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# Hyperbolic Functions and Balancing-like Sequences

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## Abstract

Motivated by the hyperbolic function representation of Fibonacci numbers, this paper develops analogous representations for balancing-like sequences. By expressing the sequence terms in terms of suitable hyperbolic functions, we derive a variety of new identities and structural relationships. This approach provides a unified analytical framework for the investigation of balancing-like sequences and yields new insights into their algebraic and structural properties.

**Keywords:** Balancing-like sequence; Lucas-balancing-like sequence; Hyperbolic function representation.

## 1. Introduction

The investigation and characterization of integer sequences constitute one of the fundamental areas of study in number theory. Within this framework, the Fibonacci sequence is considered one of the oldest and most extensively studied integer sequences [20] . Historically, the Fibonacci sequence arose from the well-known rabbit reproduction problem introduced by Fibonacci. The problem may be stated as follows [20]: A newborn pair of rabbits, consisting of one male and one female, is placed in an enclosed and safe environment. It is assumed that each pair reaches maturity after one month, that every mature pair produces a new pair of rabbits each month, and that no rabbits die. Under these assumptions, the question is to determine the total number of rabbit pairs after (n) months. The solution of this problem yields the sequence,

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765,…

which describes the number of rabbit pairs in successive months. It can be observed that, apart from the first two initial terms, each term is obtained as the sum of the two preceding terms. Hence, the number of rabbit pairs in any given month is equal to the sum of the numbers of rabbit pairs in the previous two months. The sequence obtained in this manner is known as the Fibonacci sequence, and its terms are called Fibonacci numbers [20].

Accordingly, the recurrence relation defining the Fibonacci sequence is given as follows [20] :

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2} \ (n \geq 2).$$

By using the recurrence relation, the corresponding Binet formula can be obtained and written as follows [20]:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

Here $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The connection between Fibonacci numbers and hyperbolic functions naturally follows from Binet's formula. However, the development of Fibonacci hyperbolic functions as an independent theoretical framework began with the work of Stakhov and Tkachenko, entitled "Hyperbolic Fibonacci Trigonometry," in 1993 [3]. This was later followed by Trzaska's detailed investigation of the Fibonacci hyperbolic sine and cosine functions in 1996 [21]. Within this framework, Fibonacci numbers are regarded as special values of certain continuous functions, in analogy with classical hyperbolic functions.

More recently, hyperbolic structures have also been investigated in connection with several generalized recurrence sequences and their higher-dimensional algebraic extensions. Özkan, Uysal, and Godase studied hyperbolic $k$-Jacobsthal and $k$-Jacobsthal-Lucas quaternions [8], while Özkan and Akkuş introduced hyperbolic $k$-Oresme and $k$-Oresme-Lucas quaternions [9]. Related hyperbolic quaternionic structures associated with generalized Bronze Fibonacci sequences were examined in [10]. Furthermore, hyperbolic $k$-Mersenne and $k$-Mersenne-Lucas octonions were investigated in [16]. These studies demonstrate that hyperbolic representations provide an effective framework for analyzing generalized recurrence sequences and their algebraic extensions.

Among the notable integer sequences studied in the literature, balancing numbers were introduced by Behera and Panda in 1999 [1]. The sequence of balancing numbers arose from the search for solutions to the following Diophantine equation [1].

$$1 + 2 + \cdots + (n-1) = (n+1) + \cdots + (n+r).$$

The recurrence relation for the balancing sequence is given as follows [11]:

$$B_0 = 0, B_1 = 1; \ B_{n+1} = 6B_n - B_{n-1} \ (n \geq 1).$$

Also the Binet formula for the balancing sequence is given as follows [18]:

$$B_n = \frac{\gamma_1^{2n} - \gamma_2^{2n}}{4\sqrt{2}}$$

Here $\gamma_1 = 1 + \sqrt{2}$ and $\gamma_2 = 1 - \sqrt{2}$.

Let $B_n$ denote the $n\ th$ balancing number. It is well known that $8B_n^2 + 1$ is always a perfect square [4]. The positive square root of this quantity is called the $n\ th$ Lucas balancing number [4].
The recurrence relation for the Lucas-balancing sequence is given as follows [4]:

$$C_0 = 1, C_1 = 3;\ C_{n+1} = 6C_n - C_{n-1}.$$

Also the Binet formula for Lucas-balancing sequence is given as follows:

$$C_n = \frac{\gamma_1^{2n} + \gamma_2^{2n}}{2}.$$

In 2004, the connection between balancing numbers and Fibonacci numbers was investigated, with particular emphasis on the existence of balancing numbers that are also Fibonacci numbers [13]. Lucas-balancing numbers, which are closely related to balancing numbers, were investigated in 2006 [14]. Uysal, Özkan, and Shannon introduced dual bicomplex balancing numbers [15]. Further extensions of balancing and Lucas-balancing numbers have also been considered in different algebraic settings. In particular, Uysal and Özkan introduced balancing and Lucas-balancing hybrid numbers and established several identities associated with these sequences [17].In 2018, Taşçı investigated Gaussian analogues of balancing and Lucas-balancing numbers, presenting their Binet formulas, generating functions, and various identities [6]. In 2012, Panda and Rout introduced balancing-like sequences and showed that these sequences possess properties analogous to those of balancing sequences [12]. In 2026, Özkan and Tekeoğlu Akkaya examined the properties of *k*-balancing-like sequences and defined *k*-Lucas-balancing-like sequences [7]. In addition, Gaussian extensions of balancing-like sequences have recently been studied from algebraic and geometric perspectives [5].

