A NEW APPROACH TO NARAYANA HYBRID SEQUENCES

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ABSTRACT
In this study, the known real, complex, hyperbolic and dual numbers are combined to form a hybrid number system. While creating this system, a different number sequence was created by combining Narayana numbers with real, complex, hyperbolic and dual numbers in each term. Then, some properties of this obtained sequence of numbers, Binet formula, sum formula, generating function were investigated.
Keywords: Narayana sequence, Hybrid numbers, Generating function

1. Introduction

Hybrid numbers were introduced by Özdemir in [5]. A hybrid number is given by $\mathbb{K} = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = hi = \varepsilon + i\}$ where i, ε, h are hybrid units and their multiplication'rules are in Table 1.

.	i	ε	h
i	-1	$1-h$	$\varepsilon+i$
ε	$1+h$	0	$-\varepsilon$
h	$-\varepsilon-i$	ε	1

Table 1: Multiplication Rules

Recently, many researchers [1,3-5] have studied hybrid numbers. Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas hybrid numbers were studied, and some properties were given. In addition, the results of these studies were generalized by creating a Horadam hybrid number sequence. In addition, Kızılates, Szynal-Liana and Wloch [4,9-11] generalized the special polynomials of these number sequences to form hybrinomials. The Fibonacci and Lucas hybrid numbers are defined, respectively, as:

$$FH_n = F_n + F_{n+1}i + F_{n+2}\varepsilon + F_{n+3}h$$

and

$$LH_n = L_n + L_{n+1}i + L_{n+2}\varepsilon + L_{n+3}h$$

Recently, many number sequences have been combined with hybrid number sequences in different ways [6-8].
In addition to working with many quadratic number sequences, third order recurrence number sequences have also been the subject of hybrid number sequences. Starting from this point, in this study, Narayana number sequences with third order recurrence are discussed.
The Narayana sequence denoted [2] by N_n, with 0,1,1,1 initial condition.

$$N_{n+1} = N_n + N_{n-2}$$

The first few Narayana sequence can be written as:
0,1,1,1,2,3,4,6,9,13, …
The Binet's formula and characteristic equation of this sequence are shown in the figure below:

$$N_n = \frac{r^{n+1}}{(r-s)(r-t)} + \frac{s^{n+1}}{(s-r)(s-t)} + \frac{t^{n+1}}{(t-r)(t-s)}$$

and $x^3 - x^2 - 1 = 0$. where r, s, t are the roots of the characteristic equation.
Based on the above-mentioned information, we will define Narayana hybrid numbers and give the generating functions of these numbers, the Binet's formula and some important theorems and propositions.

2. The Narayana hybrid sequences

Definition 1. The Narayana hybrid sequences defined as follows:

$$\mathcal{H}N_n = N_n + N_{n+1}i + N_{n+2}\varepsilon + N_{n+3}h$$

The first few Narayana hybrid sequence can be written as:

$$\mathcal{H}N_0 = i + \varepsilon + h, \qquad \mathcal{H}N_1 = 1 + i + \varepsilon + 2h, \qquad \mathcal{H}N_2 = 1 + i + 2\varepsilon + 3h, \ldots$$

Theorem 2. Let $n \geq 0$ be an integer. The character of the Narayana hybrid number is:

$$C(\mathcal{H}N_n) = N_{n+1}(N_{n+1} - 2N_{n+2}) - N_{n+2}(N_{n+2} + 2N_n).$$

Proof. From the definition of the Narayana sequence, we get

$$\begin{aligned}(N_n + N_{n+1}i + N_{n+2}\varepsilon + N_{n+3}h)&(N_n - N_{n+1}i - N_{n+2}\varepsilon - N_{n+3}h)\\ &= N_n^2 + N_nN_{n+1}i + N_nN_{n+2}\varepsilon + N_nN_{n+3}h - N_nN_{n+1}i + N_{n+1}^2 - N_{n+1}N_{n+2}i\varepsilon\\ &- N_{n+1}N_{n+3}ih - N_nN_{n+2}\varepsilon - N_{n+2}N_{n+1}\varepsilon i - N_{n+2}^2\varepsilon^2 - N_{n+2}N_{n+3}\varepsilon h\\ &- N_{n+3}N_nh - N_{n+3}N_{n+1}hi - N_{n+3}N_{n+2}h\varepsilon - N_{n+3}^2\end{aligned}$$

If we substitute the required values, we obtain

$$\begin{aligned}C(\mathcal{H}N_n) &= N_n^2 + N_{n+1}^2 - N_{n+3}^2 - 2N_{n+1}N_{n+2}\\ &= N_n^2 + N_{n+1}^2 - 2N_{n+1}N_{n+2} - (N_{n+2} + N_n)^2\\ &= N_{n+1}(N_{n+1} - 2N_{n+2}) - N_{n+2}(N_{n+2} + 2N_n)\end{aligned}$$

Corollary 3. The norm of the Narayana hybrid sequence is:

$$\|\mathcal{H}N_n\|^2 = |N_{n+1}(N_{n+1} - 2N_{n+2}) - N_{n+2}(N_{n+2} + 2N_n)|$$

