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ON TRUNCATED FIBONACCI SEQUENCES

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We prove some theorems concerning truncated Fibonacci sequences, which are defined below in Section 2. We also state some conjectures concerning the period (mod *m*) of truncated Fibonacci sequences, where the integer *m* is greater than or e8qual to 2. In addition, we present some computer-generated results that tend to support these conjectures.

Keywords: Fibonacci Sequence; Wall Number; Truncated Fibonacci Sequence; Multiplicative Function

1. INTRODUCTION

A sequence $\{a_i\}$ is said to be periodic with period *k* if it is satisfies $a_i = a_{i+k}$ for $i = 1, 2, \ldots$. We let s_n denote the n^{th} member of the sequence of integers $s_0 = a$, $s_1 = b, \ldots$, where $s_{n+1} = s_n + s_{n-1}$. The symbol $h(m)$ will denote the length of the period of the sequence resulting from reducing each s_n modulo *m*. An *n*-step Fibonacci sequence $\left\{f_k^{(n)}\right\}_{k=1}^{\infty}$ is defined by letting $f_k^{(n)} = 0$ for $k \leq 0, f_1^{(n)} = 1, f_2^{(n)} = 1$, and other terms according to the linear recurrence equation $f_k^{(n)} = \sum_{i=1}^{n} f_{k-i}^{(n)}$ for $k > 2$. The basic (2-step) Fibonacci sequence will be given by $f_0 = 0, f_1 = 1, \ldots$ and the Lucas sequence by $g_0 = 2, g_1 = 1, \ldots$. The symbol $k(m)$ will denote the length of the period of the 2- step or 3-step Fibonacci sequences when they are reduced modulo *m*. We sometimes call it Wall's number after a pioneer in this area. For example, let f_n be the 2-step Fibonacci sequence and $m = 5$. If we reduce each f_n modulo 5, then we obtain following periodic sequence.

$0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, \ldots$ and then repeat; so $k(5) = 20$.

From now on, let $k(m)$ denote the Wall number of the 2 and 3-step Fibonacci sequences and $k_r(m)$ denote the Wall number of truncated Fibonacci sequences (defined in section 2)

We will let $\mathbb{Z}$ and $\mathbb{Z}_m$ represent the group of integers and the group of integers modulo *m*, respectively.

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Takahashi gives a fast algorithm which is based on the product of Lucas numbers to compute large Fibonacci numbers [1]. zkan has proved two original theorems concerning the Wall number of the 3-step Fibonacci sequences and given some conjectures concerning $k(m)$ of the 3-step Fibonacci sequence [2]. We give a few of them in Conjecture 1.3.

Fibonacci sequences appear in many branches of mathematics. These include group theory, calculus, applied mathematics, linear algebra, etc.

One of the latest works in this area is [3] where Sik Choi introduces the truncated Fibonacci sequences and the truncated Lucas sequences.

In this paper, we give some interesting properties of the truncated Fibonacci sequences and we show that the function $k_r(m)$ associated with this sequence is a multiplicative function. Furthermore, we examine the relationship between m and $k_r(m)$ of the truncated Fibonacci sequences. We obtain interesting results. We also obtain results similar to the following theorems. We give them in section 3.

For the 2-step Fibonacci sequences, we gather some of the well-known properties of $k(m)$ into the following two theorems.

Theorem **1.1** — [4, 5]

(*i*) (Wall): If *m* has the prime factorization $m = \prod p_i^{e_i}$ and if $h_i = k(p_i^{e_i})$ then $k(m)$ is the least common multiple of the h_i.

(*ii*) (Wall) If $k(p) \neq k(p^2)$, then $k(p^e) = p^{e-1}k(p)$. Also, if *n* is the largest integer with $k(p) = k(p^n)$, then $k(p^e) = p^{e-1}k(p)$ for $e > n$.

(*iii*) (Wilcox) For $m > 2, k(m) = 2n$, where

$n = \min(\{n : n \text{ even and } m | f_n\} \cup \{n : n \text{ odd and } m | g_n\})$, f_n is the n^{th} member of the 2-step Fibonacci sequence and g_n is the n^{th} member of the Lucas sequence.

Remark: In all known cases, $k(p^2) = pk(p)$ for p a prime, and it appears to be open question whether the case $k(p^2) = k(p)$ ever occurs.