Let $A > 2$ be any positive integer; the balancing-like sequences $\{x_n\}$ are defined by the initial conditions $x_0 = 0, x_1 = 1$ and the recurrence relation $x_{n+1} = Ax_n - x_{n-1}$ [12]. The Binet formula is as follows [12]:

$$x_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

with $\lambda_1 = \frac{A+\sqrt{A^2-4}}{2}$ and $\lambda_2 = \frac{A-\sqrt{A^2-4}}{2}$.

Another significant integer sequence associated with balancing-like sequences is the Lucas balancing-like sequence. Let $A > 2$ be an arbitrary even integer, let $D = \frac{A^2-4}{4}$ and let $x_n$ denote the $n\ th$ balancing-like number. Then, the expression $Dx_n^2 + 1$ is a perfect square [4]. Its positive square root, denoted by $y_n = \sqrt{Dx_n^2 + 1}$ is called the $n\ th$ Lucas-balancing-like

number [4]. The recurrence relation for Lucas-balancing-like sequences are given as follows[4]:

$$y_0 = 1, y_1 = \frac{A}{2}\ ;\ y_{n+1} = Ay_n - y_{n-1}.$$

Also the Binet formula for the Lucas-balancing-like sequences are given as follows [4]:

$$y_n = \frac{\lambda_1^n + \lambda_2^n}{2}$$

The main contributions of this paper are summarized as follows. We establish explicit representations of balancing-like and Lucas-balancing-like sequences in terms of hyperbolic sine and cosine functions, thereby providing a unified analytical framework for these sequences. Based on these representations, we derive several identities, including addition and duplication formulas, and provide an alternative hyperbolic-function proof of the Cassini identity.

## 2. Hyperbolic Functions and Balancing-like Sequences

A close correspondence can be observed between formulas involving hyperbolic functions evaluated at the argument $z = \frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2}\right)$ and a balancing-like number $x_n$ or a Lucas-balancing-like number $y_n$. Throughout this section let $A$ be an even integer with $A > 2$.

By definition,

$$\sinh nz = \frac{1}{2}(e^{nz} - e^{-nz})$$

and

$$\cosh nz = \frac{1}{2}(e^{nz} + e^{-nz}).$$

Let $z = \frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2}\right)$,

$$\sinh nz = \frac{\sqrt{A^2 - 4}}{2}x_n$$

and

$$\cosh nz = y_n.$$

Consequently, many classical identities involving hyperbolic sine and cosine functions yield corresponding identities for balancing-like and Lucas-balancing-like sequences. We shall now give some examples.

$$\cosh nz + \sinh nz = \lambda_1^n$$

$$\cosh nz - \sinh nz = \lambda_2^n$$

is valid for $z = \frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2}\right)$.

Hence,

$$y_n - \frac{\sqrt{A^2-4}}{2}x_n = \lambda_2^n$$

and

$$y_n + \frac{\sqrt{A^2-4}}{2}x_n = \lambda_1^n.$$

**Theorem 2.1.** For $n, m \in \mathbb{Z}^+$ with $m \geq n$,

$$\frac{1}{2}(x_{m+n} + x_{m-n}) = x_m y_n$$

and

$$\frac{1}{2}(x_{m+n} - x_{m-n}) = x_n y_m.$$

**Proof.** Since

$$\sinh(m+n)z + \sinh(m-n)z = 2\sinh mz \cosh nz$$

and

$$\sinh(m+n)z - \sinh(m-n)z = 2\sinh nz \cosh mz$$

we get desired results.

**Theorem 2.2.** For $n, m \in \mathbb{Z}^+$ with $m \geq n$,

$$y_{m+n} + y_{m-n} = 2y_m y_n$$

$$y_{m+n} - y_{m-n} = \frac{A^2-4}{2}x_m x_n.$$

**Proof.** Since

$$\cosh(m+n)z + \cosh(m-n)z = 2\cosh mz \cosh nz$$

and

$$\cosh(m+n)z - \cosh(m-n)z = 2\sinh mz \sinh nz$$

we obtain the results.

A formula facilitating the computation of the even-indexed terms of Lucas-balancing-like sequences is presented in the following theorem.

**Theorem 2.3.** For $n \in \mathbb{Z}^+$,

$$y_{2n} = y_n^2 + \frac{A^2-4}{4} x_n^2.$$

**Proof.** Since cosh $2nz = cosh^2 nz + sinh^2 nz$ we obtain this.

The following theorem presents the Cassini identity for balancing-like sequences, a well-known identity in the literature, and provides an alternative proof based on hyperbolic functions.

**Theorem 2.4.** For $n \in \mathbb{Z}^+$,

$$x_{n+1}x_{n-1} - x_n^2 = -1.$$

**Proof.** Since $\lambda_1\lambda_2 = 1$,

$$\begin{aligned}\sinh(n+1)z \sinh(n-1)z - (\sinh nz)^2 &= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{2}\frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{2} - \left(\frac{\lambda_1^n - \lambda_2^n}{2}\right)^2 \\ &= \frac{-(\lambda_1^2+\lambda_2^2-2)}{4} \\ &= -\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 \\ &= (-1)\left(\frac{\sqrt{A^2-4}}{2}\right)^2.\end{aligned}$$

Since $\sinh nz = \frac{\sqrt{A^2-4}}{2} x_n$ we get,

$$x_{n+1}x_{n-1} - x_n^2 = -1.$$

## 3. Conclusion

In this paper, hyperbolic function representations for balancing-like and Lucas-balancing-like sequences were established by expressing the sequence terms in terms of hyperbolic sine and cosine functions. These representations provided a unified framework for deriving several identities, including addition formulas, duplication formulas, and an alternative proof of the Cassini identity.

The hyperbolic framework developed here may also be extended to generalized, polynomial, and Gaussian versions of balancing-like sequences, providing a possible basis for deriving further identities and asymptotic relations.

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