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ON A GENERALIZATION OF THE PERFECT SQUARE SEQUENCE AND ITS POLYNOMIAL

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Summary. In this study, we introduce a new recurrence relation of the perfect square sequence. We establish the relationship of perfect square sequence concerned with Fibonacci and Lucas sequences. We compute some important identities such as Catalan, Cassini, and special summation formula for this sequence. We associate the perfect square sequence with Lucas sequence and we call it the perfect square Lucas sequence. In addition, the Binet formula, generating function and summation formula of this sequence is obtained, as well as some properties are satisfied. Furthermore, we present the relationship of perfect square Lucas sequence with Fibonacci and Lucas sequences. Also, we obtain the relationship of perfect square Lucas sequence with perfect square sequence and we present matrix representation of this sequence. Besides, we described polynomials of perfect square and perfect square Lucas sequences. We get Binet formulas, generating functions, and Simpson formula for these polynomials. Eventually, satisfied some intriguing relations between these two polynomials, as well as we give the matrix representation of them.

1 INTRODUCTION

The most famous sequences of numbers are Fibonacci and Lucas sequences. For a long time, they have intrigued so many scientists. In various fields, Fibonacci sequences have been applied such as graph theory [1], Biomathematics [2], Chemistry [3], Engineering [4], etc. Many generalizations of the Fibonacci sequence have been done. The known examples of such sequences are the k-Jacobsthal, k-Oresme-Lucas, Leonardo, Gaussian Fibonacci, Mersenne sequences, etc, (see for details in [5-9]).

For $n \in \mathbb{N}$, Fibonacci numbers F_n, Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ with } F_0 = 1 \text{ and } F_1 = 1,$$

$$L_{n+2} = L_{n+1} + L_n, \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

The generating functions of Fibonacci and Lucas sequences are as follows, respectively.

$$f(z) = \sum_{n=0}^{\infty} F_n z^n = \frac{1}{1-z-z^2} \text{ and } l(z) = \sum_{n=0}^{\infty} L_n z^n = \frac{2-z}{1-z-z^2}$$

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Many number sequences were described, generalized, and examined by scientists; they found interesting relations among them. For more details [10-17].
The authors defined Mersenne Lucas numbers and examined its properties related with Mersenne numbers. They gave the recurrence relation between Mersenne and Mersenne Lucas numbers [18].

It was studied in [19], Pell Leonardo numbers and the author defined the relationship of Leonardo numbers with Pell numbers, as well as properties and matrix formula of these numbers are introduced.

Moreover, in [20], they examined Jacobsthal-Lucas numbers by matrix method. They defined E and R matrices and observed relations with Jacobsthal and Jacobsthal-Lucas numbers.

Here we will analyze the most useful numbers in mathematics and in real-life, based on studies [18-19]. Curiosity of perfect square numbers comes from its balanced, neat, and intriguing properties, which is geometrically called the mathematical equivalent of square shape. In various fields, they have applications. This essential concept encounters us in real-life situations. Such as in technology, carpentry, engineering, and designing buildings. Even more interesting, they come across at known and widely played games in a real-life such that Minecraft, Rubik's Cube, and Sudoku.

The perfect square sequence is denoted by $\{s_n\}_{n=0}^{\infty}$. Some terms of perfect square sequence $\{0, 1, 4, 9, 16, 25, \dots\}$.

2 SOME PROPERTIES OF PERFECT SQUARE NUMBERS

To indicate a perfect square number sequence, we are familiar with this linear recurrence relation below which is first order and nonhomogeneous

$$s_n = s_{n-1} + 2n - 1, \ (s_0 = 0, n \geq 1)$$

Binet Formula of perfect square sequence is $s_n = n^2$ where $n \geq 0$. The generating function of the perfect square sequence is

$$c(z) = \sum_{n=0}^{\infty} s_n z^n = \frac{z+z^2}{(1-z)^3}$$

Next, we define new recurrence relation for the perfect square sequence.

Corollary 2.1. Perfect square sequence can be defined as the first order nonhomogeneous and non-constant coefficient recurrence relation as follows:

$$s_n = \begin{cases} 0, \ n = 0 \\ 1, \ n = 1 \\ n + \frac{n s_{n-1}}{n-1}, n \geq 2 \end{cases}$$

Using perfect square recurrence relation and Binet formula of perfect square sequence, we can easily establish this result. In the following theorem, we express the relation of perfect square sequence with Fibonacci and Lucas sequences.