Theorem 1.2[4] — In $\mathbb{Z}_m$, the following hold:

(*i*) If *p* is a prime and $p = 10x \pm 1$, then $k(p)|(p-1)$.

(*ii*) If *p* is a prime and $p = 10x \pm 3$, then $k(p)|(2p+2)$.

(*iii*) If *p* is a prime and $p = 10x \pm 3$, then $h(p^e) = k(p^e)$.

Let f_n be 3-step Fibonacci sequence. Then, $f_n(mod m)$ forms periodic sequence. That is, the sequence is periodic and repeats by returning to its starting values.

We also gather some of the properties of *k*(*m*) into the following conjecture (3-step Fibonacci sequences).

Conjecture 1.3[2] —

(*i*) If *p* is a prime number and $k(p) < p$,then $k(p)|(p-1)$.

(*ii*) If *p* is a prime number and $k(p) \equiv 0(mod 2)$,then $(p \pm 1)|k(p)$.

(*iii*) If *p* is a prime number and $k(p) \equiv 1(mod 2)$,then $(p \pm 1)|(k(p)-1)$ or $p|(k(p)-1)$.

(*iv*) $p > 2$ and *k*(*p*) is the Wall number of *n*-step Fibonacci sequence such that $n = 2t$ and $t > 1$, then *k*(*p*) is an even number.

2. TRUNCATED FIBONACCI SEQUENCE

Throughout this paper, let $\mathbb{R}^{\infty}$ denote the infinite dimensional real vector space consisting all real sequences $(x_0, x_1, x_2, \ldots)^{\mathrm{T}}$, and let *P* denote The Pascal matrix $\left[\begin{pmatrix} i \\ j \end{pmatrix}\right], (i, j = 0, 1, 2, \ldots)$.

Associated with the Pascal matrix *P*, let $P^{\rightarrow}, P^{\downarrow}$ denote the matrices defined by

$$P^{\rightarrow} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\ \ldots \\ 0\,1\,1\,0\,0\,0\,0\,0\,0\ldots \\ 0\,0\,1\,2\,1\,0\,0\,0\ldots \\ 0\,0\,0\,1\,3\,3\,1\,0\ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \ \ \cdot \end{bmatrix}$$

where the row *i*, $(i = 0, 1, 2, \ldots)$, is that of the Pascal matrix *P* preceded by 0_i^{T}, where 0_i is the *i*-vector of zeros, and

$$P^{\downarrow} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & . & . & . \\ 1 & 2 & 0 & 0 & . & . & . \\ 1 & 3 & 1 & 0 & . & . & . \\ 1 & 4 & 3 & 0 & . & . & . \\ 1 & 5 & 6 & 1 & . & . & . \\ . & . & . & . & . & . & \\ . & . & . & . & . & & . \\ . & . & . & . & . & & & . \end{bmatrix}$$

where the column *j*, $(j = 0, 1, 2, \ldots)$, is that of *P* preceded by 0_j^{T} . The sequence $\mathbb{F} = (f_0, f_1, f_2, \ldots)^{\mathrm{T}}$ defined by $f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-1}, \quad (n \geq 2)$ is called 2-step Fibonacci sequence.

There are some generalizations of the notion of Fibonacci sequences. We are interested in generalization of Fibonacci sequences [3].

Suppose that we arrange *n* congruent cubic blocks in such a way that the blocks are arranged in one or two rows and each of the blocks in the top row(top blocks) is placed on a block in the bottom row(base block). We call such an arrangement a *Fibonacci arrangement* of *n* blocks. Let F_n denote the set of all Fibonacci arrangements of *n* blocks. It is well known that $|F_n|$, the number of elements of F_n, is equal to $\binom{n}{0} + \binom{n-1}{1} + \ldots + \binom{1}{n-1} + \binom{0}{n} = f_{n+1}$, the (*n* + 1)st Fibonacci number. For a nonnegative integer *r*, let $F_{r,n}$ denote the set of all Fibonacci arrangements of *n* blocks with the *'truncation'* condition that the number of top blocks is $\leq r$.

In what follows we assume, for an integer *i* and for a nonnegative integer *t*, that $\binom{i}{t} = 0$ if $i < t$.

