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On the Norms and Eigenvalues of r-Circulant Matrices with k-Mersenne and k-Mersenne–Lucas Numbers*

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Abstract—In this work, we study the r-circulant matrix $C_r = Circ_r(c_0, c_1, c_2, ..., c_{n-1})$ such that the entries of C_r are $c_i = M_{k,a+ib}$ or $c_i = R_{k,a+ib}$, where $M_{k,a+ib}$ and $R_{k,a+ib}$ are k-Mersenne and k-Mersenne–Lucas numbers, respectively. We obtain the eigenvalues and determinants for the matrices and some important identities for the k-Mersenne and k-Mersenne–Lucas numbers. Furthermore, we find norms and bounds estimation for the spectral norm for these r-circulant matrices.

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1. INTRODUCTION

The Fibonacci numbers, which emerged with the famous rabbit problem, have attracted the attention of many scientists and continue to do so. By this time, scientists have given many generalizations of Fibonacci sequences. Thus, number sequences have found application in many branches of science as well as in every branch of mathematics. In [1]–[8], one of these generalizations is Mersenne sequences. In this study, the properties of r-circulant matrices associated with k-Mersenne and k-Mersenne–Lucas numbers will be examined.

For $\lambda \geq 0$, the Mersenne sequence $\{M_\lambda\}_{\lambda \in \mathbb{N} \cup \{0\}}$ [9] is defined by $M_{\lambda+1} = 3M_\lambda - 2M_{\lambda-1}$ with $M_0 = 0$ and $M_1 = 1$. If we take initial assumption $R_0 = 2$ and $R_1 = 3$ with the recurrence relation $R_{\lambda+1} = 3R_\lambda - 2R_{\lambda-1}$, then we obtain the Mersenne–Lucas sequence [10] also called Fermat sequence. In literature, there are various types of generalization of the Mersenne and Mersenne–Lucas sequences, one can go through [11]–[15]. Some useful definitions needed in the main results are given in this section.

Definition 1 ([11]). For $k \in \mathbb{R}^+$, the k-Mersenne sequence $\{M_{k,\lambda}\}_{\lambda \geq 0}$ is defined recurrently by

$$M_{k,\lambda+1} = 3kM_{k,\lambda} - 2M_{k,\lambda-1}, \text{ with } M_{k,0} = 0,\ M_{k,1} = 1.$$

The characteristic equation for the recurrence relation given in Definition 1 is $x^2 - 3kx + 2 = 0$ and its roots are $\beta_1 = (3k + \sqrt{9k^2 - 8})/2$ and $\beta_2 = (3k - \sqrt{9k^2 - 8})/2$ which holds the following relations.

$$\beta_1 + \beta_2 = 3k,\ \beta_1 - \beta_2 = \sqrt{9k^2 - 8},\ \beta_1\beta_2 = 2. \tag{1.1}$$

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Binet's formula for the k-Mersenne numbers is given by

$$M_{k,\lambda} = \frac{\beta_1^\lambda - \beta_2^\lambda}{\beta_1 - \beta_2}, \tag{1.2}$$

and Cassini's identity for the k-Mersenne numbers is

$$M_{k,\lambda}^2 - M_{k,\lambda+1}M_{k,\lambda-1} = 2^{\lambda-1}. \tag{1.3}$$

Definition 2 ([13]). The k-Mersenne–Lucas sequence $\{R_{k,\lambda}\}_{\lambda\geq 0}$ is given by

$$R_{k,\lambda+1} = 3kR_{k,\lambda} - 2R_{k,\lambda-1}, \text{ with } R_{k,0} = 2,\ R_{k,1} = 3k.$$

The closed form formula for the k-Mersenne–Lucas numbers is $R_{k,\lambda} = \beta_1^\lambda + \beta_2^\lambda$, and Cassini's identity for the k-Mersenne–Lucas numbers is

$$R_{k,\lambda}^2 - R_{k,\lambda+1}R_{k,\lambda-1} = -(9k^2 - 8)2^{\lambda-1}. \tag{1.4}$$

Now we give some definitions and results on circulant matrices and norms that we use further in our work. Throughout the paper, we are considering $9k^2 - 8 \neq 0$.

Definition 3 ([16]). A r-circulant matrix $C_r = Circ_r(c_0, c_1, ..., c_{n-1})$ is defined as

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix}.$$

On replacing r by 1 and -1 in the above matrix, we obtain the right circulant and skew-right circulant matrix respectively.

Eigenvalues of r-circulant matrix C_r denoted by $\mu_i(C_r)$ are given by [16]

$$\mu_i(C_r) = \sum_{j=0}^{n-1} c_j(\rho\omega^{-i})^j, \quad i = 0, 1, 2, ..., n-1. \tag{1.5}$$

where ω is the nth root of unity and ρ is the nth root of r.

The Euclidean norm of matrix C_r is given by

$$||C_r||_E = \sqrt{\sum_{j=0}^{n-1} |C_j|^2[n - j(1 - |r|^2)]}. \tag{1.6}$$

Investigating the eigenvalues, the determinant, norms, and bounds for the spectral norm of a circulant matrix involving a recursive sequence has been one of the exciting areas among researchers. It becomes an interesting topic of research in matrix theory due to its special properties and wide range of applications in other branches like coding theory, linear forecast, signal and image processing, mathematical physics and many more. For example, these matrices are used in image and signal processing [17], [18] vibration analysis [19], and in solving different types of differential equations [20], [21].

