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3-Step Fibonacci sequences in nilpotent groups

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Abstract

In this paper, we have constituted 3-step Fibonacci sequences by the three generating elements of a group of exponent p (p is prime) and nilpotency class n.
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1. Introduction

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Takahashi gives a fast algorithm which is based on the product of Lucas number to compute large Fibonacci numbers [1]. Fibonacci sequences have been an interesting subject in applied mathematics. West has shown using transfer matrices that the number $|S_n(123, 3214)|$ of permutations avoiding the patterns 123 and 3214 is the Fibonacci number F_{2n} [2].

The study of Fibonacci sequences in groups began with the earlier work of Wall [3] where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid eighties Wilcox extended the problem to abelian groups [4]. Prolific co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups [5]. Aydın and Smith proved in [6] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. The theory has been generalized in [7] to the

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ordinary 3-step Fibonacci sequences in finite nilpotent groups of nilpotency class 2 and exponent p. Then it is shown in [8] that the period of 2-step general Fibonacci sequence is equal to the length of fundamental period of the 2-step general recurrence constructed by two generating elements. One of the latest works in this area is [9] where it is shown that for the 3-step general Fibonacci recurrence and any finite p-group of exponent p and nilpotency class 2, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step general Fibonacci sequence in the field $\mathrm{GF}(p)$.

Now, let $s = (s_i)$ denote the ordinary 3-step Fibonacci sequence in $\mathrm{GF}(p)$ defined by the recurrence $s_{i+3} = s_i + s_{i+1} + s_{i+2}$ and the initial data $s_0 = 0$, $s_1 = 0$ and $s_2 = 1$. This is a bi-infinite periodic sequence or loop indexed by the integers. The shortest period of this sequence is called the fundamental period and it will be denoted by k. Let G be finite group. We define a family of recurrences, parameterized by natural numbers n, via equations

$$x_i = x_{i-1}x_{i-2}\cdots x_{i-n} \quad (R_n).$$

We fix $n > 1$. We may 'start' the recurrence with any inital data $g_0, g_1, \ldots, g_{n-1}$ and then the recurrence will define a periodic bi-infinite sequence (g_i) indexed by the integers. We use the loop to describe a bi-infinite sequence satisfying the recurrence. Notice that, if $\alpha \in \mathbb{Z}$, then putting $h_i = g_{i+\alpha}$ for $0 \leqslant i < n$ gives another set of initial data. The recurrence R_n then gives us another loop (h_i) of G. Notice that (h_i) is just (g_i) shifted through α steps. We say that (h_i) is a rotation of (g_i).

The fundamental period of a loop is clearly a significant number, and sometimes is called the Wall's number. Following Wall [3] the letter k is used to describe this period. Wall investigated the recurrence R_3 in finite cyclic groups. This subject immediately reduces to the study of cyclic groups of prime power order. This paradigm is exploited elsewhere [6,10] to study recurrences in finite nilpotent groups, since they are the cartesian product of their Sylow-subgroups. Another early contributor to this field was Vinson [11], who was particularly interested in ranks of apparition, mistranslated from the French. More interest has come in the form of [5,12–14] and [4].

Let us consider the more general recurrences written multiplicatively as

$$x_i = x_{i-1}^{a_1}x_{i-2}^{a_2}\cdots x_{i-n}^{a_n}$$

or additively as

$$x_i = \sum_{r=1}^{n} a_r x_{i-r} \quad (A_n).$$

The exponents (coefficients) are deemed to be integers. Such a recurrence may not be so well behaved as R_n, for it may be singular in particular group. This

means that initial data may not define a unique bi-infinite periodic sequence; the problem arises when attempting to extend the sequence through negative integer subscripts. Certainly all is well if $a_n \in \{1, -1\}$. Equally well, if we are working in a p-group all is well provided $\mathrm{lcm}(p, a_n) = 1$. We now press on to discuss the recurrences R_3. We work in the additive group H of the a finite field $\mathrm{GF}(p^n)$. Let the standard loop s begin (0,0,1), and let its period be $k_3(p)$, written as k in this paper. As before, the period of any loop must divide k for the usual reasons.

2. Three step recurrences

The theory of general 3-step recurrences is very well understood. See [15], and Chapter 2 of [16]. The following result of this section is well-known [10,3].

Theorem 2.1. *Short loop must be geometric for the* 3*-step Fibonacci recurrence in* H*, the additive group of the finite field* $\mathrm{GF}(p^n)$ [16].