Let $\mathbb{F}_r = (f_{r,0}, f_{r,1}, f_{r,2}, \ldots)^{\mathrm{T}}$ be defined by $f_{r,0} = 0, f_{r,1} = 1$, $f_{r,n} = |F_{r,n-1}|$ for $n \geq 2$. Let *t* be the number of top blocks in an arrangement in $F_{r,n}$. Then the number of base blocks in that arrangement is $n - t$, and hence there are $\binom{n-t}{t}$ ways that the *t* top blocks can be placed. Since $0 \leq t \leq r$, we have $f_{r,n+1} = \sum_{t=0}^{r} \binom{n-t}{t}$. We see that $f_{r,n+1}$ is equal to the sum of the entries of *P* lying on the line $i + j = n$ within the range $0 \leq j \leq r$, and hence that $\mathbb{F}_r$ is the row

sum vector of $\begin{bmatrix} 0^{\mathrm{T}} \\ \mathrm{P}^{\downarrow}[0,1,2,\ldots|0,1,\ldots,r] \end{bmatrix}$.

We call $\mathbb{F}_r$ the *r* th truncated Fibonacci sequence.

The following table I shows first 19 terms of 1.,2.,3. and 4. truncated Fibonacci sequence.

TABLE I

$\mathbb{F}_{r,n+1}$	*n*=**0**	*n*=**1**	*n*=**2**	*n*=**3**	*n*=**4**	*n*=**5**	*n*=**6**	*n*=**7**	*n*=**8**	*n*=**9**	*n*=**10**	*n*=**11**	*n*=**12**	*n*=**13**	*n*=**14**	*n*=**15**	*n*=**16**	*n*=**17**	*n*=**18**	*n*=**19**
r=**1**	1	1	2	3	**4**	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
r=**2**	1	1	2	3	5	8	**12**	17	23	30	38	47	57	68	80	93	107	122	138	155
r=**3**	1	1	2	3	5	8	13	21	**33**	50	73	103	141	188	245	313	393	486	593	715
r=**4**	1	1	2	3	5	8	13	21	34	55	**88**	138	211	314	455	643	888	1201	1594	2080

Theorem2.1 — $f_{r,n+2} = f_{r-1,n} + f_{r,n+1}$

PROOF:

$$\begin{aligned} f_{r,n+2} &= \sum_{t=o}^{r} \begin{pmatrix} n+1-t \\ \mathrm{t} \end{pmatrix} = \begin{pmatrix} n+1 \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} n-1 \\ 2 \end{pmatrix} + \ldots + \begin{pmatrix} n+1-r \\ r \end{pmatrix} \end{aligned} \tag{1}$$

$$\begin{aligned} f_{r-1,n} + f_{r,n+1} &= \sum_{t=o}^{r-1} \begin{pmatrix} n-1-t \\ t \end{pmatrix} + \sum_{t=o}^{r} \begin{pmatrix} n-t \\ t \end{pmatrix} \\ &= \begin{pmatrix} n-1 \\ 0 \end{pmatrix} + \begin{pmatrix} n-2 \\ 1 \end{pmatrix} + \ldots + \begin{pmatrix} n-r \\ r-1 \end{pmatrix} \\ &\quad + \begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n-1 \\ 1 \end{pmatrix} + \ldots + \begin{pmatrix} n-r \\ r \end{pmatrix} \end{aligned} \tag{2}$$

In (2), since $\begin{pmatrix} t \\ j-1 \end{pmatrix} + \begin{pmatrix} t \\ j \end{pmatrix} = \begin{pmatrix} t+1 \\ j \end{pmatrix}$ we write following equalities;

$$\begin{aligned} \begin{pmatrix} n-1 \\ 0 \end{pmatrix} + \begin{pmatrix} n-1 \\ 1 \end{pmatrix} &= \begin{pmatrix} n \\ 1 \end{pmatrix}, \begin{pmatrix} n-2 \\ 1 \end{pmatrix} + \begin{pmatrix} n-2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} n-1 \\ 2 \end{pmatrix}, \ldots, \begin{pmatrix} n-r+1 \\ r-2 \end{pmatrix} + \begin{pmatrix} n-r+1 \\ r-1 \end{pmatrix} = \begin{pmatrix} n-r+2 \\ r-1 \end{pmatrix} \end{aligned}$$

and $\begin{pmatrix} n+1 \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ 0 \end{pmatrix}$, so we show that $f_{r,n+2} = f_{r-1,n} + f_{r,n+1}$.

Result 2.2 — $f_{1,n} = n-1$.