Recently, Solak et al. [22] have obtained the norm of r-circulant matrices involving Fibonacci and Lucas numbers whereas Shen et al. [23] found the bound for the norms of these matrices. In continuation, Shen et al. [24] presented the study of bounds for the spectral norms for the r-circulant matrices with k-Fibonacci and k-Lucas numbers. Some recent works on the r-circulant matrices involving different number sequences like Mersenne, Jacobsthal, Pell, Pell-Lucas, etc. can be seen in [25]–[29].

Lemma 1. *For any square matrix $H = [h_{ij}]$ of order n, we have*

$$||H||_1 = \max_{1\le j\le n}\sum_{i=1}^{n}|h_{ij}|,\ ||H||_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|h_{ij}|,$$

$$||H||_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|h_{ij}|^2},\ \textit{and}\ ||H||_2 = \sqrt{\max_{1\le i\le n}\mu_i(H^*H)},$$

where $\mu_i(H^*H)$ denotes the eigenvalues of the matrix (H^*H) with H^* as the conjugate transpose of H. The norm $||H||_2$ and $||H||_E$ satisfy the following relation

$$\frac{1}{\sqrt{n}} \le ||H||_E \le ||H||_2 \le ||H||_E. \tag{1.7}$$

Lemma 2. *Let M be the Hadamard product of matrices $P = [p_{ij}] \in \mathbb{M}_n(\mathbb{C})$ and $Q = [q_{ij}] \in \mathbb{M}_n(\mathbb{C})$, then*

$$||M||_2 \le \mu(P)\nu(Q),$$

where $\mu(P) = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|p_{ij}|^2}$ and $\nu(Q) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|q_{ij}|^2}$.

Lemma 3. *Let ρ be the nth root of r and ω be the nth root of unity, then for any x and y, we have*

$$\prod_{j=0}^{n-1}(x - y\rho\omega^{-j}) = x^n - ry^n. \tag{1.8}$$

Proof. Because ω is a n-th root of unity, so

$$\prod_{j=0}^{n-1}(x - \omega^j) = x^n - \omega^n = x^n - 1. \tag{1.9}$$

Replacing x by $x/y\rho$ in (1.9), we get

$$\prod_{j=0}^{n-1}\Big(\frac{x}{y\rho} - \omega^j\Big) = \Big(\frac{x}{y\rho}\Big)^n - \omega^n = \frac{x^n - y^n\rho^n}{y^n\rho^n}.$$

Since, $\prod_{j=0}^{n-1}(x-\omega^j) = \prod_{j=0}^{n-1}(x-\omega^{n-j})$ and for $1 \le j \le n-1$, $\omega^{n-j} = \omega^{-j}$. So,

$$\prod_{j=0}^{n-1}\Big(\frac{x}{y\rho} - \omega^{-j}\Big) = \frac{x^n - y^n\rho^n}{y^n\rho^n}, \qquad \text{hence} \qquad \prod_{j=0}^{n-1}\Big(x - y\rho\omega^{-j}\Big) = x^n - ry^n. \quad \square$$

Now we define r-circulant matrices involving k-Mersenne and k-Mersenne–Lucas numbers denoted as A_r and B_r, respectively, given in the following definitions. Throughout the paper it is assumed $a, b \in \mathbb{Z}$.

Definition 4. The r-circulant matrix with k-Mersenne numbers is given as

$$A_r = \begin{bmatrix} M_{k,a} & M_{k,a+b} & M_{k,a+2b} & \cdots & M_{k,a+(n-2)b} & M_{k,a+(n-1)b} \\ rM_{k,a+(n-1)b} & M_{k,a} & M_{k,a+b} & \cdots & M_{k,a+(n-3)b} & M_{k,a+(n-2)b} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rM_{k,a+2b} & rM_{k,a+3b} & rM_{k,a+4b} & \cdots & M_{k,a} & M_{k,a+b} \\ rM_{k,a+b} & rM_{k,a+2b} & rM_{k,a+3b} & \cdots & rM_{k,a+(n-1)b} & M_{k,a} \end{bmatrix}.$$

Definition 5. The r-circulant matrix with k-Mersenne–Lucas numbers is given as

$$B_r = \begin{bmatrix} R_{k,a} & R_{k,a+b} & R_{k,a+2b} & \cdots & R_{k,a+(n-2)b} & R_{k,a+(n-1)b} \\ rR_{k,a+(n-1)b} & R_{k,a} & R_{k,a+b} & \cdots & R_{k,a+(n-3)b} & R_{k,a+(n-2)b} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rR_{k,a+2b} & rR_{k,a+3b} & rR_{k,a+4b} & \cdots & R_{k,a} & R_{k,a+b} \\ rR_{k,a+b} & rR_{k,a+2b} & rR_{k,a+3b} & \cdots & rR_{k,a+(n-1)b} & R_{k,a} \end{bmatrix}.$$

In this study, we investigate formulas for eigenvalues, determinants, and norms of the above matrices and give some important identities for k-Mersenne and k-Mersenne–Lucas numbers. At last, we give upper and lower bounds for the spectral norm of these matrices.

2. MAIN RESULTS

Before we proceed to obtain eigenvalues for the above matrices, we first give the following lemmas that are further needed to prove the main results.