Theorem 2.1. Let $\alpha = \frac{1-\sqrt{5}}{2}$, $\beta = \frac{1+\sqrt{5}}{2}$ and $A_{i,n} = \left(\frac{(1+n-i)(2+n-i)}{2} - s_{n-i}\right)$,

The following equation among F_i , L_i *and* s_n is satisfied.

$$s_n = 2\sum_{i=0}^{n} F_i A_{i,n} - \sum_{i=0}^{n} L_i\, A_{i,n} + \sum_{i=1}^{n} \left(\frac{\alpha^{\mathrm{i}}\beta - \alpha\beta^{\mathrm{i}}}{\mathrm{B}-\alpha}\right) A_{i,n} \quad , n > 0$$

$$s_n = 2\sum_{i=0}^{n} F_i A_{i,n} - \sum_{i=0}^{n} L_i\, A_{i,n}\, , n = 0$$

Proof. Here we will use generating functions of Fibonacci, Lucas and perfect square sequence, and some properties of the sum symbol.

Note that,

$$\frac{z^2}{1-z-z^2} = \sum_{n=-\infty}^{\infty} \left(\begin{cases} \frac{\alpha^n\beta - \alpha\beta^n}{\beta-\alpha}, & n > 0 \\ 0 \quad , & otherwise \end{cases} \right) z^n, \ \alpha = \frac{1-\sqrt{5}}{2}, \beta = \frac{1+\sqrt{5}}{2}$$

$$\sum_{n=0}^{\infty} L_n z^n = \frac{2-z}{1-z-z^2} = \frac{z+z^2}{1-3z+3z^2-z^3}\frac{1-3z+3z^2-z^3}{1-z-z^2} + \frac{2-2z-z^2}{1-z-z^2}$$

$$= \frac{\sum_{n=0}^{\infty} s_n z^n}{\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} z^n - \sum_{n=0}^{\infty} s_n z^n} - 2\sum_{n=0}^{\infty} F_n z^n + 2\sum_{n=0}^{\infty} L_n z^n - \sum_{n=1}^{\infty} \frac{\alpha^n\beta - \alpha\beta^n}{\beta-\alpha} z^n$$

$$\sum_{n=0}^{\infty} s_n z^n = \left(2\sum_{n=0}^{\infty} F_n z^n - \sum_{n=0}^{\infty} L_n z^n + \sum_{n=1}^{\infty} \frac{\alpha^n\beta - \alpha\beta^n}{\beta-\alpha} z^n\right) \sum_{n=0}^{\infty} \left(\frac{(n+1)(n+2)}{2} - s_n\right) z^n$$

$$= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n} F_i\, A_{i,n} z^n - \sum_{n=0}^{\infty}\sum_{i=0}^{n} L_i\, A_{i,n} z^n + \sum_{n=0}^{\infty}\sum_{i=1}^{n} \left(\frac{\alpha^i\beta - \alpha\beta^i}{\beta-\alpha}\right) A_{i,n} z^n \quad .$$

Thus we obtain the desired result,

$$s_n = 2\sum_{i=0}^{n} F_i A_{i,n} - \sum_{i=0}^{n} L_i\, A_{i,n} + \sum_{i=1}^{n} \left(\frac{\alpha^{\mathrm{i}}\beta - \alpha\beta^{\mathrm{i}}}{\mathrm{B}-\alpha}\right) A_{i,n} \quad , n > 0 \qquad \square$$

Proposition 2.1. For $k \geq 1$ and $j \geq 0$,

$$\sum_{i=0}^{n} s_{ki+j} = s_n\left(\frac{nk^2}{3} + \frac{k^2}{2} + kj\right) + n\left(\frac{k^2}{6} + kj + j^2\right) + j^2$$

Proof. Using Binet formula of perfect square sequence and the partial sum formula, we get

$$\sum_{i=0}^{n} s_{ki+j} = \sum_{i=0}^{n} (ki+j)^2 = \sum_{i=0}^{n} (ki)^2 + 2kj\sum_{i=0}^{n} i + \sum_{i=0}^{n} j^2$$

After substituting special partial sum and Binet formula for perfect square number sequence on the equation, we obtain,

$$\sum_{i=0}^{n} s_{ki+j} = \frac{k^2}{6}(2ns_n + 3s_n + n) + kj(s_n + n) + (n+1)j^2$$

$$= s_n\left(\frac{nk^2}{3} + \frac{k^2}{2} + kj\right) + n\left(\frac{k^2}{6} + kj + j^2\right) + j^2 \qquad \square$$

Next, we find some special identities for a given sequence perfect square, which these identities are significant for many fields of mathematics, such as Cassini is mentioned in Curry's Paradox. For more details see in [8].