The situation here is more complex than in the two step case; it is possible for all three roots of the $x^3 - x^2 - x - 1$ to give rise to genuine short loops. This kind of loops are called golden loops. The first few instances of this phenomenon are given in [16] as follows:

Prime	*Golden Loop Uzunluğu*	*Wall number*
4481	2240	4480
	640	
	896	
4621	84	4620
	660	
	2310	

Definition 2.2. F/R is called to be relatively free group if F is a free group and R is a fully invariant subgroup of F.

Definition 2.3. Let $H \triangleleft G, K \triangleleft G$ and $K \leqslant H$. If H/K is contained in the center of G/K, then H/K is called a central factor of G. A group G is called nilpotent if it has finite series of normal subgroups

$$G = G_0 \geqslant G_1 \geqslant \cdots G_r = 1,$$

such that G_{i-1}/G_i is a central factor of G for each $i = 1, 2, \ldots, r$. The smallest possible r is called the nilpotency class of G.

Consider a group H with the following presentation: $H =< h_1, h_2, h_3, h_4 : (h_2, h_1) = h_3, (h_3, h_1) = h_4, \text{explaw} = p >$ (p is a prime number) where pairs of generators with unspecified commutator are implicitly deemed to commute. H has two generators x and y. Thus H is a copy of C_p^3 extended by a cyclic group of order p.

Let G be the 3-generator relatively free exponent class 3 group on g_1, g_2 and g_3.

Thus G has order p^{14} and a power commutator presentation of G is given by

$$(g_2, g_1) = g_4, \quad (g_3, g_1) = g_5, \quad (g_3, g_2) = g_6, \quad (g_4, g_1) = g_7,$$
$$(g_4, g_2) = g_8, \quad (g_4, g_3) = g_9, \quad (g_5, g_1) = g_{10}, \quad (g_5, g_2) = g_{11},$$
$$(g_5, g_3) = g_{12}, \quad (g_6, g_1) = g_9^{-1} g_{11}, \quad (g_6, g_2) = g_{13}, \quad (g_6, g_3) = g_{14}.$$

Once again we have the convention that pairs of generators with unspecified commutator are implicitly deemed to commute.

In GF(p)-vector notation, we put $g_i = (\delta_{ij}) \in G$, where δ_{ij} in the Krönecker symbol, and j ranges from 1 to 14.

The group G is relatively free and so admits an automorphism f, which we call the 3-step Fibonacci automorphism, defined by $g_1 f = g_2, \quad g_2 f = g_3, \quad g_3 f = g_1 g_2 g_3$.

We define two maps $\Pi_i : G \to H$ via $(i = 1, 2)$

$$g_1 \Pi_1 = 1, \quad g_2 \Pi_1 = h_1 \quad \text{and} \quad g_3 \Pi_1 = h_2$$

and

$$g_1 \Pi_2 = h_1, \quad g_2 \Pi_2 = h_2 \quad \text{and} \quad g_3 \Pi_2 = 1.$$

Let $g_i = (\delta_{ij}) \in G$, where δ_{ij} in the Krönecker symbol, and j ranges from 1 to 14.

$$\text{Ker}\Pi_1 = (*, 0, 0, *, *, 0, *, *, *, *, *, *, 0, *),$$

$$\text{Ker}\Pi_2 = (0, 0, *, 0, *, *, 0, *, *, *, *, *, *, *).$$

Let $M = \text{Ker}\Pi_1 \cap \text{Ker}\Pi_2$, so that

$$M = (0, 0, 0, 0, *, 0, 0, *, *, *, *, *, 0, *),$$

in the sense that each $*$ can independently be any element of GF(p). Now M is an elementary abelian group of order p^7 and is therefore a GF(p)-space of dimension 7.

A basis of M is $(g_5, g_8, g_9, g_{10}, g_{11}, g_{12}, g_{14})$.

Computer aided calculations [13] yield that

$$M \cap Mf = (g_8 g_{11}, g_9, g_{10} g_{11} g_{12}, g_{14}),$$
$$M \cap Mf \cap Mf^2 = (g_9 (g_{14})^{-2}, g_{10} g_{11} g_{12})$$

and

$$M \cap Mf \cap Mf^2 \cap Mf^3 = 1.$$

Thus we have a monomorphism

$$s : G \to G/K_1xG/K_2xG/K_1fxG/K_2fxG/K_1f^2xG/K_2f^2xG/K_1f^3xG/K_2f^3,$$

where the codomain is isomorfic to $x_{i=1}^{8}H$ where $K_1 = \mathrm{Ker}(\Pi_1)$ and $K_2 = \mathrm{Ker}(\Pi_2)$ The automorphism f and its powers induce isomorphisms $G/K_if^j \to G/K_i$ which can be composed co-ordinatewise with Π to form a group monomorphism

$$\overline{\Pi} : G \to x_{j=1}^{4}(G/K_1xG/K_2),$$

defined by

$$x \to (K_1x, K_2x, K_1(xf^{-1})), \quad K_2(xf^{-1}), \quad K_1(xf^{-2}),$$
$$K_2(xf^{-2}), \quad K_1(xf^{-3}), \quad K_2(xf^{-3}).$$

