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A New Approach to k–Jacobsthal Lucas Sequences

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Abstract

In this study, Catalan transformation $CS_{k,n}$ of k–Jacobsthal-Lucas sequences $S_{k,n}$ is defined. In addition, the transformation of $CS_{k,n}$ is written as the product of the Catalan matrix C which is the lower triangular matrix and the S_k matrix of type $n\ x\ 1$, and the Hankel transformations of some k–Jacobsthal-Lucas numbers is found.

Keywords: k–Pell sequences, k–Lucas sequence, k–Fibonacci sequence, Catalan Transform, Hankel Transform

1. INTRODUCTION

For any integer n, it is called a generalized Fibonacci-type sequence in the following form $G(n+1)=aG(n)+bG(n-1)$, $G(0)=m, G(1)=t$,where m,t,a and b are any complex numbers. There is an extensive work in the literature concerning Fibonacci-type sequences and their applications in modern science (see e.g.[1-8]). The known Jacobstahal-Lucas numbers have some applications in many branches of mathematics such as group theory, calculus, applied mathematics, linear algebra, etc. [9-12]. There exist generalizations of the Jacobsthal-Lucas numbers. This study is an extension of the papers [13-15].

In this paper, we put in for Catalan transform to the k–Jacobsthal-Lucas sequence and present application of the Catalan transform of the k–Jacobsthal-Lucas sequence. In section 2, we introduce some fundamental definitions of k–Jacobsthal-Lucas sequences and some basic theorems. In Theorem 2.1, we obtain Binet's formula of k–Jacobsthal-Lucas sequences and in Theorem 2.2, we give the relationship between positive and negative terms of k–Jacobsthal-Lucas numbers. In Theorem 2.3, we get Cassini identity for this sequence. In section 3, Catalan transform of k–Jacobsthal-Lucas sequence is given. Hankel transform of Catalan transformation of k–Jacobsthal-Lucas sequence is obtained in section 4.

2. k–JACOBSTHAL-LUCAS SEQUENCES

Let k be any positive real number. Then the k-Jacobsthal-Lucas sequences is defined

$S_{k,n+1} = S_{k,n} + 2k.S_{k,n-1}$ for $n \geq 1$ with the initial values $S_{k,0} = 2$ and $S_{k,1} = 1$.

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When k=1, the known Jacobsthal-Lucas sequences is obtained. Characteristic equation of the sequence is

$r^2-r-2k=0$.

Its characteristic roots are

$r_1 = \frac{1+\sqrt{1+8k}}{2}$

and

$r_2 = \frac{1-\sqrt{1+8k}}{2}$

Characteristic roots verify the properties

$r_1 + r_2 = 1, \qquad r_1.r_2 = -2k$

$r_1 - r_2 = \sqrt{1+8k}$

Binet's formula for $S_{k,n}$ is

$S_{k,n} = r_1^n + r_2^n$.

k−Jacobsthal-Lucas sequences as numbered;

$S_{k,n+1} = S_{k,n} + 2k.S_{k,n-1}$

$S_{k,0} = 2$

$S_{k,1} = 1$

$S_{k,2} = S_{k,1} + 2k.S_{k,0} = 4k+1$

$S_{k,3} = S_{k,2} + 2k.S_{k,1}$

$= 4k+1+2k.1$

$= 6k+1.$

$S_{k,4} = S_{k,3} + 2k.S_{k,2}$

$= 6k+1+2k.(4k+1)$

$= 8k^2 + 8k+1.$

$S_{k,5} = S_{k,4} + 2k.S_{k,3}$

$= 8k^2 + 8k + 1 + 2k.(6k+1)$

$= 20k^2 + 10k + 1.$

$S_{k,6} = S_{k,5} + 2k.S_{k,4}$

$= (20k^2 + 10k + 1) + 2k.(8k^2 + 8k+1)$

$= 16k^3 + 36k^2 + 12k + 1.$

$S_{k,7} = S_{k,6} + 2kS_{k,5}$

$= 16k^3 + 36k^2 + 12k + 1 + 2k.(20k^2 + 10k + 1)$

$= 56k^3 + 56k^2 + 14k + 1.$

Theorem 2.1. Binet's formula of k−Jacobsthal-Lucas sequences are obtained from the relations.

$S_{k,n} = r_1^n + r_2^n$

Proof.

The solutions of the characteristic equation are

$r^2 - r - 2k = 0$,

$r_1 = \frac{1+\sqrt{1+8k}}{2}$ and $r_2 = \frac{1-\sqrt{1+8k}}{2}$.