Proposition 2.2. (Cassini's Identity) For $n \geq 1$,

$$s_{n-1}s_{n+1} - s_n^2 = 1 - 2n^2$$

Proof. Using Binet Formula for perfect square sequence we get,

$$s_{n-1}s_{n+1} - s_n^2 = (n^2-1)^2 - n^4$$
$$= 1 - 2n^2$$ □

Corollary 2.2. For $n \geq 1$, we obtain this property for the perfect square sequence

$$s_{n-1}s_{n+1} = (s_n - 1)^2.$$

This easily can be obtain by using Cassini's Identity.

Proposition 2.3. (Catalan Identity) For $n, r \in \mathbb{N}$

$$s_{n-r}s_{n+r} - s_n^2 = r^4 - 2r^2 s_n$$

Proof. Using Binet Formula for Perfect Square Numbers we obtain,

$$s_{n-r}s_{n+r} - s_n^2 = (n-r)^2(n+r)^2 - \mathrm{n}^4$$
$$= r^4 - 2r^2 s_n$$ □

Proposition 2.4. (d'Ocagne Identity) For $n, r \in \mathbb{N}$, and $r < n$, we obtain

$$s_{n+1}s_r - s_n s_{r+1} = (r-n)(r+n+2nr)$$

Proof. It can be proved using the Binet Formula for perfect square sequence, same as above.□

Proposition 2.5. (Vajda Identity) For $n, r, i \in \mathbb{N}$, and $r, i < n$, we obtain

$$s_{n+i}s_{n+j} - s_n s_{n+i+j} = 2nij(n+i+j) + i^2 j^2$$

Proof. It can be proved similar as proposition 2.2. □

Proposition 2.6. (Halton Identity) For $n, r \in \mathbb{N}$, and $r, i < n$, we obtain

$$s_{n+r}s_{n-r} - s_{n-i}s_{n+i} = n^2(i^2 - r^2) + r^4 - i^4$$

Proof. It can be proved similar as proposition 2.2. □

Now, let's define a new number sequence obtained from perfect square sequence and Lucas sequence. We associate these two sequences and call it the perfect square Lucas sequence.

3 PERFECT SQUARE LUCAS SEQUENCE

Now we introduce perfect square Lucas sequence and obtain Binet's formula and generating function for this sequence. Furthermore, we ensure some properties concerned with the perfect square Lucas numbers and square numbers. Our aim is to indicate these numbers in terms of Lucas numbers, as well. Before giving a definition of recurrence relation of perfect square Lucas sequence, let's do some calculations on recurrence relation of perfect square sequence.

Because of the recurrence relation of the perfect square sequence is nonhomogeneous, to convert homogeneous, substitute $n+1$ instead of *n.*

$$s_{n+1} - 2s_n = 2 - s_{n-1}$$

Doing some calculation on the above recurrence relation and perfect square recurrence

relation, below we obtain homogeneous third order recurrence relation of $\{s_n\}_{n=0}^{\infty}$

$$s_{n+2} = 3s_{n+1} - 3s_n + s_{n-1} \quad (n \geq 1\,;\; s_0 = 0, s_1 = 1\,, s_2 = 4\,)$$

Definition 3.1. The recurrence relation of perfect square Lucas sequence, which is denoted as $\{LS_n\}_{n=0}^{\infty}$ is given as follows:

$$LS_{n+2} = 3LS_{n+1} - 3LS_n + LS_{n-1} \;\; (n \geq 1\,;\; LS_0 = 2, LS_1 = 1\,, LS_2 = 4\,)$$

Let's indicate some terms of the perfect square Lucas numbers:

$$\{2, 1, 4, 11, 22, 37, 56, 79, \ldots\}$$

The characteristic equation is $x^3 - 3x^2 + 3x - 1 = 0$ which has three same characteristic real roots $x_1 = x_2 = x_3 = 1$.

$$x_1 + x_2 + x_3 = 3, x_1x_2x_3 = 1,\; x_1x_2 + x_1x_3 + x_2x_3 = 3$$

Theorem 3.1. Binet's formula for perfect square Lucas sequence is as follows:

$$LS_n = 2 - 3n + 2n^2$$

Proof. From the characteristic equation of LS_n we have the same three roots so the formula will be:

$$LS_n = \alpha x_1^n + \beta n x_1^n + \gamma n^2 x_1^n$$

$$LS_n = \alpha + \beta n + \gamma n^2$$

From initial conditions, we find $\alpha = 2\,, \beta = -3\; and\; \gamma = 2.$ □

Theorem 3.2. The generating function of perfect square Lucas sequence is given by

$$g(z) = \sum_{n=0}^{\infty} LS_n z^n = \frac{2 - 5z + 7z^2}{1 - 3z + 3z^2 - z^3}$$