PROOF: $f_{1,n} = \sum_{t=0}^{1} \begin{pmatrix} n-1-t \\ t \end{pmatrix} = \begin{pmatrix} n-1 \\ 0 \end{pmatrix} + \begin{pmatrix} n-2 \\ 1 \end{pmatrix} = 1 + (n-2) = n-1$.

Result 2.3 — $f_{0,n}=1$.

PROOF: $f_{0,n}=\sum_{t=0}^{0}\begin{pmatrix} n-1-t \\ t \end{pmatrix}=\begin{pmatrix} n-1 \\ 0 \end{pmatrix}=1$.

We suppose that $f_{r,n+1}=f^0_{r,n+1}$ and for $n\geq r$, $i\leq r-1$ and $i,r,n\in\mathbb{Z}^+$, $f^i_{r,n+1}=f^{i-1}_{r,n+1}-f^{i-1}_{r,n}$.

Theorem 2.4 — $f^{r-1}_{r,n+1}=\sum_{i=0}^{r-1}(-1)^i\begin{pmatrix} r-1 \\ i \end{pmatrix}f^0_{r,n+1-i}$, ($n+1\geq 2r$). Furthermore, notice that coefficients of term $f^0_{r,n+1}$ are the same as those of binomial expansion of the term $(x-y)^{r-1}$.

PROOF: We will prove the theorem by induction on *r*.

For $r=1$,

$$f^0_{r,n+1}=\sum_{i=0}^{0}(-1)^i\begin{pmatrix} r-1 \\ i \end{pmatrix}f^0_{r,n+1-i}=(-1)^0\begin{pmatrix} r-1 \\ 0 \end{pmatrix}f^0_{r,n+1}=f^0_{r,n+1}.$$

For $r=k$, we suppose that $f^{k-1}_{k,n+1}=\sum_{i=0}^{k-1}(-1)^i\begin{pmatrix} k-1 \\ i \end{pmatrix}f^0_{k,n+1-i}$. That is, we have

$$\begin{aligned} f^{k-1}_{k,n+1} &= (-1)^0\begin{pmatrix} k-1 \\ 0 \end{pmatrix}f^0_{k,n+1}+(-1)^1\begin{pmatrix} k-1 \\ 1 \end{pmatrix}f^0_{k,n}+(-1)^2\begin{pmatrix} k-1 \\ 2 \end{pmatrix}f^0_{k,n-1}+\dots \\ &\quad +(-1)^{k-2}\begin{pmatrix} k-1 \\ k-2 \end{pmatrix}f^0_{k,n+1-(k-2)}+(-1)^{k-1}\begin{pmatrix} k-1 \\ k-1 \end{pmatrix}f^0_{k,n+1-(k-1)}. \end{aligned}$$

For $r=k+1$, let's find

$$\begin{aligned} f^k_{k+1,n+1} &= \sum_{i=0}^{k}(-1)^i\begin{pmatrix} k \\ i \end{pmatrix}f^0_{k+1,n+1-i}=(-1)^0\begin{pmatrix} k \\ 0 \end{pmatrix}f^0_{k+1,n+1}+(-1)^1\begin{pmatrix} k \\ 1 \end{pmatrix}f^0_{k+1,n}+\dots \\ &\quad +(-1)^{k-1}\begin{pmatrix} k \\ k-1 \end{pmatrix}f^0_{k+1,n+1-(k-1)}+(-1)^k\begin{pmatrix} k \\ k \end{pmatrix}f^0_{k+1,n+1-k}. \end{aligned}$$

By using this equality that $f^k_{k+1,n+1}=f^{k-1}_{k+1,n+1}-f^{k-1}_{k+1,n}$,

We obtain $f^k_{k+1,n+1}=f^{k-1}_{k+1,n+1}-f^{k-1}_{k+1,n}$

$$\begin{aligned} &= (-1)^0\begin{pmatrix} k-1 \\ 0 \end{pmatrix}f^0_{k+1,n+1}+(-1)^1\begin{pmatrix} k-1 \\ 1 \end{pmatrix}f^0_{k+1,n}+(-1)^2\begin{pmatrix} k-1 \\ 2 \end{pmatrix}f^0_{k+1,n-1}+\dots \\ &\quad +(-1)^{k-2}\begin{pmatrix} k-1 \\ k-2 \end{pmatrix}f^0_{k+1,n+1-(k-2)}+(-1)^{k-1}\begin{pmatrix} k-1 \\ k-1 \end{pmatrix}f^0_{k+1,n+1-(k-1)} \\ &\quad -(-1)^0\begin{pmatrix} k-1 \\ 0 \end{pmatrix}f^0_{k+1,n}-(-1)^1\begin{pmatrix} k-1 \\ 1 \end{pmatrix}f^0_{k+1,n-1}-(-1)^2\begin{pmatrix} k-1 \\ 2 \end{pmatrix}f^0_{k+1,n-2}+\dots \end{aligned}$$