$S_{k,n} = c.r_1^n + d.r_2^n$

for n = 0, it is $S_{k,0}$= 2 and for n=1, it is $S_{k,1} = 1$. Thus c = 1 and d =1 are obtained. So, the proof is completed.

Theorem 2.2. For the k−Jacobsthal-Lucas numbers, the following identity holds for:

$$S_{k,n} = (-2k)^n.S_{k,-n} .$$

Proof. By virtue of Binet's formula, we find that

$S_{k,-n} = r_1^{-n} + r_2^{-n}$

$= \frac{1}{r_1^n} + \frac{1}{r_2^n}$

$= \frac{r_1^n + r_2^n}{(r_1.r_2)^n}$ $\quad ((r_1.r_2)^n = (-2k)^n)$

$= \frac{S_{k,n}}{(-2k)^n}$.

$S_{k,n} = (-2k)^n.S_{k,-n}$.

Theorem 2.3. (Cassini identity) For the k−Jacobsthal-Lucas numbers, the following equailty holds:

$$S_{k,n+1}.S_{k,n-1}- S_{k,n}^2= (-2k)^{n-1}.(8k+1)$$

Proof. By using the Binet's formula, we have $S_{k,n+1}.S_{k,n-1}-S_{k,n}^2=$

$(a^{n+1} + b^{n+1}).(a^{n-1} + b^{n-1})- (a^n + b^n)^2$

$= a^{2n} + a^{n+1}.b^{n-1} + a^{n-1}.b^{n+1} + b^{2n} - a^{2n} - 2a^n b^n - b^{2n}$

$= (ab)^n \frac{a}{b} + (ab)^n \frac{b}{a}$ -2.$(ab)^n$

$= (ab)^n[\frac{a}{b} + \frac{b}{a} - 2]$

$= (-2k)^n(\frac{4k+1}{-2k} - 2)$

$= (-2k)^{n-1}.(8k + 1).$

3. CATALAN NUMBERS

For n ≥0, the n^{th} Catalan number [13] is defined as follows

$C_n = \frac{1}{n+1}$(2n,n) or $C_n = \frac{(2n)!}{(n+1)!n!}$

Its generating function is given by

C(x) = $\frac{1-\sqrt{1-4x}}{2x}$.

The first Catalan numbers are {1,1,2,5,14,132,429,1430,4862, …}.

3.1. Catalan Transform of the k−Jacobsthal-Lucas sequences

We define the Catalan transform of the k−Jacobsthal-Lucas sequences { $S_{k,n}$} as

C$S_{k,n}$= $\sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} S_{k,i}$, $n \geq 1$ with

C$S_{k,0}$ = 0.

We can give the first few of Catalan transform of the first k−Jacobsthal-Lucas numbers. These are the polynomials in k:

$$CS_{k,1} = \sum_{i=0}^{1} \frac{i}{2-i}\binom{2-i}{1-i} S_{k,i}$$

$=1.S_{k,1} = 1,$

$$CS_{k,2} = \sum_{i=0}^{2} \frac{i}{4-i}\binom{4-i}{2-i} S_{k,i}$$

$= \frac{1}{3}\binom{3}{1} S_{k,1} + \binom{2}{0} S_{k,2}$

$= 4k +2.$

$$CS_{k,3} = \sum_{i=0}^{3} \frac{i}{6-i}\binom{6-i}{3-i} S_{k,i}$$

$= \frac{1}{5}\binom{5}{2} S_{k,1} + \frac{2}{4}\binom{4}{1} S_{k,2} + \frac{3}{3}\binom{3}{0} S_{k,3}$

$= \frac{1}{5} .10.1 + \frac{2}{4}.4.(4k + 1) + \frac{3}{3}.1.(6k + 1)$

$= 14k + 5.$

$$CS_{k,4} = \sum_{i=0}^{4} \frac{i}{8-i}\binom{8-i}{4-i} S_{k,i}$$

$= \frac{1}{7}\binom{7}{3} S_{k,1} + \frac{2}{6}\binom{6}{2} S_{k,2} + \frac{3}{5}\binom{5}{3} S_{k,3} + \frac{4}{4}\binom{4}{0} S_{k,4}$

$= \frac{1}{7}.35.1 + \frac{2}{6} .15 .(4k + 1) + \frac{3}{5} .5.(6k + 1) + \frac{4}{4}.1.(8k^2 + 8k + 1)$