Lemma 4. *Let $a, b \in \mathbb{Z}$, then for the k-Mersenne numbers, we have*

$$\frac{\beta_1^a\beta_2^b - \beta_2^a\beta_1^b}{\beta_1 - \beta_2} = \begin{cases} 2^b M_{k,a-b} & : a > b, \\ 0 & : a = b, \\ -2^a M_{k,b-a} & : a < b. \end{cases}$$

Proof. Consider,

$$\frac{\beta_1^a\beta_2^b - \beta_2^a\beta_1^b}{\beta_1 - \beta_2} = \begin{cases} \beta_1^b\beta_2^b\Big(\dfrac{\beta_1^{a-b} - \beta_2^{a-b}}{\beta_1 - \beta_2}\Big) & : a > b, \\ \dfrac{\beta_1^a\beta_2^a - \beta_2^a\beta_1^a}{\beta_1 - \beta_2} & : a = b, \\ \beta_1^a\beta_2^a\Big(\dfrac{\beta_1^{b-a} - \beta_2^{b-a}}{\beta_1 - \beta_2}\Big) & : a < b. \end{cases}$$

Now by using Binet's formula for the k-Mersenne numbers and the fact that $\beta_1\beta_2 = 2$, we get

$$\frac{\beta_1^a\beta_2^b - \beta_2^a\beta_1^b}{\beta_1 - \beta_2} = \begin{cases} 2^b M_{k,a-b} & : a > b, \\ 0 & : a = b, \\ -2^a M_{k,b-a} & : a < b. \end{cases}$$

This completes the proof. □

Lemma 5. *For the k-Mersenne–Lucas numbers, we have*

$$\beta_1^a\beta_2^b + \beta_2^a\beta_1^b = \begin{cases} 2^b R_{k,a-b} & : a > b, \\ 2^{a+1} & : a = b, \\ 2^a R_{k,b-a} & : a < b. \end{cases}$$

Proof. The proof follows the same argument of Lemma 4. □

For simplicity, we write $K_M(a,b) := (\beta_1^a\beta_2^b - \beta_2^a\beta_1^b)/(\beta_1 - \beta_2)$ and $K_R(a,b) := \beta_1^a\beta_2^b + \beta_2^a\beta_1^b$. Now we give the closed form formula for eigenvalues of the matrices A_r and B_r in the following theorems.

Theorem 1. *The eigenvalues of A_r are*

$$\mu_i(A_r) = \frac{M_{k,a} - rM_{k,a+nb} - \rho\omega^{-i}[K_M(a,b) - r2^bM_{k,a+(n-1)b}]}{1 - R_{k,b}\rho\omega^{-i} + 2^b\rho^2\omega^{-2i}}.$$

Proof. Using Eq. (1.5), we can write the eigenvalues of the matrix A_r as

$$\begin{aligned}
\mu_i(A_r) &= \sum_{j=0}^{n-1} M_{k,a+jb}(\rho\omega^{-i})^j,\ i=0,1,2,...,n-1\\
&= \sum_{j=0}^{n-1}\Big(\frac{\beta_1^{a+jb}-\beta_2^{a+jb}}{\beta_1-\beta_2}\Big)(\rho\omega^{-i})^j\\
&= \frac{\beta_1^a}{\beta_1-\beta_2}\sum_{j=0}^{n-1}(\beta_1^b\rho\omega^{-i})^j - \frac{\beta_2^a}{\beta_1-\beta_2}\sum_{j=0}^{n-1}(\beta_2^b\rho\omega^{-i})^j\\
&= \frac{\beta_1^a}{\beta_1-\beta_2}\Big(\frac{1-(\beta_1^b\rho\omega^{-i})^n}{1-\beta_1^b\rho\omega^{-i}}\Big) - \frac{\beta_2^a}{\beta_1-\beta_2}\Big(\frac{1-(\beta_2^b\rho\omega^{-i})^n}{1-\beta_2^b\rho\omega^{-i}}\Big)\\
&= \frac{1}{\beta_1-\beta_2}\Big[\beta_1^a\Big(\frac{1-r\beta_1^{nb}}{1-\beta_1^b\rho\omega^{-i}}\Big) - \beta_2^a\Big(\frac{1-r\beta_2^{nb}}{1-\beta_2^b\rho\omega^{-i}}\Big)\Big]\\
&= \frac{(\beta_1^a-\beta_2^a)-r(\beta_1^{a+nb}-\beta_2^{a+nb})-\rho\omega^{-i}[(\beta_1^a\beta_2^b-\beta_2^a\beta_1^b)-r(\beta_1^{a+nb}\beta_2^b-\beta_2^{a+nb}\beta_1^a)]}{(\beta_1-\beta_2)(1-\beta_1^b\rho\omega^{-i})(1-\beta_2^b\rho\omega^{-i})}\\
&= \frac{M_{k,a}-rM_{k,a+nb}-\rho\omega^{-i}\Big[\frac{(\beta_1^a\beta_2^b-\beta_2^a\beta_1^b)}{(\beta_1-\beta_2)}-r2^bM_{k,a+(n-1)b}\Big]}{(1-\beta_1^b\rho\omega^{-i})(1-\beta_2^b\rho\omega^{-i})} \quad \text{(using Eqs. (1.1) and (1.2))}\\
&= \frac{M_{k,a}-rM_{k,a+nb}-\rho\omega^{-i}[K_M(a,b)-r2^bM_{k,a+(n-1)b}]}{(1-\beta_1^b\rho\omega^{-i})(1-\beta_2^b\rho\omega^{-i})} \quad \text{(using Lemma 4)}\\
&= \frac{M_{k,a}-rM_{k,a+nb}-\rho\omega^{-i}[K_M(a,b)-r2^bM_{k,a+(n-1)b}]}{1-R_{k,b}\rho\omega^{-i}+2^b\rho^2\omega^{-2i}}
\end{aligned}$$