$$-(-1)^{k-2}\begin{pmatrix} k-1 \\ k-2 \end{pmatrix} f^0_{k+1,n-(k-2)} - (-1)^{k-1}\begin{pmatrix} k-1 \\ k-1 \end{pmatrix} f^0_{k+1,n-(k-1)}.$$

We can write

$$(-1)^0\begin{pmatrix} k-1 \\ 0 \end{pmatrix} f^0_{k+1,n+1} = (-1)^0\begin{pmatrix} k \\ 0 \end{pmatrix} f^0_{k+1,n+1},$$

$$-(-1)^{k-1}\begin{pmatrix} k-1 \\ k-1 \end{pmatrix} f^0_{k+1,n-(k-1)} = (-1)^k\begin{pmatrix} k \\ k \end{pmatrix} f^0_{k+1,n+1-k}$$

and

$$\begin{pmatrix} k+1 \\ j \end{pmatrix} = \begin{pmatrix} k \\ j \end{pmatrix} + \begin{pmatrix} k \\ j-1 \end{pmatrix}.$$

Together with the last three equalities, the proof is completed.

Theorem 2.5 — We suppose $f_{r,n+1} = f^0_{r,n+1}$ and $f^i_{r,n+1} = f^{i-1}_{r,n+1} - f^{i-1}_{r,n}$ for $n \geq r$, $i \leq r-1$ and $i, r, n \in \mathbb{Z}^+$. Then $(f^{r-1}_{r,2r}, f^{r-1}_{r,2r+1}, f^{r-1}_{r,2r+2}, \ldots, f^{r-1}_{r,2r+s}, \ldots) = (1, 2, 3, \ldots, s+1, \ldots)$.

PROOF: We will, in fact, prove that

$$\begin{aligned}(f^{r-1}_{r,2r}, f^{r-1}_{r,2r+1}, f^{r-1}_{r,2r+2}, \ldots, f^{r-1}_{r,2r+s}, \ldots) &= (1, 2, 3, \ldots, s+1, \ldots) \\ &= (f^0_{1,2}, f^0_{1,3}, f^0_{1,4}, \ldots, f^0_{1,s+2}, \ldots)\end{aligned}$$

Here we give a proof of $f^{r-1}_{r,2r} = f^0_{1,2} = 1$. For generals, the proof is similar and is left to the reader.

If $r = 1$, then $f^{r-1}_{r,2r} = f^0_{1,2}$. Because, $f^0_{1,n} = n-1, f^0_{1,2} = 1$ by result 2.2.

For $r = 2$, $f^{r-1}_{r,2r} = f^1_{2,4} = f^0_{2,4} - f^0_{2,3} = f^0_{1,2} = 1$.

For $r = 3$, $f^{r-1}_{r,2r} = f^2_{3,6} = f^1_{3,6} - f^1_{3,5} = f^0_{3,6} - f^0_{3,5} - f^0_{3,5} + f^0_{3,4}$

$$= f^0_{2,4} - f^0_{2,3} = f^0_{1,2} = 1.$$

For $r = 4$, $f^{r-1}_{r,2r} = f^3_{4,8} = f^2_{4,8} - f^2_{4,7} = f^1_{4,8} - f^1_{4,7} - f^1_{4,7} + f^1_{4,6}$