Proof. By the definition of character and norm for hybrid numbers

$$\|\mathcal{H}N_n\| = \sqrt{C(\mathcal{H}N_n)} = \sqrt{|N_{n+1}(N_{n+1} - 2N_{n+2}) - N_{n+2}(N_{n+2} + 2N_n)|}$$

$$\|\mathcal{H}N_n\|^2 = |N_{n+1}(N_{n+1} - 2N_{n+2}) - N_{n+2}(N_{n+2} + 2N_n)|$$

Theorem 4. Let $n \geq 0$. Then

$$\mathcal{H}N_n = \frac{r^{n+1}(1 + ri + r^2\varepsilon + r^3h)}{(r-s)(r-t)} + \frac{s^{n+1}(1 + si + s^2\varepsilon + s^3h)}{(s-r)(s-t)} + \frac{t^{n+1}(1 + ti + t^2\varepsilon + t^3h)}{(t-r)(t-s)}$$

Proof. From Binet's formula for Narayana sequence in the Narayana hybrid number, we get

$$\mathcal{H}N_n = \frac{r^{n+1}}{(r-s)(r-t)} + \frac{s^{n+1}}{(s-r)(s-t)} + \frac{t^{n+1}}{(t-r)(t-s)} + \left(\frac{r^{n+2}}{(r-s)(r-t)} + \frac{s^{n+2}}{(s-r)(s-t)} + \frac{t^{n+2}}{(t-r)(t-s)}\right) i + \left(\frac{r^{n+3}}{(r-s)(r-t)} + \frac{s^{n+3}}{(s-r)(s-t)} + \frac{t^{n+3}}{(t-r)(t-s)}\right) \varepsilon + \left(\frac{r^{n+4}}{(r-s)(r-t)} + \frac{s^{n+4}}{(s-r)(s-t)} + \frac{t^{n+4}}{(t-r)(t-s)}\right) h$$
$$= \frac{r^{n+1}}{(r-s)(r-t)}(1 + ri + r^2\varepsilon + r^3h) + \frac{s^{n+1}}{(s-r)(s-t)}(1 + si + s^2\varepsilon + s^3h) + \frac{t^{n+1}}{(t-r)(t-s)}(1 + ti + t^2\varepsilon + t^3h)$$

Theorem 5. Let $n \geq 0$. Then

$$\sum_{k=0}^{n} \mathcal{H}N_k = \mathcal{H}N_{n+1} + \mathcal{H}N_n + \mathcal{H}N_{n-1} - 1 - i - \varepsilon - 2h$$

Proof.

$$\sum_{k=0}^{n} \mathcal{H}N_k = \mathcal{H}N_0 + \mathcal{H}N_1 + \mathcal{H}N_2 + \cdots + \mathcal{H}N_n$$

thus, we obtain

$$\sum_{k=0}^{n} \mathcal{H}N_k = \mathcal{H}N_0 + \mathcal{H}N_1 + \mathcal{H}N_2 + \cdots + \mathcal{H}N_n$$
$$= (N_0 + N_1 i + N_2\varepsilon + N_3 h) + (N_1 + N_2 i + N_3\varepsilon + N_4 h) + \cdots + (N_n + N_{n+1} i + N_{n+2}\varepsilon + N_{n+3} h)$$
$$= (N_0 + N_1 + N_2 + \cdots + N_n) + i(N_1 + N_2 + N_3 + \cdots + N_{n+1}) + \varepsilon(N_2 + N_3 + N_4 + \cdots + N_{n+2}) + h(N_3 + N_4 + N_5 + \cdots + N_{n+3})$$
$$= \sum_{k=0}^{n} N_k + i\sum_{k=0}^{n+1} N_k + \varepsilon\sum_{k=0}^{n+2}(N_k - 1) + h\sum_{k=0}^{n+3}(N_k - 2)$$
$$= (N_{n+1} + N_n + N_{n-1} - 1) + i(N_{n+2} + N_{n+1} + N_n - 1) + \varepsilon(N_{n+3} + N_{n+2} + N_{n+1} - 1) + h(N_{n+4} + N_{n+3} + N_{n+2} - 2)$$
$$= \mathcal{H}N_{n+1} + \mathcal{H}N_n + \mathcal{H}N_{n-1} - 1 - i - \varepsilon - 2h$$

Theorem 6. The generating function for Narayana hybrid sequence $\mathcal{H}N_n$ is

$$\sum_{n=0}^{\infty} \mathcal{H}N_n x^n = \frac{i + \varepsilon + h + x(1+h) + x^2(\varepsilon + h)}{1 - x - x^3}$$