as required. □

By setting $r=1$ and $r=-1$ in Theorem 1, we obtain eigenvalues for the k-Mersenne right circulant and k-Mersenne skew-right circulant matrices given in the following corollary.

Corollary 1.

$$\mu_i(A_1) = \frac{M_{k,a}-M_{k,a+nb}-\omega^{-i}[K_M(a,b)-2^bM_{k,a+(n-1)b}]}{1-R_{k,b}\omega^{-i}+2^b\omega^{-2i}},$$

$$\mu_i(A_{-1}) = \frac{M_{k,a}+M_{k,a+nb}-\zeta\omega^{-i}[K_M(a,b)+2^bM_{k,a+(n-1)b}]}{1-R_{k,b}\zeta\omega^{-i}+2^b\zeta^2\omega^{-2i}}.$$

where ζ is the nth root of -1.

Theorem 2. *The eigenvalues of B_r made with k-Mersenne–Lucas numbers are given by*

$$\mu_i(B_r) = \frac{R_{k,a}-rR_{k,a+nb}+\rho\omega^{-i}[r2^bR_{k,a+(n-1)b}-K_R(a,b)]}{1-R_{k,b}\rho\omega^{-i}+2^b\rho^2\omega^{-2i}}.$$

Proof. The proof follows the same argument of Theorem 1 with Lemma 5 and Eq. (1.1). □

Corollary 2. *Let ζ be the nth root of* -1. *The eigenvalues for the right circulant matrix B_1 and skew-right circulant matrix B_{-1} are, respectively,*

$$\mu_i(B_1) = \frac{R_{k,a}-R_{k,a+nb}+\omega^{-i}[2^bR_{k,a+(n-1)b}-K_R(a,b)]}{1-R_{k,b}\omega^{-i}+2^b\omega^{-2i}},$$

$$\textit{and}\quad \mu_i(B_{-1}) = \frac{R_{k,a}+R_{k,a+nb}-\zeta\omega^{-i}[r2^bR_{k,a+(n-1)b}+K_R(a,b)]}{1-R_{k,b}\zeta\omega^{-i}+2^b\zeta^2\omega^{-2i}}.$$

Proof. It can be proved easily by substituting $r = 1$ and $r = -1$ in Theorem 2. □

Theorem 3. *The sum of the eigenvalues of the k-Mersenne circulant matrices is*

$$\sum_{i=0}^{n-1} \frac{M_{k,a} - rM_{k,a+nb} - \rho\omega^{-i}[K_M(a,b) - r2^bM_{k,a+(n-1)b}]}{1 - R_{k,b}\rho\omega^{-i} + 2^b\rho^2\omega^{-2i}} = nM_{k,a}.$$

Proof. We use the fact that the sum of the eigenvalues of a matrix gives a trace of that matrix and the trace is also defined as the sum of the diagonal elements of the matrix (here A_r). In our case, the matrix A_r is a diagonal constant and each diagonal entry is $M_{k,a}$.

Hence $trace(A_r) = \sum_{l=1}^{n} M_{k,a} = nM_{k,a}$.

It gives,

$$\sum_{i=0}^{n-1} \frac{M_{k,a} - rM_{k,a+nb} - \rho\omega^{-i}[K_M(a,b) - r2^bM_{k,a+(n-1)b}]}{1 - R_{k,b}\rho\omega^{-i} + 2^b\rho^2\omega^{-2i}} = \sum_{i=0}^{n-1} \mu_i(A_r) = nM_{k,a}$$

as required. □

Since $M_{k,a} = 0$ for $a = 0$, we obtain

$$\sum_{i=0}^{n-1} \frac{rM_{k,nb} + \rho\omega^{-i}[K_M(0,b) - r2^bM_{k,(n-1)b}]}{1 - R_{k,b}\rho\omega^{-i} + 2^b\rho^2\omega^{-2i}} = 0.$$

Theorem 4. *For the r-circulant matrices with k-Mersenne–Lucas number, the following identity is verified*:

$$\sum_{i=0}^{n-1} \frac{R_{k,a} - rR_{k,a+nb} + \rho\omega^{-i}[r2^bR_{k,a+(n-1)b} - K_R(a,b)]}{1 - R_{k,b}\rho\omega^{-i} + 2^b\rho^2\omega^{-2i}} = nR_{k,a}.$$

Proof. Consider $K_R(a,b)$ as defined in Lemma 5. Then proceeding as Theorem 3 with k-Mersenne Lucas numbers $R_{k,n}$, we get the desired result. □

If $a = 0$ then $R_{k,a} = 2$, hence from the above theorem, we have

$$\sum_{i=0}^{n-1} \frac{2 - rR_{k,nb} + \rho\omega^{-i}[r2^bR_{k,(n-1)b} - K_R(0,b)]}{1 - R_{k,b}\rho\omega^{-i} + 2^b\rho^2\omega^{-2i}} = 2n.$$