Let us examine the image of the loop $r = (r_i)$ beginning $r_0 = g_1, r_1 = g_2, r_2 = g_3$ under $\overline{\Pi}$. We have

$$\overline{\Pi} : r_i \to (r_i\Pi_1, r_i\Pi_2, r_{i-1}\Pi_1, r_{i-1}\Pi_2, r_{i-2}\Pi_1, r_{i-2}\Pi_2, r_{i-3}\Pi_1, r_{i-3}\Pi_2).$$

The sequence in the odd positions are just rotations of $(r_i\Pi_1)$ and the sequences in the even positions are rotations of $(r_i\Pi_2)$. Thus, if we can show that $(r_i\Pi_1)$ and $(r_i\Pi_2)$ both have Wall number k, it will follow that r has Wall number k.

Let H be a nilpotent group with nilpotency class 3 and exponent p for a prime number $p > 3$. H has two generators x and y. A presentation of H is

$$H = \langle x, y, z, t : (y, x) = z, (z, x) = t\rangle,$$

where pairs of generators with unspecified commutator are implicitly deemed to commute. The subgroup $\langle y, z, t\rangle$ is abelian and v is in the centre of H. Every element of H has a unique representation as

$$x^a y^b z^c t^d,$$

where the exponents are elements of $\mathrm{GF}(p)$. Having distinguished this way of writing elements we can even think of groups elements as vectors of dimension 4 over $\mathrm{GF}(p)$, i.e. as (a, b, c, d). So triple multiplication is determined by the following rules;

That is, $(a_0, b_0, c_0, d_0)(a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (a_3, b_3, c_3, d_3)$.

As to knowledge in above we can give the following lemma.

Lemma 2.4. *If G is the group of exponent p, nilpotency class 3 and with the presentation*

$$G = \langle x, y, z, t : [y, x] = z, [z, x] = t \rangle,$$

then according to the group law $(a_0, b_0, c_0, d_0)(a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (a_3, b_3, c_3, d_3)$, *the entries of the* 3*-step Fibonacci sequence constructed by three elements of G has the following form*

$$\begin{aligned}
a_3 &= a_0 + a_1 + a_2 \\
b_3 &= b_0 + b_1 + b_2 \\
c_3 &= c_0 + c_1 + c_2 + b_0 a_1 + (b_0 + b_1) a_2 \\
d_3 &= d_0 + d_1 + d_2 + c_0 a_1 + b_0 \binom{a_1}{2} + a_2 (c_0 + c_1 + b_0 a_1) + \binom{a_2}{2} (b_0 + b_1)
\end{aligned}$$

and finally

$$d_3 = d_0 + d_1 + d_2 + c_0 a_1 + b_0 \binom{a_1}{2} + a_2 (c_0 + c_1 + b_0 a_1) + \binom{a_2}{2} (b_0 + b_1)$$

[17].

3. Main result

Now we want to generalize the formulas in the above lemma for a group of exponent p and nilpotency class n.

Theorem 3.1. *If G is a group of exponent p, nilpotency class n and with the presentation*

$$G = \langle x_1, x_2, \ldots, x_n, x_{n+1} : [x_2, x_1] = x_3, [x_3, x_1] = x_4, \ldots, [x_n, x_1] = x_{n+1} \rangle,$$

then entries of the 3*-step Fibonacci sequences formed by three elements of G have the following form*:

$$\begin{aligned}
d_1 &= a_1 + b_1 + c_1, \\
d_2 &= a_2 + b_2 + c_2, \\
d_3 &= a_3 + b_3 + c_3 + b_1 a_2 + (a_2 + b_2) c_1, \\
d_4 &= a_4 + b_4 + c_4 + b_1 a_3 + \binom{b_1}{2} a_2 + \binom{c_1}{1} (a_3 + b_3 + b_1 a_2) \\
&\quad + (a_2 + b_2) \binom{c_1}{2},
\end{aligned}$$

