Palestine Journal of Mathematics

Vol 14(3)(2025) , 1016–1027 © Palestine Polytechnic University-PPU 2025

GENERALIZATION OF THE k-WOODALL SEQUENCE

Hakan Akkuş and Engin Özkan

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B39, 11B83; Secondary 05A15.

Keywords and phrases: Woodall numbers, Binet formulas, Cassini Identity, Cullen numbers, Generating function.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Hakan Akkuş

Abstract *In this study, we define the k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences, and some terms of these sequences are given. Then we obtain the Binet formulas, generating functions, sum formulas, etc. Also, the relations among the terms of the k-Woodall, k-Woodall-Lucas, modified k-Woodall, Woodall, Woodall-Lucas, Modified Woodall, Cullen, Cullen-Lucas, and Modified Cullen sequences are presented. In addition, we calculate Vajda's identity, D'ocagne identity, etc. Finally, we associate the terms of these sequences with matrices.*

1 Introduction

In mathematics, sequences play a crucial role in gaining a clearer understanding of a problem that arises under specific conditions. One of the most well-known examples of such problems is the Rabbit problem, featured in Leonardo Fibonacci's renowned book Liber Abaci. From the outcomes of this problem, the Fibonacci sequence is defined, and the relationships between consecutive terms of this sequence are established.

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields such as Engineering [1], Cryptology [2], Biomathematics [3], Phylotaxis [4], Chemistry [5], etc. New sequences have been obtained by changing the generalized Fibonacci sequence recurrence relation and initial conditions. The known examples of such sequences are the Woodall, Woodall-Lucas, Horadam, Leonardo, k-Jacobsthal, k-Narayana, Gaussian Fibonacci, k-Jacobsthal-Lucas, Olivier, Bronze Fibonacci, Copper Fibonacci, Cullen, Edouard-Lucas sequences, etc. (see for details in [6]-[25]).

For $n \in \mathbb{N}$, Fibonacci numbers F_n, Lucas numbers L_n, Pell numbers p_n, and Pell-Lucas numbers q_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, with\ F_0 = 0\ and\ F_1 = 1, L_{n+2} = L_{n+1} + L_n, with\ L_0 = 2\ and\ L_1 = 1,$$

$$p_{n+2} = 2p_{n+1} + p_n, with\ p_0 = 0\ and\ p_1 = 1, q_{n+2} = 2q_{n+1} + q_n, with\ q_0 = 2\ and\ q_1 = 2.$$

For F_n, L_n, p_n, and q_n the Binet formulas are given by the following relations, respectively,

$$F_n = \frac{\varphi^n - \theta^n}{\varphi - \theta}, L_n = \varphi^n + \theta^n,\ p_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, and\ q_n = \alpha^n + \beta^n$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, $\theta = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + \sqrt{2}$, and $\beta = 1 - \sqrt{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$ and $r^2 - 2r - 1 = 0$, respectively. Here the numbers φ and α are the known golden ratio and silver ratio, respectively.

In addition, for $n \in \mathbb{N}$, the Woodall numbers R_n, Woodall-Lucas numbers D_n, Modified Woodall numbers W_n, Cullen numbers C_n, Cullen-Lucas numbers E_n, and Modified Cullen-Lucas numbers H_n are defined by the recurrence relations, respectively,

$$R_{n+2} = 4R_{n+1} - 4R_n - 1, with\ R_0 = -1\ and\ R_1 = 1,$$

$$D_{n+2} = 4D_{n+1} - 4D_n - 1, with\ D_0 = 1\ and\ D_1 = 4,$$

$$W_{n+2} = 4W_{n+1} - 4W_n + 1, with\ W_0 = 0\ and\ W_1 = 1,$$

$$C_{n+2} = 4C_{n+1} - 4C_n + 1, with\ C_0 = 1\ and\ C_1 = 3,$$

$$E_{n+2} = 4E_{n+1} - 4E_n + 1, with\ E_0 = 1\ and\ E_1 = 4,$$

$$H_{n+2} = 4H_{n+1} - 4H_n - 1, with\ H_0 = -1\ and\ H_1 = 2.$$

Moreover, for $n \in \mathbb{N}$, the third-order recurrence relations of the sequences R_n, D_n, W_n, C_n, E_n and H_n are as follows, respectively,

$$R_{n+3} = 5R_{n+2} - 8R_{n+1} + 4R_n, with\ R_0 = -1,\ R_1 = 1, and\ R_2 = 7,$$

$$D_{n+3} = 5D_{n+2} - 8D_{n+1} + 4D_n, with\ D_0 = 1, D_1 = 4, and\ D_2 = 11,$$

$$W_{n+3} = 5W_{n+2} - 8W_{n+1} + 4W_n, with\ W_0 = 0, W_1 = 1, and\ W_2 = 5,$$

$$C_{n+3} = 5C_{n+2} - 8C_{n+1} + 4C_n, with\ C_0 = 1, C_1 = 3, and\ C_2 = 9,$$

$$E_{n+3} = 5E_{n+2} - 8E_{n+1} + 4E_n, with\ E_0 = 1, E_1 = 4, and\ E_2 = 13,$$

$$H_{n+3} = 5H_{n+2} - 8H_{n+1} + 4H_n, with\ H_0 = -1, H_1 = 2, and\ H_2 = 11.$$

The authors found many properties of these sequences and made many applications related to these sequences. (see for details in [26]-[32]).