Theorem 5. *The determinant of the matrix A_r is given by*

$$\det(A) = \frac{(M_{k,a} - rM_{k,a+nb})^n - r[K_M(a,b) - r2^bM_{k,a+(n-1)b}]^n}{1 - rR_{k,nb} + 2^{nb}r^2}.$$

Proof. We implement the fact that the determinant of a matrix is the product of its eigenvalues. Thus, we have

$$\begin{aligned}
\det(A_r) &= \prod_{i=0}^{n-1} \mu_i(A_r) \\
&= \prod_{i=0}^{n-1} \frac{(M_{k,a} - rM_{k,a+nb}) - \rho\omega^{-i}[K_M(a,b) - r2^bM_{k,a+(n-1)b}]}{(1 - \beta_1^b\rho\omega^{-i})(1 - \beta_2^b\rho\omega^{-i})} \\
&= \frac{(M_{k,a} - rM_{k,a+nb})^n - r[K_M(a,b) - r2^bM_{k,a+(n-1)b}]^n}{(1 - r\beta_1^{nb})(1 - r\beta_2^{nb})} \quad \text{(using Eq. (1.8))} \\
&= \frac{(M_{k,a} - rM_{k,a+nb})^n - r[K_M(a,b) - r2^bM_{k,a+(n-1)b}]^n}{1 - rR_{k,nb} + 2^{nb}r^2}
\end{aligned}$$

as required. □

Corollary 3. *The determinants of the right and skew-right circulant matrix with k-Mersenne numbers are given as*

$$\det(A_1) = \frac{(M_{k,a} - M_{k,a+nb})^n - [K_M(a,b) - 2^b M_{k,a+(n-1)b}]^n}{1 - R_{k,nb} + 2^{nb}},$$
$$\det(A_{-1}) = \frac{(M_{k,a} + M_{k,a+nb})^n + [K_M(a,b) + 2^b M_{k,a+(n-1)b}]^n}{1 + R_{k,nb} + 2^{nb}}.$$

Proof. On replacing $r = 1$ and $r = -1$ in Theorem 5, respectively, we obtain the results. □

Theorem 6. *The determinant of r-circulant matrix with k-Mersenne–Lucas numbers is given as,*

$$\det(B_r) = \frac{(R_{k,a} - rR_{k,a+nb})^n - r[K_R(a,b) - r2^b M_{k,a+(n-1)b}]^n}{1 - rR_{k,nb} + 2^{nb}r^2}.$$

Proof. Consider the eigenvalues given in Theorem 2 and result of Eq. (1.8). On proceeding as Theorem 5, we get the desired result. □

Corollary 4.

$$\det(B_1) = \frac{(R_{k,a} - R_{k,a+nb})^n - [K_R(a,b) - 2^b M_{k,a+(n-1)b}]^n}{1 - R_{k,nb} + 2^{nb}},$$
$$\det(B_{-1}) = \frac{(R_{k,a} + R_{k,a+nb})^n + [K_R(a,b) + 2^b M_{k,a+(n-1)b}]^n}{1 + R_{k,nb} + 2^{nb}}.$$

Proof. It can be achieved by setting $r = 1$ and -1 in Theorem 6, respectively. □

Sum Identities

For k-Mersenne and k-Mersenne–Lucas numbers having arithmetic indices, we have the following sum formula.

Theorem 7. *Let $M_{k,n}$ and $R_{k,n}$ are the k-Mersenne and k-Mersenne–Lucas numbers, then we have*

$$\sum_{j=0}^{n-1} M_{k,a+jb} = \frac{M_{k,a} - M_{k,a+nb} - [K_M(a,b) - 2^b M_{k,a+(n-1)b}]}{1 - R_{k,b} + 2^b},$$
$$\sum_{j=0}^{n-1} R_{k,a+jb} = \frac{R_{k,a} - R_{k,a+nb} + [2^b R_{k,a+(n-1)b} - K_R(a,b)]}{1 - R_{k,b} + 2^b}.$$

Proof. It can be easily proved by considering the result of Theorem 1 and Theorem 2 for $i = 0$ and $r = 1$. □

On taking $a = 0, b = 1$ in the above theorem, for $k \neq 1$ we get the sum of the first n terms of the k-Mersenne and k-Mersenne–Lucas sequence, respectively.

Corollary 5.

$$\sum_{j=0}^{n-1} M_{k,j} = \frac{2M_{k,n-1} - M_{k,n} + 1}{3(1-k)},$$
$$\sum_{j=0}^{n-1} R_{k,j} = \frac{2R_{k,n-1} - R_{k,n} - 3k + 2}{3(1-k)}.$$

Proof. It is particular case of Theorem 7 for $a = 0$ and $b = 1$. □

3. NORMS AND BOUNDS ON THE SPECTRAL NORM

In this section, we give different norms and bounds for the spectral norm of the matrices given in Definitions 4 and 5 for the particular values $a = 0$ and $b = 1$.