$= f^0_{4,8} - f^0_{4,7} - f^0_{4,7} + f^0_{4,6} - f^0_{4,7} + f^0_{4,6} + f^0_{4,6} - f^0_{4,5} = f^0_{3,6} - f^0_{3,5} - f^0_{3,5} + f^0_{3,4}$

$$= f^0_{2,4} - f^0_{2,3} = f^0_{1,2} = 1$$

Similarly,

$$f^{r-1}_{r,2r} = f^{r-2}_{r,2r} - f^{r-2}_{r,2r-1}$$

$$
\begin{aligned}
&= f_{r,2r}^{r-3} - f_{r,2r-1}^{r-3} - f_{r,2r-1}^{r-3} + f_{r,2r-2}^{r-3} \\
&\vdots \\
&= (-1)^0 \begin{pmatrix} r-1 \\ 0 \end{pmatrix} f_{r,2r}^0 + (-1)^1 \begin{pmatrix} r-1 \\ 1 \end{pmatrix} f_{r,2r-1}^0 + (-1)^2 \begin{pmatrix} r-1 \\ 2 \end{pmatrix} f_{r,2r-2}^0 + \dots \\
&\quad + (-1)^{r-2} \begin{pmatrix} r-1 \\ r-2 \end{pmatrix} f_{r,r+2}^0 + (-1)^{r-1} \begin{pmatrix} r-1 \\ r-1 \end{pmatrix} f_{r,r+1}^0 \\
&= (-1)^0 \begin{pmatrix} r-1 \\ 0 \end{pmatrix} f_{r,2r}^0 + (-1)^1 \left[\begin{pmatrix} r-2 \\ 1 \end{pmatrix} + \begin{pmatrix} r-2 \\ 0 \end{pmatrix} \right] f_{r,2r-1}^0 \\
&\quad + (-1)^2 \left[\begin{pmatrix} r-2 \\ 2 \end{pmatrix} + \begin{pmatrix} r-2 \\ 1 \end{pmatrix} \right] f_{r,2r-2}^0 + \dots \\
&\quad + (-1)^{r-2} \left[\begin{pmatrix} r-2 \\ r-2 \end{pmatrix} + \begin{pmatrix} r-2 \\ r-3 \end{pmatrix} \right] f_{r,r+2}^0 + (-1)^{r-1} \begin{pmatrix} r-1 \\ r-1 \end{pmatrix} f_{r,r+1}^0 \\
&= (-1)^0 \begin{pmatrix} r-1 \\ 0 \end{pmatrix} f_{r,2r}^0 + (-1)^1 \begin{pmatrix} r-2 \\ 0 \end{pmatrix} f_{r,2r-1}^0 + (-1)^1 \begin{pmatrix} r-2 \\ 1 \end{pmatrix} f_{r,2r-1}^0 \\
&\quad + (-1)^2 \begin{pmatrix} r-2 \\ 1 \end{pmatrix} f_{r,2r-2}^0 + (-1)^2 \begin{pmatrix} r-2 \\ 2 \end{pmatrix} f_{r,2r-2}^0 + \dots \\
&\quad + (-1)^{r-2} \begin{pmatrix} r-2 \\ r-3 \end{pmatrix} f_{r,r+2}^0 + (-1)^{r-2} \begin{pmatrix} r-2 \\ r-2 \end{pmatrix} f_{r,r+2}^0 + (-1)^{r-1} \begin{pmatrix} r-1 \\ r-1 \end{pmatrix} f_{r,r+1}^0 \\
&= (-1)^0 f_{r-1,2r-2}^0 + (-1)^1 \begin{pmatrix} r-2 \\ 1 \end{pmatrix} f_{r-1,2r-3}^0 + \dots + (-1)^{r-1} f_{r-1,r}^0
\end{aligned}
$$

and we go on similarly, we obtain

$$= f_{3,6}^0 - f_{3,5}^0 - f_{3,5}^0 + f_{3,4}^0$$

$$= f_{2,4}^0 - f_{2,3}^0 = f_{1,2}^0 = 1.$$

Thus, one can show easily that $f_{r,2r+s}^{r-1} = f_{1,s+2}^0 = s+1$.

We consider 2-step Fibonacci sequence $f_n = f_{n-1} + f_{n-1}$, for $f_0 = 0, f_1 = 1$ and $n \geq 2$ and terms of the truncated Fibonacci $f_{r,n+1} = \sum_{t=0}^{r} \begin{pmatrix} n-t \\ t \end{pmatrix}$. As we see Table I, firstly $f_{r,n+1} = f_{n+1}$ but, for n sufficiently large we see that $f_{r,n+1} \neq f_{n+1}$, (showed black bold in Table I).

We clarify this assertion via the following theorem.

Theorem **2.6** — $f_{r,2r+2} = f_{2r+2}$ and $f_{r,2r+3} \neq f_{2r+3}$.