Binet formulas for R_n, D_n, W_n, C_n, E_n and H_n are given by the following relations, respectively,

$$R_n = n2^n - 1, D_n = (4+n)2^{n-1} - 1, W_n = (n-1)\,2^n + 1,$$

$$C_n = n2^n + 1, E_n = 3n2^{n-1} + 1, and\ H_n = 3n2^{n-1} - 1.$$

By changing the initial values and recurrence relation of Fibonacci sequence, many new sequences were introduced and these sequences took an important place in number theory such as k-Fibonacci and k-Lucas sequences. In [33], Falcon introduced k-Fibonacci sequence and found different identities and special transformations of this sequence. Akkuş and Özkan introduced k-Leonardo sequence and they found many properties [34]. In [35]-[37], authors found many properties of k-Fibonacci sequence and related them to known sequences.

We separate the article into three parts.

In chapter 2, we define the k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences. Then we give the characteristic equation, Binet formulas, the summation formulas, and some properties for these sequences. Also, we obtain the generating functions and Simpson formulas.

In chapter 3, we present the relations among the terms of the k-Woodall, k-Woodall-Lucas, Modified k-Woodall, Woodall, Woodall-Lucas, and Modified Woodall sequences. Then we associate the k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences with special number sequences. In addition, we calculate the specific identities of these sequences. Finally, we associate the terms of these sequences with matrices.

2 k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences

For $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the k-Woodall $\mathcal{O}_{k,n}$, k-Woodall-Lucas $\mathcal{S}_{k,n}$, and Modified k-Woodall $\mathcal{B}_{k,n}$ are defined by, respectively,

$$\mathcal{O}_{k,n+2} = 4\mathcal{O}_{k,n+1} - 4\mathcal{O}_{k,n} - k, with\ \mathcal{O}_{k,0} = -1\ and\ \mathcal{O}_{k,1} = 1,$$

$$\mathcal{S}_{k,n+2} = 4\mathcal{S}_{k,n+1} - 4\mathcal{S}_{k,n} - k, with\ \mathcal{S}_{k,0} = 1\ and\ \mathcal{S}_{k,1} = 4,$$

$$\mathcal{B}_{k,n+2} = 4\mathcal{B}_{k,n+1} - 4\mathcal{B}_{k,n} + k, with\ \mathcal{B}_{k,0} = 0\ and\ \mathcal{B}_{k,1} = 1.$$

Also, for $n \in \mathbb{N}$, the third-order recurrence relations of the sequences $\mathcal{O}_{k,n}$, $\mathcal{S}_{k,n}$, and $\mathcal{B}_{k,n}$ are as follows,

$$\mathcal{O}_{k,n+3} = 5\mathcal{O}_{k,n+2} - 8\mathcal{O}_{k,n+1} + 4\mathcal{O}_{k,n}, with\ \mathcal{O}_{k,0} = -1,\ \mathcal{O}_{k,1} = 1, and\ \mathcal{O}_{k,2} = 8 - k,$$

$$\mathcal{S}_{k,n+3} = 5\mathcal{S}_{k,n+2} - 8\mathcal{S}_{k,n+1} + 4\mathcal{S}_{k,n}, with\ \mathcal{S}_{k,0} = 1, \mathcal{S}_{k,1} = 4, and\ \mathcal{S}_{k,2} = 12 - k,$$

$$\mathcal{B}_{k,n+3} = 5\mathcal{B}_{k,n+2} - 8\mathcal{B}_{k,n+1} + 4\mathcal{B}_{k,n}, with\ \mathcal{B}_{k,0} = 0, \mathcal{B}_{k,1} = 1, and\ \mathcal{B}_{k,2} = k + 4,$$

respectively.

Then, let's give some information about the equations of these sequences.

The characteristic equation of the k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences is

$$r^3 - 5r^2 + 8r - 4 = \left(r^2 - 4r + 4\right)(r - 1) = 0.$$

The roots of this equation are;

$$r_1 = r_2 = 2, and\ r_3 = 1.$$

The relationship between these roots is given below;

$$r_1 + r_2 + r_3 = 5, r_1r_2 + r_1r_3 + r_2r_3 = 8, and r_1r_2r_3 = 4.$$

For $n \in \mathbb{N}$, some terms of these sequences are as follows:

$$\mathcal{O}_{k,n} = \{-1, 1, 8 - k, 28 - 5k, 80 - 17k,\ 208 - 49k,\ \dots\}$$

$$S_{k,n} = \{1,\ 4,\ 12 - k, 32 - 5k, 80 - 17k,\ 192 - 49k,\ \dots\}$$

$$\mathcal{B}_{k,n} = \{0,\ 1,\ 4 + k, 12 + 5k, 32 + 17k, 80 + 49k,\ \dots\}$$

In the following theorems, the Binet formulas of the k-Woodall $\mathcal{O}_{k,n}$, k-Woodall-Lucas $\mathcal{S}_{k,n}$, and Modified k-Woodall $\mathcal{B}_{k,n}$ sequences are expressed. In addition, we find the generating functions of these sequences.