Proof. Narayana hybrid sequence $\mathcal{H}N_n$ has the following form

$$f(x) = \sum_{n=0}^{\infty} \mathcal{H}N_n x^n = \mathcal{H}N_0 + \mathcal{H}N_1 x + \mathcal{H}N_2 x^2 + \mathcal{H}N_3 x^3 + \cdots$$
$$xf(x) = \mathcal{H}N_0 x + \mathcal{H}N_1 x^2 + \mathcal{H}N_2 x^3 + \mathcal{H}N_3 x^4 + \cdots$$
$$x^3 f(x) = \mathcal{H}N_0 x^3 + \mathcal{H}N_1 x^4 + \mathcal{H}N_2 x^5 + \mathcal{H}N_3 x^6 + \cdots$$
$$f(x) - xf(x) - x^3 f(x) = \mathcal{H}N_0 + x(\mathcal{H}N_1 - \mathcal{H}N_0) + x^2(\mathcal{H}N_2 - \mathcal{H}N_1) + x^3(\mathcal{H}N_3 - \mathcal{H}N_2 - \mathcal{H}N_0) + 0 + 0 + \cdots$$

$$f(x)(1-x-x^3)$$
$$= i+\varepsilon+h+x(1+i+\varepsilon+2h-i-\varepsilon-h)+x^2(1+i+2\varepsilon+3h-1-i-\varepsilon-2h)$$
$$f(x)=\frac{i+\varepsilon+h+x(1+h)+x^2(\varepsilon+h)}{1-x-x^3}$$

Definition 7. Let $n \geq 0$, $n \in \mathbb{Z}$. Let us define
$$\mathcal{HS}_n = \mathcal{HN}_n + \mathcal{HN}_{-n}\,, \qquad \mathcal{HS}'_n = \mathcal{HN}_n - \mathcal{HN}_{-n}$$
where
$$\mathcal{HN}_{-n} = N_{-n} + N_{-(n+1)}i + N_{-(n+2)}\varepsilon + N_{-(n+3)}h$$

According to this new sequence, table 2 can be created.

n	1	2	3	4
$\mathcal{HN}_{-n}$	$i-h$	$1-\varepsilon+h$	$-i+\varepsilon+h$	$-1+i+\varepsilon-2h$
$\mathcal{HS}_n$	$1+2i+\varepsilon+h$	$2+i+\varepsilon+4h$	$1+i+4\varepsilon+5h$	$1+4i+5\varepsilon+4h$
$\mathcal{HS}'_n$	$1+\varepsilon+3h$	$i+3\varepsilon+2h$	$1+3i+2\varepsilon+3h$	$3+2i+3\varepsilon+8h$

Table 2. Sum and difference values of negative index Narayana sequence

Theorem 8. Let $n \geq 0$, $n \in \mathbb{Z}$. Then
$$\mathcal{HN}_{n-2} + \mathcal{HN}_{n-1} = \mathcal{HS}_n + \mathcal{HS}_{n-2} - \mathcal{HS}_{n-3}$$
Proof. By the definition of $\mathcal{HS}_n$, we get
$$\mathcal{HS}_n + \mathcal{HS}_{n-2} - \mathcal{HS}_{n-3} = \mathcal{HN}_n + \mathcal{HN}_{-n} + \mathcal{HN}_{n-2} + \mathcal{HN}_{-(n-2)} - \left(\mathcal{HN}_{n-3} + \mathcal{HN}_{-(n-3)}\right)h$$
$$= (N_n + N_{n+1}i + N_{n+2}\varepsilon + N_{n+3}h) + \left(N_{-n} + N_{-(n+1)}i + N_{-(n+2)}\varepsilon + N_{-(n+3)}h\right)$$
$$+ (N_{n-2} + N_{n-1}i + N_n\varepsilon + N_{n+1}h)$$
$$+ \left(N_{-(n-2)} + N_{-(n-1)}i + N_{-n}\varepsilon + N_{-(n+1)}h\right)$$
$$- (N_{n-3} + N_{n-2}i + N_{n-1}\varepsilon + N_n h)$$
$$- \left(N_{-(n-3)} + N_{-(n-2)}i + N_{-(n-1)}\varepsilon + N_{-n}h\right)$$
$$= \left[N_n + N_{-n} + N_{n-2} + N_{-(n-2)} - (N_{n-3} + N_{-(n-3)})\right]$$
$$+ \left[N_{n+1} + N_{-(n+1)} + N_{n-1} + N_{-(n-1)} - (N_{n-2} + N_{-(n-2)})\right]i$$
$$+ \left[N_{n+2} + N_{-(n+2)} + N_n + N_{-n} - (N_{n-1} + N_{-(n-1)})\right]\varepsilon$$
$$+ \left[N_{n+3} + N_{-(n+3)} + N_{n+1} + N_{-(n+1)} - (N_n + N_{-n})\right]h$$
Then we get from definition of Narayana sequence
$$\mathcal{HS}_n + \mathcal{HS}_{n-2} - \mathcal{HS}_{n-3}$$
$$= (N_{n-2} + N_{n-1}) + (N_{n-1} + N_n)i + (N_n + N_{n+1})\varepsilon + (N_{n+1} + N_{n+2})h$$
$$= \mathcal{HN}_{n-2} + \mathcal{HN}_{n-1}$$
As desired.

3. Conclusion

In this study, we examined the hybrid numbers in Narayana sequence then we have expressed some important features, theorem, and definition of Narayana hybrid sequence. This work can be similarly defined on many number sequences.

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