Proof. We use the definition of the generating function, properties of the sum symbol, and definition of recurrence relation of perfect square Lucas sequence;

$$\begin{aligned}
&\textstyle\sum_{n=0}^{\infty} LS_n z^n = LS_0 + LS_1 + LS_2 + \sum_{n=3}^{\infty} LS_n z^n \\
&= 2 + z + 4z^2 + \textstyle\sum_{n=3}^{\infty}(3LS_{n-1} - 3LS_{n-2} + LS_{n-3})z^n \\
&= 2 + z + 4z^2 + (-6z - 3z^2) + 3z\textstyle\sum_{n=0}^{\infty} LS_n z^n + 6z^2 \\
&\qquad -3z^2\textstyle\sum_{n=0}^{\infty} LS_n z^n + z^3\sum_{n=0}^{\infty} LS_n z^n.
\end{aligned}$$

Thus, we obtain

$$\textstyle\sum_{n=0}^{\infty} LS_n z^n = \frac{2-5z+7z^2}{1-3z+3z^2-z^3}.$$ □

Next, we extend this definition of perfect square Lucas numbers to negative index, and we get some equalities of even and odd indexes.

Proposition 3.1.

i. $LS_{-n} = LS_n + 6n$

ii. $LS_{2n} = 2(\, LS_n - (s_{n-1}s_{n+1} - s_n^2))$

iii. $LS_{2n+1} = 8s_n + 2n + 1$

Proof. From Binet formula for perfect square Lucas sequence we can obtain,

i. $LS_{-n} = 2n^2 + 3n + 2 = LS_n + 6n$

ii. $LS_{2n} = 8n^2 - 6n + 2 = 2\big(LS_n - (1 - 2n^2)\big) = 2(\, LS_n - (s_{n-1}s_{n+1} - s_n^2)\,)$

iii. $LS_{2n+1} = 8n^2 + 2n + 1 = 8s_n + 2n + 1$

Note that, here at ii. We used Cassini formula for the perfect square sequence. □

By using generating functions of $s_n \; and \; LS_n$, we give the relationship between perfect square Lucas sequence and perfect square sequence.

Theorem 3.3.

$$LS_n = 4s_{n-2} - 2s_{n-1} + 9n - 12 \;\text{ where }\; n \geq 2$$

Proof. By the generating functions of the perfect square Lucas sequence and the perfect square sequence we get,

$$\sum_{n=0}^{\infty} LS_{n+2}z^n = \frac{z^2 - z + 4}{1 - 3z + 3z^2 - z^3} = \frac{z^2 + z}{1 - 3z + 3z^2 - z^3} + \frac{4 - 2z}{1 - 3z + 3z^2 - z^3}$$

$$= \sum_{n=0}^{\infty} s_n z^n - 2\sum_{n=0}^{\infty} s_{n+1}z^n + 3\sum_{n=0}^{\infty}(n+1)(n+2)z^n$$

$$= 4\sum_{n=0}^{\infty} s_n z^n - 2\sum_{n=0}^{\infty} s_{n+1}z^n + \sum_{n=0}^{\infty}(9n+6)z^n$$

$$= \textstyle\sum_{n=0}^{\infty}(4s_n - 2s_{n+1} + 9n + 6\,)z^n.$$

So, we get

$$LS_{n+2} = 4s_n - 2s_{n+1} + 9n + 6.$$ □

Next, we obtain perfect square Lucas sequence in terms of Fibonacci and Lucas sequences.

Theorem 3.4. Let $\alpha = \frac{1-\sqrt{5}}{2}$, $\beta = \frac{1+\sqrt{5}}{2}$ and $A_{i,n} = \left(\frac{(1+n-i)(2+n-i)}{2} - s_{n-i}\right)$,

The following equation among F_i , $L_i \; and \; LS_n$ is satisfied.

$$LS_{n+2} = \sum_{i=0}^{n} L_i A_{i,n} + 2\sum_{i=0}^{n} F_i\, A_{i,n} + \sum_{i=1}^{n}\left(\frac{\alpha^i\beta - \alpha\beta^i}{\mathrm{B} - \alpha}\right)A_{i,n} \;\; , n > 0$$

$$LS_{n+2} = \sum_{i=0}^{n} L_i A_{i,n} + 2\sum_{i=0}^{n} F_i\, A_{i,n}\,, n = 0$$

Proof. Here we will use generating functions of Fibonacci, Lucas, perfect square, and perfect

square Lucas sequence and some properties of the sum symbol.