$$d_5 = a_5 + b_5 + c_5 + b_1a_4 + \binom{b_1}{2}a_3 + \binom{b_1}{3}a_2 + \binom{c_1}{1}\left(a_4 + b_4 + b_1a_3 + a_2\binom{b_1}{2}\right) + \binom{c_1}{2}(a_3 + b_3 + b_1a_2) + \binom{c_1}{3}(a_2 + b_2)$$

$$d_6 = a_6 + b_6 + c_6 + b_1a_5 + \binom{b_1}{2}a_4 + \binom{b_1}{3}a_3 + \binom{b_1}{4}a_2 + \binom{c_1}{1} \times \left(a_5 + b_5 + b_1a_4 + \binom{b_1}{2}a_3 + \binom{b_1}{3}a_2\right) + \binom{c_1}{2}\left(a_4 + b_4 + b_1a_3 + a_2\binom{b_1}{2}\right) + \binom{c_1}{3}(a_3 + b_3 + b_1a_2) + \binom{c_1}{4}(a_2 + b_2)$$

$$\vdots$$

$$d_{n+1} = a_{n+1} + b_{n+1} + c_{n+1} + \binom{b_1}{1}a_n + \binom{b_1}{2}a_{n-1} + \binom{b_1}{3}a_{n-2} + \cdots + \binom{b_1}{n-1}a_2 + \binom{c_1}{1}\left(a_n + b_n + b_1a_{n-1} + \binom{b_1}{2}a_{n-2} + \cdots + \binom{b_1}{n-2}a_2\right) + \binom{c_1}{2}\left(a_{n-1} + b_{n-1} + b_1a_{n-2} + \binom{b_1}{2}a_{n-3} + \cdots + \binom{b_1}{n-3}a_2\right) + \cdots + \binom{c_1}{n-2}\left(a_3 + b_3 + \binom{b_1}{1}a_2\right) + \binom{c_1}{n-1}(a_2 + b_2).$$

Proof. Let G be the group which has the following presentation

$$G = \langle x_1, x_2, \ldots, x_n, x_{n+1} : [x_2, x_1] = x_3, [x_3, x_1] = x_4, \ldots, [x_n, x_1] = x_{n+1} \rangle,$$

where those pairs of generators with unspecified commutators are implicitly deemed to commute. This is a group of nilpotency class n and exponent p. Each element of the group will have a unique representation as $x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n+1}^{a_{n+1}}$ where $a_1, a_2, a_3, \cdots a_{n+1} \in Z/pZ$. The element $x_{n+1}^{a_{n+1}}$ is central and the group $H = \langle x_2, x_3, \cdots, x_n, x_{n+1} \rangle$ is abelian. Let $x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n+1}^{a_{n+1}}$, $x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots x_{n+1}^{b_{n+1}}$ and $x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}}$ be elements of G . Then, suppose that

$$(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n+1}^{a_{n+1}})(x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots x_{n+1}^{b_{n+1}})(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}}) = (x_1^{d_1}x_2^{d_2}x_3^{d_3}\cdots x_{n+1}^{d_{n+1}})$$

Under the group law, we can calculate the entries of 3-step Fibonacci sequences as follows:

$$
\begin{aligned}
&[(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n+1}^{a_{n+1}})(x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots x_{n+1}^{b_{n+1}})](x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_n^{a_n}x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots x_{n+1}^{a_{n+1}+b_{n+1}})(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}}x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}-1}x_{n-1}x_1x_1^{b_1-1}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}-1}x_1x_{n-1}x_nx_1^{b_1-1}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}-1}x_1x_{n-1}x_nx_{n+1}^{b_1-1}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}-1}x_1x_{n-1}x_1x_1^{b_1-2}x_nx_{n+1}^{b_1-1}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}-1}x_1^2x_{n-1}x_nx_1^{b_1-2}x_nx_{n+1}^{b_1-1}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}-1}x_1^2x_{n-1}x_1^{b_1-2}x_n^2x_{n+1}^{b_1-1+b_1-2}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\qquad\vdots\\
&\quad=(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-1}^{a_{n-1}-1}x_1^{b_1}x_{n-1}x_n^{b_1}x_{n+1}^{b_1-1+\cdots+1}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1})\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\qquad\vdots\\
&\quad=\left(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-2}^{a_{n-2}}x_1^{b_1}x_{n-1}^{a_{n-1}}x_n^{b_1a_{n-1}}x_{n+1}^{\binom{b_1}{2}a_{n-1}}x_2^{b_2}x_3^{b_3}\cdots x_n^{b_n+a_n}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1}\right)\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}})\\
&\quad=\left(x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{n-2}^{a_{n-2}}x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots x_{n-1}^{a_{n-1}+b_{n-1}}x_n^{b_n+a_n+b_1a_{n-1}}x_{n+1}^{a_{n+1}+b_{n+1}+a_nb_1+\binom{b_1}{2}a_{n-1}}\right)\\
&\qquad\times(x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots x_{n+1}^{c_{n+1}}).
\end{aligned}
$$

