k}^{m}\vec{Q}_{n+1}^{m} = \begin{cases} 2P_{m}Q_{k+m+n-1}, & m\ even \\ Q_{m}P_{k+m+n-1}, & m\ odd \end{cases}.$$

Proof.

$i)\ \vec{P}_{k-1}^{m}.\vec{P}_{n}^{m} + \vec{P}_{k}^{m}.\vec{P}_{n+1}^{m}$

$$= \frac{1}{(a-b)^{2}}\left(a^{n+k-1}\vec{a}.\vec{a} - a^{k-1}b^{n}\vec{a}.\vec{b} - b^{k-1}a^{n}\vec{b}.\vec{a} + a^{n+k-1}\vec{b}.\vec{b}\right.$$

$$\left. + a^{n+k+1}\vec{a}.\vec{a} - a^{k}b^{n+1}\vec{a}.\vec{b} - b^{k}a^{n+1}\vec{b}.\vec{a} + b^{n+k+1}\vec{b}.\vec{b}\right)$$

If m is even:

$$= \frac{1}{(a-b)^{2}}\left(a^{n+k-1}a^{m-1}\sqrt{2}P_{m} + a^{k+n+1}a^{m-1}\sqrt{2}P_{m} - b^{k+n-1}b^{m-1}\sqrt{2}P_{m} - b^{n+k+1}b^{m-1}\sqrt{2}P_{m}\right)$$

$$= \frac{\sqrt{2}P_m[a^{k+m+n-2} - b^{k+m+n-2}) + (a^{k+m+n} - b^{k+m+n}]}{(a-b)^2}$$

$$= \frac{P_m(P_{k+m+n-2} + P_{k+m+n})}{2}$$

from, we have

$$\vec{P}_{k-1}^m . \vec{P}_n^m + \vec{P}_k^m . \vec{P}_{n+1}^m = P_m Q_{k+m+n-1}.$$

If is m is odd:

$$= \frac{1}{(a-b)^2}\left(Q_m \frac{Q_{k+n+m-2}}{2} + Q_m \frac{Q_{k+n+m}}{2} - a^{k-1}b^n(ab+1) - a^n b^{k-1}(ab+1)\right).$$

$$= \frac{2Q_m(Q_{k+n+m-2} + Q_{k+n+m})}{(a-b)^2}$$

From, we have

$$\vec{P}_{k-1}^m . \vec{P}_n^m + \vec{P}_k^m . \vec{P}_{n+1}^m = Q_m P_{k+m+n-1}.$$

Consequently, the proof is complete. The remaining part can be derived in an analogous manner.

3. RESULTS

In this paper, a vector-oriented framework based on the Pell and Pell–Lucas number sequences is proposed and investigated in detail. The primary objective of the study is to extend these classical sequences, originally defined in a scalar setting, to higher-dimensional vector spaces and to examine the resulting structures from both algebraic and sequence-theoretic viewpoints. In this sense, the sequences are analyzed not only through their numerical characteristics but also through their behavior within the context of linear algebra.

As an initial step, Pell and Pell–Lucas numbers are represented as vectors in two- and three-dimensional spaces. These constructions are then generalized to vectors of arbitrary dimension. For the resulting vector forms, fundamental concepts of linear algebra—including inner products, vector products, norms, and matrix representations—are derived explicitly. In particular, the matrix representations obtained from the associated recurrence relations are shown to effectively characterize the transition between successive vector terms.

Furthermore, the introduced vectors are treated as vector sequences, and the corresponding recurrence relations are formulated. Vector analogues of Binet-type closed formulas are established, extending the classical results known for Pell and Pell–Lucas sequences. These findings are complemented by the construction of suitable generating functions, which provide additional insight into the analytical properties of the vector sequences.

To emphasize the geometric aspects of the proposed framework, the angles between Pell and Pell–Lucas vectors are examined using inner products and norms. This approach allows the relationships between different vector terms to be interpreted geometrically, offering an alternative perspective alongside the algebraic analysis. As a result, classical connections between number sequences acquire a clear geometric meaning within vector spaces.

Finally, several well-known identities from the theory of integer sequences, including the Cassini, Catalan, Vajda, d'Ocagne, and Honsberger identities, are extended to the vector setting. The obtained results demonstrate that many fundamental properties of Pell and Pell–Lucas numbers remain valid when these sequences are embedded in higher-dimensional vector structures.

Overall, the proposed approach provides a unified framework that naturally links number theory and linear algebra. Moreover, the methodology developed in this study can be adapted to other recurrence-based sequences, suggesting potential directions for further research.

This work could also be applied to different number sequences or polynomial sequences.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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