Circulant matrices for $a = 0$ and $b = 1$ denoted as A'_r and B'_r are given, respectively, by

$$A'_r = \begin{bmatrix} M_{k,0} & M_{k,1} & M_{k,2} & \cdots & M_{k,n-2} & M_{k,n-1} \\ rM_{k,n-1} & M_{k,0} & M_{k,1} & \cdots & M_{k,n-3} & M_{k,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rM_{k,2} & rM_{k,3} & rM_{k,4} & \cdots & M_{k,0} & M_{k,1} \\ rM_{k,1} & rM_{k,2} & rM_{k,3} & \cdots & rM_{k,n-1} & M_{k,0} \end{bmatrix}$$

and

$$B'_r = \begin{bmatrix} R_{k,0} & R_{k,1} & R_{k,2} & \cdots & R_{k,n-2} & R_{k,n-1} \\ rR_{k,n-1} & R_{k,0} & R_{k,1} & \cdots & R_{k,n-3} & R_{k,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rR_{k,2} & rR_{k,3} & rR_{k,4} & \cdots & R_{k,0} & R_{k,1} \\ rR_{k,1} & rR_{k,2} & rR_{k,3} & \cdots & rR_{k,n-1} & R_{k,0} \end{bmatrix}.$$

The norms $||.||_1$ and $||.||_\infty$ are also known as the maximum absolute column sum and maximum absolute row sum matrix norm, respectively.

Lemma 6. *Let M_r be any r-circulant square matrix, then $||M_r||_1 = ||M_r||_\infty$.*

Proof. For a r-circulant square matrix M_r, the maximum of absolute column sums is equal to the maximum of absolute row sums, therefore $||M_r||_1 = ||M_r||_\infty$. □

Theorem 8. *Let $k \neq 1$, then for the matrix A'_r, two types of norms $||A'_r||_1$ and $||A'_r||_\infty$ are given by*

$$||A'_r||_1 = ||A'_r||_\infty = \begin{cases} |r|\Big(\dfrac{2M_{k,n-1} - M_{k,n} + 1}{3(1-k)}\Big) & : |r| \geq 1, \\ \dfrac{2M_{k,n-1} - M_{k,n} + 1}{3(1-k)} & : |r| \leq 1. \end{cases}$$

Proof. When $|r| \geq 1$, considering the results of Lemma 1 and Corollary 5, we can write

$$\begin{aligned} ||A'_r||_1 = ||A'_r||_\infty &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |m_{ij}| \\ &= |M_{k,0}| + |r| \sum_{l=1}^{n-1} |M_{k,l}| \\ &= |r| \sum_{l=0}^{n-1} |M_{k,l}| \\ &= |r|\Big(\frac{2M_{k,n-1} - M_{k,n} + 1}{3(1-k)}\Big). \end{aligned}$$

When $|r| \leq 1$,

$$\begin{aligned} ||A'_r||_1 = ||A'_r||_\infty &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |m_{ij}| \\ &= \sum_{l=0}^{n-1} |M_{k,l}| \end{aligned}$$

$$= \frac{2M_{k,n-1} - M_{k,n} + 1}{3(1-k)}.$$

This completes the proof. □

Theorem 9. *Let $k \neq 1$, then norms $||B'_r||_1$ and $||B'_r||_\infty$ for matrix B'_r, are given by*

$$||B'_r||_1 = ||B'_r||_\infty = \begin{cases} 2(1-|r|) + |r|\Big(\dfrac{2R_{k,n-1} - R_{k,n} - 3k + 2}{3(1-k)}\Big) & : |r| \geq 1, \\ \dfrac{2R_{k,n-1} - R_{k,n} - 3k + 2}{3(1-k)} & : |r| \leq 1. \end{cases}$$

Proof. For $|r| \geq 1$, using the results of Lemma 1, we can write

$$\begin{aligned} ||B'_r||_1 = ||B'_r||_\infty &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |r_{ij}| \\ &= |R_{k,0}| + |r| \sum_{l=1}^{n-1} |R_{k,l}| \\ &= |R_{k,0}|(1-|r|) + |r| \sum_{l=0}^{n-1} |R_{k,l}|. \end{aligned}$$

Now using sum formula of Corollary 5 for the k-Mersenne–Lucas numbers, we have

$$||B'_r||_1 = ||B'_r||_\infty = 2(1-|r|) + |r|\Big(\frac{2R_{k,n-1} - R_{k,n} - 3k + 2}{3(1-k)}\Big).$$

For $|r| \leq 1$, we have

$$\begin{aligned} ||B'_r||_1 = ||B'_r||_\infty &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |r_{ij}| \\ &= \sum_{l=0}^{n-1} |R_{k,l}| \\ &= \frac{2R_{k,n-1} - R_{k,n} - 3k + 2}{3(1-k)}, \end{aligned}$$

as required. □

Theorem 10. *The Euclidean norm $||A'_r||_E$ is given by*

$$||A'_r||_E = \sqrt{\frac{1}{9k^2-8} \sum_{j=0}^{n-1} (R_{k,2j} - 2^{j+1})[n - j(1-|r|^2)]}.$$