Note that,

$$\frac{z^2}{1-z-z^2} = \sum_{n=-\infty}^{\infty}\left(\begin{cases}\frac{\alpha^n\beta-\alpha\beta^n}{\beta-\alpha}, & n>0\\ 0, & otherwise\end{cases}\right)z^n,\ \alpha=\frac{1-\sqrt{5}}{2}, \beta=\frac{1+\sqrt{5}}{2}$$

We obtain,

$$\sum_{n=0}^{\infty} L_n z^n = \frac{2-z}{1-z-z^2} = \frac{z^2-z+4}{1-z-z^2} - \frac{z^2+2}{1-z-z^2}$$

$$= \frac{z^2-z+4}{\left((1-3z+3z^2-z^3)(1+\frac{2z-4z^2+z^3}{1-3z+3z^2-z^3})\right)} - \frac{z^2+2}{1-z-z^2}$$

$$= \frac{\sum_{n=0}^{\infty} LS_{n+2}z^n}{\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}z^n - \sum_{n=0}^{\infty} s_n z^n} - 2\sum_{n=0}^{\infty} F_n z^n - \sum_{n=1}^{\infty}\frac{\alpha^n\beta-\alpha\beta^n}{\beta-\alpha}z^n$$

$$\sum_{n=0}^{\infty} LS_{n+2}z^n =$$

$$= \left(\sum_{n=0}^{\infty} L_n z^n + 2\sum_{n=0}^{\infty} F_n z^n + \sum_{n=1}^{\infty}\frac{\alpha^n\beta-\alpha\beta^n}{\beta-\alpha}z^n\right)\sum_{n=0}^{\infty}\left(\frac{(n+1)(n+2)}{2} - s_n\right)z^n$$

After doing some calculations and using properties of the sum symbol we get,

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} L_i\, A_{i,n} z^n + 2\sum_{n=0}^{\infty}\sum_{i=0}^{n} F_i\, A_{i,n} z^n + \sum_{n=0}^{\infty}\sum_{i=1}^{n}\left(\frac{\alpha^i\beta-\alpha\beta^i}{\beta-\alpha}\right)A_{i,n} z^n \ .$$

$$LS_{n+2} = \sum_{i=0}^{n} L_i A_{i,n} + 2\sum_{i=0}^{n} F_i\, A_{i,n} + \sum_{i=1}^{n}\left(\frac{\alpha^i\beta-\alpha\beta^i}{\beta-\alpha}\right)A_{i,n}\ , n>0$$ □

Thus, we associated perfect square Lucas numbers with Lucas and Fibonacci numbers.

Theorem 3.5. The sum of the perfect square Lucas numbers is

$$\sum_{k=0}^{n} LS_k = \frac{(n+1)(4n^2-7n+12)}{6} \text{ where } n \geq 0$$

Proof. By using Binet's formula for perfect square Lucas sequence,

$$\sum_{k=0}^{n}(2k^2-3k+2) = 2\sum_{k=0}^{n} k^2 - 3\sum_{k=0}^{n} k + 2\sum_{k=0}^{n} 1$$

$$= \frac{(n+1)(4n^2-7n+12)}{6}$$ □

Let's examine some important identities such as Catalan:

Proposition 3.2. For $n, k \in \mathbb{N}$,

$$LS_{n-k}LS_{n+k} - LS_n^2 = -8s_n k^2 + 12nk^2 + 4k^4 - k^2.$$

Proof. We use Binet's formula for perfect square Lucas sequence. After some calculations, we obtain:

$$LS_{n-k}LS_{n+k} - LS_n^2 = -8n^2k^2 + 12nk^2 + 4k^4 - k^2$$

$$= -8s_n\, k^2 + 12nk^2 + 4k^4 - k^2.$$

If we substitute $k = 1$ on Catalan Identity, we get Cassini Identity.

$$LS_{n-1}LS_{n+1} - LS_n^2 = -8s_n + 12n + 3 \qquad \square$$

Next, we introduce a matrix representation for perfect square Lucas numbers.

Theorem 3.6. Let $n \in \mathbb{N}$. We get

i. $\begin{bmatrix} LS_{n+2} \\ LS_{n+1} \\ LS_n \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} LS_2 \\ LS_1 \\ LS_0 \end{bmatrix}$, **ii.** $\begin{bmatrix} LS_{n+2} \\ LS_{n+1} \\ LS_n \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} LS_{n+1} \\ LS_n \\ LS_{n-1} \end{bmatrix}$.

Proof. i. Proving by Principle of Mathematical Induction. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain,

$$\begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} LS_{n+1} \\ LS_n \\ LS_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} LS_{n+2} \\ LS_{n+1} \\ LS_n \end{bmatrix}.$$

From the last equation, for n, the equality is true.