$= 8k^2 + 46k + 14.$

$$CS_{k,5} = \sum_{i=0}^{5} \frac{i}{10-i}\binom{10-i}{5-i} S_{k,i}$$

$= \frac{1}{9}\binom{9}{4} S_{k,1} + \frac{2}{8}\binom{8}{3} S_{k,2} + \frac{3}{7}\binom{7}{2} S_{k,3} + \frac{4}{6}\binom{6}{1} S_{k,4} + \frac{5}{5}\binom{5}{0} S_{k,5}$

$= \frac{1}{9}.126.1 + \frac{2}{8}.56.(4k + 1) + \frac{3}{7}.21.(6k+1) + \frac{4}{6} .6.(8k^2 + 8k + 1) + \frac{5}{5}.1.(20k^2 + 10k + 1)$

$=52k^2+152k+42.$

$$CS_{k,6}=\sum_{i=0}^{6}\frac{i}{12-i}\binom{12-i}{6-i}S_{k,i}$$

$$=\frac{1}{11}\binom{11}{5}S_{k,1}+\frac{2}{10}\binom{10}{4}S_{k,2}+\frac{3}{9}\binom{9}{3}S_{k,3}+\frac{4}{8}\binom{8}{2}S_{k,4}+\frac{5}{7}\binom{7}{1}S_{k,5}+\frac{6}{6}\binom{6}{0}S_{k,6}$$

$=\frac{1}{11}.462.1+\frac{2}{10}.210.(4k+1)+\frac{3}{9}.84.(6k+1)+\frac{4}{8}.28.(8k^2+8k+1)+\frac{5}{7}.7.(20k^2+10k+1)+\frac{6}{6}.1.(16k^3+56k^2+14k+1)$

$=16k^3+268k^2+512k+132.$

We can show $\{S_{k,n}\}$ as the *n* x 1 matrix S_k and the product of the lower triangular matrix C as

$$\begin{bmatrix}CS_{k,1}\\CS_{k,2}\\CS_{k,3}\\CS_{k,4}\\\vdots\end{bmatrix}=\begin{bmatrix}1 & & & & \cdots\\1 & 1 & & & \cdots\\2 & 2 & 1 & & \cdots\\5 & 5 & 3 & 1 & \cdots\\\vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}S_{k,1}\\S_{k,2}\\S_{k,3}\\S_{k,4}\\\vdots\end{bmatrix}$$

So, we have

$$\begin{bmatrix}1\\4k+2\\14k+5\\8k^2+46k+14\\\vdots\end{bmatrix}$$

$$=\begin{bmatrix}1 & & & & \cdots\\1 & 1 & & & \cdots\\2 & 2 & 1 & & \cdots\\5 & 5 & 3 & 1 & \cdots\\\vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}1\\4k+1\\6k+1\\8k^2+8k+1\\\vdots\end{bmatrix}.$$

4. HANKEL DETERMINANT OF THE CATALAN k−JACOBSTHAL-LUCAS SEQUENCES

The Hankel matrix H of the integer sequence $A=\{a_0,a_1,a_2\ldots\}$ is the infinite matrix

$$H_n=\begin{vmatrix}a_0 & a_1 & a_2 & a_3 & \ldots\\a_1 & a_2 & a_3 & a_4 & \ldots\\a_2 & a_3 & a_4 & a_5 & \ldots\\a_3 & a_4 & a_5 & a_6 & \cdots\\\vdots & \vdots & \vdots & \vdots & \ddots\end{vmatrix}$$

with elements $h_{i,j}=a_{i+j-1}$. The Hankel matrix H_n of order *n* of A is the upper-left nxn submatrix of H, and h_n, the Hankel determinant of order n of A, is the determinant of the corresponding Hankel matrix of order *n*, h_n = det (H_n) [14,15].

In addition, by applying Hankel determinant return to the CS_k polynomials we obtain;

$HCS_1=Det[1]=1$

$$HCS_2=\begin{vmatrix}1 & 4k+2\\4k+2 & 6k+1\end{vmatrix}$$

$= 6k+1-(16k^2+16k+4)$

$= -16k^2-10k-3.$

$$HCS_3=\begin{vmatrix}1 & 4k+2 & 14k+5\\4k+2 & 14k+5 & 8k^2+16k+14\\14k+5 & 8k^2+16k+14 & 52k^2+152k+42\end{vmatrix}$$

$=-384k^4-3232k^3+2024k^2+2282k-255.$

5. CONCLUSION

We introduced Catalan transformation of k−Jacobsthal-Lucas sequences and Hankel determinant of the Catalan k−Jacobsthal-Lucas sequences.

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