Proof. From Eq. (1.6), we can write

$$\begin{aligned} ||A'_r||_E^2 &= \sum_{j=0}^{n-1} |M_{k,j}|^2 [n - j(1-|r|^2)] \\ &= \sum_{j=0}^{n-1} \Big(\frac{\beta_1^j - \beta_2^j}{\beta_1 - \beta_2}\Big)^2 [n - j(1-|r|^2)] \\ &= \frac{1}{(\beta_1 - \beta_2)^2} \sum_{j=0}^{n-1} (\beta_1^{2j} + \beta_2^{2j} - 2\beta_1^j \beta_2^j)[n - j(1-|r|^2)] \end{aligned}$$

$$= \frac{1}{9k^2-8}\sum_{j=0}^{n-1}(R_{k,2j} - 2^{j+1})[n - j(1-|r|^2)]$$

as required. □

Theorem 11. *The Euclidean norm of matrix B'_r is*

$$||B'_r||_E = \sqrt{\sum_{j=0}^{n-1}(R_{k,2j} + 2^{j+1})[n - j(1-|r|^2)]}.$$

Proof. The proof follows the same argument as of Theorem 10. □

To calculate bounds estimation for the spectral norm, we first need the sum of squares of the k-Mersenne and k-Mersenne–Lucas numbers, which are given in the following lemmas, respectively.

Lemma 7. *For $k \neq \pm 1$, we have*

$$\sum_{j=0}^{n-1} M_{k,j}^2 = \frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}.$$

Proof. Let, $S = \sum_{j=0}^{n-1} M_{k,j}^2$. Then we have

$$\begin{aligned}
S = \sum_{j=0}^{n-1} M_{k,j}^2 &= M_{k,0}^2 + \sum_{j=1}^{n-1} M_{k,j}^2 \\
&= \sum_{j=1}^{n-1}\Big(\frac{M_{k,j+1} + 2M_{k,j-1}}{3k}\Big)^2 \qquad \text{(using Definition 1)} \\
&= \frac{1}{9k^2}\Big[\sum_{j=1}^{n-1} M_{k,j+1}^2 + 4\sum_{j=1}^{n-1} M_{k,j-1}^2 + 4\sum_{j=1}^{n-1} M_{k,j+1}M_{k,j-1}\Big] \\
&= \frac{1}{9k^2}\Big[\sum_{j=1}^{n-1} M_{k,j+1}^2 + 4\sum_{j=1}^{n-1} M_{k,j-1}^2 + 4\sum_{j=1}^{n-1}(M_{k,j}^2 - 2^{j-1})\Big] \text{ using Eq. (1.3)} \\
&= \frac{1}{9k^2}\Big[S - M_{k,0}^2 - M_{k,1}^2 + M_{k,n}^2 + 4\Big(S - M_{k,n-1}^2\Big) + 4\Big(S - M_{k,0}^2 - (2^{n-1}-1)\Big)\Big] \\
\implies 9k^2S - 9S &= M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1} \\
\implies S &= \frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}.
\end{aligned}$$

This completes the proof. □

By the same argument of the above lemma, we have the following result which gives the sum of the squares of k-Mersenne–Lucas numbers.

Lemma 8. *For $k \neq \pm 1$, we have*

$$\sum_{j=0}^{n-1} R_{k,j}^2 = \frac{R_{k,n}^2 - 4R_{k,n-1}^2 + 2^{n+1}(9k^2-8) + (12-9k^2)}{9(k^2-1)}.$$

Proof. It is easy to prove the result by using the definition of k-Mersenne–Lucas sequence and result of Eq. (1.4), then proceeding as the above lemma. □

Theorem 12. *The lower and upper bounds for the spectral norm of the matrix A'_r are*

$$\begin{cases} \sqrt{\frac{M_{k,n}^2-4M_{k,n-1}^2+3-2^{n+1}}{9(k^2-1)}} \le ||A'_r||_2 \le \frac{|r|\sqrt{(M_{k,n}^2-4M_{k,n-1}^2+3-2^{n+1})+(M_{k,n}^2-4M_{k,n-1}^2+3-2^{n+1})^2}}{9(k^2-1)}, & |r| \ge 1 \\ |r|\sqrt{\Big(\frac{M_{k,n}^2-4M_{k,n-1}^2+3-2^{n+1}}{9(k^2-1)}\Big)} \le ||A'_r||_2 \le \sqrt{n\frac{M_{k,n}^2-4M_{k,n-1}^2+3-2^{n+1}}{9(k^2-1)}}, & |r| \ge 1. \end{cases}$$

Proof. From Eq. (1.6), we find

$$||A'_r||_E^2 = \sum_{j=0}^{n-1} |M_{k,j}|^2[n - j(1-|r|^2)].$$

For $|r| \ge 1$: using Lemma 7, we have

$$\begin{aligned} ||A'_r||_E^2 &= \sum_{j=0}^{n-1}(n-j)|M_{k,j}|^2 + |r|^2\sum_{j=0}^{n-1} j|M_{k,j}|^2 \\ &\ge \sum_{j=0}^{n-1}(n-j)|M_{k,j}|^2 + \sum_{j=0}^{n-1} j|M_{k,j}|^2 \\ &\ge n\Big(\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}\Big) \end{aligned}$$

Thus, we get

$$\frac{||A'_r||_E}{\sqrt{n}} \ge \sqrt{\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}.$$

And, using inequality given in Eq. (1.7), we obtain

$$||A'_r||_2 \ge \sqrt{\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}.$$