We can do the same operations between $x_{n-2}^{a_{n-2}}$ and $x_1^{b_1}$. Then the last equation is

$$= \left(x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-3}^{a_{n-3}} x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots x_{n-2}^{a_{n-2}+b_{n-2}} x_{n-1}^{b_{n-1}+a_{n-1}+b_1 a_{n-2}} x_n^{a_n+b_n+a_{n-1}b_1+\binom{b_1}{2}^{a_{n-1}}} \right.$$

$$\left. \times x_{n+1}^{a_{n+1}+b_{n+1}+a_n b_1+\binom{b_1}{2}a_{n-1}+\binom{b_1}{3}a_{n-2}} \right) (x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots x_{n+1}^{c_{n+1}}).$$

So, we can do the same operations between $x_{n-3}^{a_{n-3}}, x_{n-4}^{a_{n-4}}, \ldots, x_2^{a_2}$ and $x_1^{b_1}$. Then last equation is

$$= \left(x_1^{a_1+b_1} x_2^{a_2+b_2} x_3^{a_3+b_3+b_1 a_2} \cdots x_n^{a_n+b_n+a_{n-1}b_1+\binom{b_1}{2}a_{n-1}+\cdots+\binom{b_1}{n-2}a_2} \right.$$

$$\left. \times x_{n+1}^{a_{n+1}+b_{n+1}+a_n b_1+\binom{b_1}{2}a_{n-1}+\binom{b_1}{3}a_{n-2}+\cdots+\binom{b_1}{n-1}a_2} \right) (x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots x_{n+1}^{c_{n+1}}).$$

So we can also the same operations between

$$\left(x_1^{a_1+b_1} x_2^{a_2+b_2} x_3^{a_3+b_3+b_1 a_2} \cdots x_n^{a_n+b_n+a_{n-1}b_1+\binom{b_1}{2}a_{n-1}+\cdots+\binom{b_1}{n-2}a_2} \right.$$

$$\left. \times x_{n+1}^{a_{n+1}+b_{n+1}+a_n b_1+\binom{b_1}{2}a_{n-1}+\binom{b_1}{3}a_{n-2}+\cdots+\binom{b_1}{n-1}a_2} \right)$$

and

$$(x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots x_{n+1}^{c_{n+1}}).$$

Thus the last equation is

$$\left(x_1^{a_1+b_1+c_1}x_2^{a_2+b_2+c_2}x_3^{a_3+b_3+c_3+b_1a_2+(a_2+b_2)c_1}\cdots\right.$$

$$\times x_{n+1}^{a_{n+1}+b_{n+1}+c_{n+1}+\binom{b_1}{1}a_n+\binom{b_1}{2}a_{n-1}+\binom{b_1}{3}a_{n-2}+\cdots+\binom{b_1}{n-1}a_2}$$

$$\times x_{n+1}^{\binom{c_1}{1}\left(a_n+b_n+\binom{b_1}{1}a_{n-1}+\binom{b_1}{2}a_{n-2}+\cdots+a_2\binom{b_1}{n-2}\right)}$$

$$\times x_{n+1}^{+\binom{c_1}{2}\left(a_{n-1}+b_{n-1}+\binom{b_1}{1}a_{n-2}+\binom{b_1}{2}a_{n-3}+\cdots+a_2\binom{b_1}{n-3}\cdots\binom{c_1}{n-2}\right)\left(a_3+b_3+\binom{b_1}{1}a_2\right)}$$

$$\left.x_{n+1}^{+\binom{c_1}{n-1}(a_2+b_2)}\right).$$

We are done. □

Now we use these formulas to calculate the Fibonacci loop $l(x,y,z,\ldots)$. For example let $n=4$. The first few terms are as follows (reduce modulo p of your choice): (1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (1,1,1,0,0), (1,2,3,1,0), (2,3,6,4,1), (4,6,17,19,15), (7,11,50,114,119).

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