PROOF:

$$f_{r,2r+2} = \sum_{t=0}^{r} \left(\begin{array}{c} 2r+1-t \\ t \end{array} \right) = \left(\begin{array}{c} 2r+1 \\ 0 \end{array} \right) + \left(\begin{array}{c} 2r \\ 1 \end{array} \right) + \left(\begin{array}{c} 2r-1 \\ 2 \end{array} \right) + \cdots + \left(\begin{array}{c} r+2 \\ r-1 \end{array} \right) + \left(\begin{array}{c} r+1 \\ r \end{array} \right) \tag{3}$$

and since $f_{n+1} = \left(\begin{array}{c} n \\ 0 \end{array} \right) + \left(\begin{array}{c} n-1 \\ 1 \end{array} \right) + \ldots + \left(\begin{array}{c} 1 \\ n-1 \end{array} \right) + \left(\begin{array}{c} 0 \\ n \end{array} \right)$,

we write

$$f_{2r+2} = \binom{2r+1}{0} + \binom{2r}{1} + \ldots + \binom{r+2}{r-1} + \binom{r+1}{r} + \binom{r}{r+1} + \ldots + \binom{1}{2r} + \binom{0}{2r+1} \tag{4}$$

For an integer *i* and for a nonnegative integer *t*, since that $\left(\begin{array}{c} i \\ t \end{array} \right) = 0$ if $i < t$, and equation (1) and (2), we see easily that $f_{r,2r+2} = f_{2r+2}$.

Now we show that $f_{r,2r+3} \neq f_{2r+3}$.

$$f_{r,2r+3} = \sum_{t=0}^{r} \binom{2r+2-t}{t} = \binom{2r+2}{0} + \binom{2r+1}{1} + \binom{2r}{2} + \ldots + \binom{r+3}{r-1} + \binom{r+2}{r} \tag{5}$$

and

$$f_{2r+3} = \underbrace{\left(\begin{array}{c} 2r+2 \\ 0 \end{array} \right) + \left(\begin{array}{c} 2r+1 \\ 1 \end{array} \right) + \ldots + \left(\begin{array}{c} r+2 \\ r \end{array} \right)}_{f_{r,2r+3}} + \underbrace{\left(\begin{array}{c} r+1 \\ r+1 \end{array} \right)}_{1} + \underbrace{\left(\begin{array}{c} r \\ r+2 \end{array} \right) + \ldots + \left(\begin{array}{c} 0 \\ 2r+2 \end{array} \right)}_{0} \tag{6}$$

Since equation (5), we write $f_{2r+3} = f_{r,2r+3} + 1$. That is, $f_{r,2r+3} \neq f_{2r+3}$.

3.THE FUNCTION $k_r(m)$

This section is concerned with determining the length of the recurring sequence obtained by reducing a Fibonacci sequence by a modulo *m*.

Let $\mathbb{F}_r = (f_{r,0}, f_{r,1}, f_{r,2}, \ldots)^{\mathrm{T}}$ be the *r* th *truncated Fibonacci sequence*. Then, $\mathbb{F}_r(modm)$ does not form a periodic sequence. But if we take $\mathbb{F}_{r,n+1} = (f_{r,n+1}, f_{r,n+2}, f_{r,n+3}, \ldots)^{\mathrm{T}}$, ($n = r, r+1, r+2, \ldots$), then $\mathbb{F}_{r,n+1}(modm)$ forms a periodic sequence. Now we can talk about the function $k_r(m)$ of the sequence $\mathbb{F}_{r,n+1}(modm)$. We make the following conjectures.

Conjecture 3.1 — $k_r(m.n) = k_r(m).k_r(n)$ on condition that $(m, n) = 1$.

That is, the function $k_r(m)$ is a multiplicative function.

Conjecture 3.2 — If *p* is a prime number and $t \in \mathbb{Z}^+$, then $k_r(p^t) = p^{t-1}k_r(p)$.

Conjecture 3.3 — If *p* is a prime number and $p > r$, then $k_r(p) = p$.

Conjecture 3.4 — If *m* is not a prime number, then $m \,|k_r(m)$ and $m \neq k_r(m)$.

For several values of the parameters, these conjectures seem to be validated by a computer program written in MAPLE and FORTRAN.

In addition, Table II shows Wall number of $2., 3., \ldots, 7.$ truncated Fibonacci sequence for some numbers. Validities of above conjectures can be checked with Table II. We have lots of data for various of m and r.