The proof of the ii. can be established in a similar manner. $\square$

4 PERFECT SQUARE AND PERFECT SQUARE LUCAS POLYNOMIALS

Definition 4.1. For $n \in \mathbb{N}$, the perfect square polynomials $S_n(x)$, and perfect square Lucas polynomials $LS_n(x)$ are defined, respectively, as

$$S_{n+3}(x) = (x+2)S_{n+2}(x) - (2x+1)S_{n+1}(x) + xS_n(x)$$

with $S_0(x) = 0$, $S_1(x) = 1$, and $S_2(x) = x + 3$,

$$LS_{n+3}(x) = (x+2)LS_{n+2}(x) - (2x+1)LS_{n+1}(x) + xLS_n(x)$$

with $LS_0(x) = 2$, $LS_1(x) = 1$ and $LS_2(x) = x + 3$.

Perfect square and perfect square Lucas numbers are obtained by substituting $x = 1$ in $S_n(x)$ and $LS_n(x)$.

The characteristic equation of the perfect square and perfect square Lucas polynomials is

$$r^3 - (x+2)r^2 + (2x+1)r - x = 0.$$

The roots of this equation

$$r_1 = x,\ r_2 = 1, \text{ and } r_3 = 1.$$

Next, we give the relationships between these roots below:

$$r_1 + r_2 + r_3 = x + 2,\ r_1r_2r_3 = x, \text{ and } r_1r_2 + r_1r_3 + r_2r_3 = 2x + 1.$$

Some terms of the $S_n(x)$ and $LS_n(x)$ polynomials are given in the (Tab. 1)

$S_0(x)$	0	$LS_0(x)$	2
$S_1(x)$	1	$LS_1(x)$	1
$S_2(x)$	$x+3$	$LS_2(x)$	$x+3$
$S_3(x)$	x^2+3x+5	$LS_3(x)$	x^2+5x+5
$S_4(x)$	x^3+3x^2+5x+7	$LS_4(x)$	x^3+5x^2+9x+7
$S_5(x)$	$x^4+3x^3+5x^2+7x+9$	$LS_5(x)$	$x^4+5x^3+9x^2+13x+9$
⋮	⋮	⋮	⋮

Table 1. Perfect Square and Perfect Square Lucas Polynomials

Now, we can give the Binet formulas of the perfect square and perfect square Lucas polynomials.

Theorem 4.1. Let $n \in \mathbb{N}$.

$$\textbf{i. } S_n(x) = \frac{x+1}{(x-1)^2}(x^n - 1) + \frac{2}{1-x}n, \qquad \textbf{ii. } LS_n(x) = \frac{x+3}{(x-1)^2}x^n + \frac{2x^2-5x-1}{(x-1)^2} + \frac{2x+2}{1-x}n.$$

Proof. i. The Binet form of a sequence is as follows (if two roots are the same)

$$S_n(x) = A_1r_1^n + (A_2 + A_3n)r_2^n.$$

For these n values, we obtain;

$$S_0(x) = 0 = A_1 + A_2,$$
$$S_1(x) = 1 = A_1x + A_2 + A_3,$$
$$S_2(x) = x + 3 = A_1x^2 + A_2 + 2A_3.$$

We find

$$A_1 = \frac{x+1}{(x-1)^2},\ A_2 = -\frac{x+1}{(x-1)^2}, \text{ and } A_3 = \frac{2}{1-x}.$$

Thus, we obtain

$$S_n(x) = \frac{x+1}{(x-1)^2}(x^n - 1) + \frac{2}{1-x}n.$$

The proof of the other can be established in a similar manner. □

Next, we obtain the generating function of the perfect square and perfect square Lucas polynomials.

Theorem 4.2. We have

$$\textbf{i. } \mathfrak{H}(t) = \sum_{n=0}^{\infty} S_n(x)t^n = \frac{t+t^2}{1-t(x+2)+(2x+1)t^2-xt^3},$$

$$\textbf{ii. } l(t) = \sum_{n=0}^{\infty} LS_n(x)t^n = \frac{2-3t-2tx+6t^2+4t^2x}{1-t(x+2)+(2x+1)t^2-xt^3}.$$

Proof. i. If we use the definition of the generating function, the properties of the sum symbol,

and the recurrence relation of perfect square polynomials, we obtain

$$\mathfrak{H}(t) = \sum_{n=0}^{\infty} S_n(x)t^n = t + (x+3)t^2 + \sum_{n=3}^{\infty} S_n(x)t^n$$
$$= t + (x+3)t^2 + \sum_{n=3}^{\infty}((x+2)S_{n-1}(x) - (2x+1)S_{n-2}(x) + xS_{n-3}(x))t^n$$
$$= t + t^2(x+3) + t(x+2)(-t + \sum_{n=0}^{\infty} S_n(x)t^n) - t^2(2x+1)\sum_{n=0}^{\infty} S_n(x)t^n$$
$$+t^3 x \sum_{n=0}^{\infty} S_n(x)t^n.$$

Thus, we have

$$\mathfrak{H}(t) = \sum_{n=0}^{\infty} S_n(x)t^n = \frac{t+t^2}{1-t(x+2)+(2x+1)t^2-xt^3}.$$

A similar method can be used to prove the other. □

In the following theorem, relations between perfect square and perfect square Lucas polynomials are obtained.