Let

$$S = \begin{bmatrix} M_{k,0} & 1 & 1 & \cdots & 1 \\ rM_{k,n-1} & M_{k,0} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rM_{k,2} & rM_{k,3} & rM_{k,4} & \cdots & 1 \\ rM_{k,1} & rM_{k,2} & rM_{k,3} & \cdots & M_{k,0} \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & M_{k,1} & M_{k,2} & \cdots & M_{k,n-1} \\ 1 & 1 & M_{k,1} & \cdots & M_{k,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & M_{k,1} \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then we can write the matrix A'_r as the Hadamard product of S and T. Also, we have

$$\begin{aligned} \mu(S) &= \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|s_{ij}|^2} = \sqrt{M_{k,0}^2 + |r|^2\sum_{j=1}^{n-1}M_{k,j}^2} \\ &= \sqrt{|r|^2\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}, \end{aligned}$$

and

$$\nu(T) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|t_{ij}|^2} = \sqrt{1 + \sum_{i=1}^{n-1}M_{k,i}^2} = \sqrt{1 + \frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}.$$

Hence from Lemma 2 we obtain

$$||A'_r||_2 \le \mu(S)\nu(T) = \sqrt{|r|^2\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}\sqrt{1 + \frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}.$$

When $|r| < 1$: using Eq. (1.6) and Lemma 7, we write

$$\begin{aligned}||A'_r||_E^2 &= \sum_{j=0}^{n-1}(n-j)|M_{k,j}|^2 + \sum_{j=0}^{n-1} j|r|^2|M_{k,j}|^2\\ &\ge \sum_{j=0}^{n-1}|r|^2(n-j)|M_{k,j}|^2 + \sum_{j=0}^{n-1} j|r|^2|M_{k,j}|^2\\ &\ge n|r|^2\Big(\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}\Big).\end{aligned}$$

Thus, we obtain

$$\frac{||A'_r||_E}{\sqrt{n}} \ge |r|\sqrt{\Big(\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}\Big)}.$$

And, on using inequality of Eq. (1.7), we obtain

$$||A'_r||_2 \ge |r|\sqrt{\Big(\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}\Big)}.$$

To calculate the upper bound for the spectral norm of A'_r, we first consider two matrices S', T' such that A'_r is the Hadamard product of these two matrices.

Let these two matrices be

$$S' = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1\\ r & 1 & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ r & r & r & \cdots & 1\\ r & r & r & \cdots & 1\end{bmatrix} \text{ and } T' = \begin{bmatrix} M_{k,0} & M_{k,1} & M_{k,2} & \cdots & M_{k,n-1}\\ M_{k,n-1} & M_{k,0} & M_{k,1} & \cdots & M_{k,n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ M_{k,2} & M_{k,3} & M_{k,4} & \cdots & M_{k,1}\\ M_{k,1} & M_{k,2} & M_{k,3} & \cdots & M_{k,0}\end{bmatrix}.$$

Then

$$\mu(S') = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|s'_{ij}|^2} = \sqrt{n}$$

and

$$\nu(T') = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|t'_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1}M_{k,i}^2} = \sqrt{\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}.$$

Thus, from Lemma 2 we obtain

$$||A'_r||_2 \le \mu(S')\nu(T') = \sqrt{n\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}},$$

and hence

$$|r|\sqrt{\Big(\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}\Big)} \le ||A'_r||_2 \le \sqrt{n\frac{M_{k,n}^2 - 4M_{k,n-1}^2 + 3 - 2^{n+1}}{9(k^2-1)}}.$$

Thus, this completes the proof. □

Theorem 13. *The lower and upper bounds for the spectral norm of the matrix B'_r are as follows.*

For $|r| \geq 1$,

- $\sqrt{\dfrac{R^2_{k,n} - 4R^2_{k,n_1} + 2^{n+1}(9k^2-8) + (12-9k^2)}{9(k^2-1)}} \leq ||B'_r||_2,$
- $||B'_r||_2 \leq \sqrt{|r|^2 \dfrac{R^2_{k,n} - 4R^2_{k,n_1} + 2^{n+1}(9k^2-8) + (12-9k^2)}{9(k^2-1)}} \sqrt{1 + \dfrac{R^2_{k,n} - 4R^2_{k,n_1} + 2^{n+1}(9k^2-8) + (12-9k^2)}{9(k^2-1)}},$

For $|r| < 1$,

- $|r| \sqrt{\dfrac{R^2_{k,n} - 4R^2_{k,n_1} + 2^{n+1}(9k^2-8) + (12-9k^2)}{9(k^2-1)}} \leq ||B'_r||_2,$
- $||B'_r||_2 \leq \sqrt{n \dfrac{R^2_{k,n} - 4R^2_{k,n_1} + 2^{n+1}(9k^2-8) + (12-9k^2)}{9(k^2-1)}},$

Proof. Considering the result of Lemma 8 and proceeding with the above theorem, we get the desired result. □

4. CONCLUSIONS

In this paper, we have defined and investigated the r-circulant matrices with entries from the k-Mersenne and k-Mersenne–Lucas sequences. Then we have investigated its algebraic properties like eigenvalues, trace, determinants, etc, and obtained some important identities like finite sum formula for these sequences having arithmetic indices. We have also investigated these properties for the right circulant and skew-right circulant matrices involving k-Mersenne and k-Mersenne–Lucas numbers. Moreover, we have found the norms $||.||_1$, $||.||_\infty$ and the Euclidean norm and give bounds estimation for the spectral norm for these r-circulant matrices.

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