TABLE II

$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
$k(2)=4$	$k(2)=4$	$k(2)=8$	$k(2)=8$	$k(2)=8$	$k(2)=8$
$k(3)=3$	$k(3)=9$	$k(3)=9$	$k(3)=9$	$k(3)=9$	$k(3)=9$
$k(4)=8$	$k(4)=8$	$k(4)=16$	$k(4)=16$	$k(4)=16$	$k(4)=16$
$k(5)=5$	$k(5)=5$	$k(5)=5$	$k(5)=25$	$k(5)=25$	$k(5)=25$
$k(6)=12$	$k(6)=36$	$k(6)=72$	$k(6)=72$	$k(6)=72$	$k(6)=72$
$k(7)=7$	$k(7)=7$	$k(7)=7$	$k(7)=7$	$k(7)=7$	$k(7)=49$
$k(8)=16$	$k(8)=16$	$k(8)=32$	$k(8)=32$	$k(8)=32$	$k(8)=32$
$k(9)=9$	$k(9)=27$	$k(9)=27$	$k(9)=27$	$k(9)=27$	$k(9)=27$
$k(10)=20$	$k(10)=20$	$k(10)=40$	$k(10)=200$	$k(10)=200$	$k(10)=200$
$k(11)=11$	$k(11)=11$	$k(11)=11$	$k(11)=11$	$k(11)=11$	$k(11)=11$
$k(12)=24$	$k(12)=72$	$k(12)=144$	$k(12)=144$	$k(12)=144$	$k(12)=144$
$k(13)=13$	$k(13)=13$	$k(13)=13$	$k(13)=13$	$k(13)=13$	$k(13)=13$
$k(14)=28$	$k(14)=28$	$k(14)=56$	$k(14)=56$	$k(14)=56$	$k(14)=392$
$k(15)=15$	$k(15)=45$	$k(15)=45$	$k(15)=225$	$k(15)=225$	$k(15)=225$
$k(16)=32$	$k(16)=32$	$k(16)=64$	$k(16)=64$	$k(16)=64$	$k(16)=64$
$k(17)=17$	$k(17)=17$	$k(17)=17$	$k(17)=17$	$k(17)=17$	$k(17)=17$
$k(18)=36$	$k(18)=108$	$k(18)=216$	$k(18)=216$	$k(18)=216$	$k(18)=216$
$k(19)=19$	$k(19)=19$	$k(19)=19$	$k(19)=19$	$k(19)=19$	$k(19)=19$
$k(20)=40$	$k(20)=40$	$k(20)=80$	$k(20)=400$	$k(20)=400$	$k(20)=400$
$k(21)=21$	$k(21)=63$	$k(21)=63$	$k(21)=63$	$k(21)=63$	$k(21)=441$
$k(22)=44$	$k(22)=44$	$k(22)=88$	$k(22)=88$	$k(22)=88$	$k(22)=88$
$k(23)=23$	$k(23)=23$	$k(23)=23$	$k(23)=23$	$k(23)=23$	$k(23)=23$
$k(24)=48$	$k(24)=144$	$k(24)=288$	$k(24)=288$	$k(24)=288$	$k(24)=288$

Notice that by using Conjecture 3.3, We can now rewrite some items of Theorem 1.1, Theorem 1.2 and Conjecture 1.3. So we can see a relationship between $k(p)$ and $k_r(p)$.

Result 3.5 —

If $k(p) \neq k(p^2)$, then $k(p^e) = k_r(p)^{e-1}k(p)$. Also, if *n* is the largest integer with $k(p) = k(p^n)$, then $k(p^e) = k_r(p)^{e-1}k(p)$ for $e > n$.

Result 3.6: In $\mathbb{Z}_m$, the following hold:

(*i*) If *p* is a prime and $p = 10x \pm 1$,then $k(p)|(k_r(p) - 1)$.

(*ii*) If *p* is a prime and $p = 10x \pm 3$,then $k(p)|(2k_r(p) + 2)$.

Result 3.7 —

(*i*) If *p* is a prime number and $k(p) < p$, then $k(p)|(k_r(p) - 1)$.

(*ii*) If *p* is a prime number and $k(p) \equiv 0(mod 2)$,then $(k_r(p) \pm 1)|k(p)$.

(*iii*) If *p* is a prime number and $k(p) \equiv 1(mod 2)$,then $(k_r(p) \pm 1)|(k(p)-1)$ or $k_r(p)|(k(p)-1)$.

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