Theorem 4.3. Let $n \in \mathbb{N}$. The following equations are true:

i. $S_n(x) = \frac{-x^2-2x+2}{2x^3+10x^2+14x+6} LS_{n+2}(x) + \frac{3x+8}{2x^2+8x+6} LS_{n+1}(x) - \frac{x+4}{2x^3+10x^2+14x+6} LS_n(x)$

ii. $LS_n(x) = \frac{3x+4}{2x+2} S_{n+2}(x) - \frac{3x^2+9x+8}{2x+2} S_{n+1}(x) + \frac{5x^2+10x+6}{2x+2} S_n(x),$

Proof. i. The following relation is used for proofs;

$$S_n(x) = axLS_{n+2}(x) + bxLS_{n+1}(x) + cxLS_n(x).$$

For these n values, we obtain;

$$S_0(x) = a\ x\ LS_2(x) + b\ x\ LS_1(x) + c\ x\ LS_0(x),$$
$$S_1(x) = a\ x\ LS_3(x) + b\ x\ LS_2(x) + c\ x\ LS_1(x),$$
$$S_2(x) = a\ x\ LS_4(x) + b\ x\ LS_3(x) + c\ x\ LS_2(x).$$

If the values are substituted, the following equation system is obtained.

$$0 = a(x+3) + b\ + 2c,$$
$$1 = a(x^2+5x+5) + b(x+3) + c,$$
$$x+3 = a(x^3+5x^2+9x+7) + b(x^2+5x+5) + c(x+3).$$

We find

$$a = \frac{-x^2-2x+2}{2x^3+10x^2+14x+6},\ b = \frac{3x+8}{2x^2+8x+6},\ \text{and}\ c = -\frac{x+4}{2x^3+10x^2+14x+6}.$$

Thus, we have

$$S_n(x) = \frac{-x^2-2x+2}{2x^3+10x^2+14x+6} LS_{n+2}(x) + \frac{3x+8}{2x^2+8x+6} LS_{n+1}(x) - \frac{x+4}{2x^3+10x^2+14x+6} LS_n(x).$$

The proof of the other is shown similarly. □

Next, we associate the terms of the perfect square and perfect square Lucas polynomials with matrices.

Theorem 4.4. Let $n \in \mathbb{N}$. We have

i. 1. $$\begin{bmatrix} S_{n+2}(x) \\ S_{n+1}(x) \\ S_n(x) \end{bmatrix} = \begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} x+3 \\ 1 \\ 0 \end{bmatrix},$$

2. $$\begin{bmatrix} LS_{n+2}(x) \\ LS_{n+1}(x) \\ LS_n(x) \end{bmatrix} = \begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} x+3 \\ 1 \\ 2 \end{bmatrix}.$$

ii. 1. $$\begin{bmatrix} S_n(x) \\ S_{n+1}(x) \\ S_{n+2}(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x+2 & -2x-1 & x \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ x+3 \end{bmatrix},$$

2. $$\begin{bmatrix} LS_n(x) \\ LS_{n+1}(x) \\ LS_{n+2}(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x+2 & -2x-1 & x \end{bmatrix}^n \begin{bmatrix} 2 \\ 1 \\ x+3 \end{bmatrix}.$$

iii. 1. $$\begin{bmatrix} S_{n+2}(x) \\ S_{n+1}(x) \\ S_n(x) \end{bmatrix} = \begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_{n+1}(x) \\ S_n(x) \\ S_{n-1}(x) \end{bmatrix},$$

2. $$\begin{bmatrix} LS_{n+2}(x) \\ LS_{n+1}(x) \\ LS_n(x) \end{bmatrix} = \begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} LS_{n+1}(x) \\ LS_n(x) \\ LS_{n-1}(x) \end{bmatrix}.$$

Proof. i. 1. Let show the proof by induction over n. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain

$$\begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} x+3 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} x+3 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x+2 & -2x-1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_{n+1}(x) \\ S_n(x) \\ S_{n-1}(x) \end{bmatrix}$$

$$= \begin{bmatrix} S_{n+2}(x) \\ S_{n+1}(x) \\ S_n(x) \end{bmatrix}.$$

From the last equation, for n, it can be seen that the equality is true.
Similarly, the proof of the other can be derived. □

Theorem 4.5. (Simpson Formula) Let $n \in \mathbb{N}$. The following equations are satisfied:

i. $det\begin{bmatrix} S_{n+2}(x) & S_{n+1}(x) & S_n(x) \\ S_{n+1}(x) & S_n(x) & S_{n-1}(x) \\ S_n(x) & S_{n-1}(x) & S_{n-2}(x) \end{bmatrix} = -(4x+4)x^{n-2},$

ii. $det\begin{bmatrix} LS_{n+2}(x) & LS_{n+1}(x) & LS_n(x) \\ LS_{n+1}(x) & LS_n(x) & LS_{n-1}(x) \\ LS_n(x) & LS_{n-1}(x) & LS_{n-2}(x) \end{bmatrix} = \frac{x^{n-1}}{9}(-16x^4-24x^3+111x^2-57x-135)$

Proof. i. Let show the proof by induction over n. For $n = 1$, the equality is true. For $n-1$, assume the equality is true. We obtain

$$det\begin{bmatrix} S_{n+2}(x) & S_{n+1}(x) & S_n(x) \\ S_{n+1}(x) & S_n(x) & S_{n-1}(x) \\ S_n(x) & S_{n-1}(x) & S_{n-2}(x) \end{bmatrix}$$

$$= det\begin{bmatrix} (x+2)S_{n+1}(x)-(2x+1)S_n(x)+xS_{n-1}(x) & S_{n+1}(x) & S_n(x) \\ (x+2)S_n(x)-(2x+1)S_{n-1}(x)+xS_{n-2}(x) & S_n(x) & S_{n-1}(x) \\ (x+2)S_{n-1}(x)-(2x+1)S_{n-2}(x)+xS_{n-3}(x) & S_{n-1}(x) & S_{n-2}(x) \end{bmatrix}$$

$$= det\begin{bmatrix} (x+2)S_{n+1}(x) & S_{n+1}(x) & S_n(x) \\ (x+2)S_n(x) & S_n(x) & S_{n-1}(x) \\ (x+2)S_{n-1}(x) & S_{n-1}(x) & S_{n-2}(x) \end{bmatrix}$$

$$+det\begin{bmatrix} -(2x+1)S_n(x) & S_{n+1}(x) & S_n(x) \\ -(2x+1)S_{n-1}(x) & S_n(x) & S_{n-1}(x) \\ -(2x+1)S_{n-2}(x) & S_{n-1}(x) & S_{n-2}(x) \end{bmatrix} + det\begin{bmatrix} xS_{n-1}(x) & S_{n+1}(x) & S_n(x) \\ xS_{n-2}(x) & S_n(x) & S_{n-1}(x) \\ xS_{n-3}(x) & S_{n-1}(x) & S_{n-2}(x) \end{bmatrix}$$

$$= xdet\begin{bmatrix} S_{n+1}(x) & S_n(x) & S_{n-1}(x) \\ S_n(x) & S_{n-1}(x) & S_{n-2}(x) \\ S_{n-1}(x) & S_{n-2}(x) & S_{n-3}(x) \end{bmatrix}$$

$$= -(4x+4)x^{n-2}.$$

From the last equation, for n, it can be seen that the equality is true.

The proof for the other follows a similar approach. □

5 CONCLUSIONS

In this study, we introduced a new recurrence relation for the perfect square sequence and examined its relationship with Fibonacci and Lucas sequences. Through detailed analysis, we derived several significant properties, including generating functions, Binet formulas, and summation identities for both the perfect square sequence and its extension, the perfect square Lucas sequence. Furthermore, we explored the polynomial versions of these sequences, establishing their matrix representations and Simpson formulas. The results provide a new perspective on the interconnections between classical number sequences and perfect square-based structures, potentially leading to further mathematical discoveries.

Several potential research directions can be inspired by this study:

Further exploration of similar recurrence relations in other generalized sequences, such as

Pell, Tribonacci, or Padovan numbers, could lead to new insights.

Investigating combinatorial interpretations of the perfect square and perfect square Lucas sequences in graph theory, tiling problems, or coding theory.

Extending the matrix representations and eigenvalue analysis of these sequences to study their properties in higher dimensions.

Analyzing continued fraction expansions related to perfect square sequences and their applications in number theory.

Examining potential cryptographic applications of perfect square and perfect square Lucas sequences, similar to Fibonacci-based cryptographic methods.

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