Theorem 2.1. *Let $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. We obtain*

i. $\mathcal{O}_{k,n} = \left(k - 1 + \frac{3-k}{2}n\right)2^n - k$,

ii. $\mathcal{S}_{k,n} = \left(k + 1 + \frac{2-k}{2}n\right)2^n - k$,

iii. $\mathcal{B}_{k,n} = \left(-k + \frac{k+1}{2}n\right)2^n + k$.

Proof. i. The Binet form of a sequence is as follows (two distinct roots case $r_1 = r_2 = 2$ and $r_3 = 1$)

$$\mathcal{O}_{k,n} = (x + yn)r_1^n + zr_3^n.$$

For these n values, we obtain;

$$\mathcal{O}_{k,0} = -1 = x + z,$$

$$\mathcal{O}_{k,1} = 1 = (x + y)2 + z,$$

$$\mathcal{O}_{k,2} = 8 - k = (x + 2y)4 + z.$$

We find

$x = k - 1$, $y = \frac{3-k}{2}$, and $z = -k$.

Thus, we obtain

$$\mathcal{O}_{k,n} = \left(k - 1 + \frac{3-k}{2}n\right)2^n - k.$$

The proofs of the others are shown similarly. □

Theorem 2.2. ***(Generating Functions)*** *The generating functions for k-Woodall $\mathcal{O}_{k,n}$, k-Woodall-Lucas $\mathcal{S}_{k,n}$, and Modified k-Woodall $\mathcal{B}_{k,n}$ sequences are given as follows, respectively,*

i. $o(t) = \sum_{n=0}^{\infty} \mathcal{O}_{k,n}t^n = \frac{-1+6t-(k+5)t^2}{-4t^3+8t^2-5t+1}$,

ii. $s(t) = \sum_{n=0}^{\infty} \mathcal{S}_{k,n}t^n = \frac{1-t-kt^2}{-4t^3+8t^2-5t+1}$,

iii. $b(t) = \sum_{n=0}^{\infty} \mathcal{B}_{k,n}t^n = \frac{t+(k-1)t^2}{-4t^3+8t^2-5t+1}$.

Proof. i. For the k-Woodall sequence, we have

$$o(t) = \sum_{n=0}^{\infty} \mathcal{O}_{k,n}t^n = -1 + t + (8-k)t^2 + \sum_{n=3}^{\infty} \mathcal{O}_{k,n}t^n$$

$$= -1 + t + (8-k)t^2 + 5\sum_{n=3}^{\infty} \mathcal{O}_{k,n-1}t^n - 8\sum_{n=3}^{\infty} \mathcal{O}_{k,n-2}t^n + 4\sum_{n=3}^{\infty} \mathcal{O}_{k,n-3}t^n$$

$$= -1 + t + (8-k)t^2 + 5t\sum_{n=2}^{\infty} \mathcal{O}_{k,n}t^n - 8t^2\sum_{n=1}^{\infty} \mathcal{O}_{k,n}t^n + 4t^3\sum_{n=0}^{\infty} \mathcal{O}_{k,n}t^n.$$

Thus, we have

$$o(t) = \sum_{n=0}^{\infty} \mathcal{O}_{k,n}t^n = \frac{-1 + 6t - (k+5)t^2}{-4t^3 + 8t^2 - 5t + 1}.$$

The proofs of the others are shown similarly. □

Next, we give special sum formulas of the k-Woodall $\mathcal{O}_{k,n}$, k-Woodall-Lucas $\mathcal{S}_{k,n}$, and Modified k-Woodall $\mathcal{B}_{k,n}$ sequences.

Theorem 2.3. *Let $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. We obtain*

i. $\sum_{s=0}^{n} \mathcal{O}_{k,n} = 4\mathcal{O}_{k,n-1} - (n-1)k + 4$,

ii. $\sum_{s=0}^{n} \mathcal{S}_{k,s} = 4\mathcal{S}_{k,n-1} - (n-1)k + 1$,

iii. $\sum_{s=0}^{n} \mathcal{B}_{k,n} = 4\mathcal{B}_{k,n-1} + (n-1)k + 1$.

Proof. ii. From the definition of the k-Woodall-Lucas sequence, we obtain

$$\mathcal{S}_{k,2} = 4\mathcal{S}_{k,1} - 4\mathcal{S}_{k,0} - k,$$

$$\mathcal{S}_{k,3} = 4\mathcal{S}_{k,2} - 4\mathcal{S}_{k,1} - k,$$

$$\vdots$$

$$\mathcal{S}_{k,n} = 4\mathcal{S}_{k,n-1} - 4\mathcal{S}_{k,n-2} - k.$$

So, we have

$$-5+\sum_{s=0}^{n}\mathcal{S}_{k,s} =4\sum_{s=1}^{n-1}\mathcal{S}_{k,s}-4\sum_{s=0}^{n-2}\mathcal{S}_{k,s}-(n-1)k,$$

$$-5+\sum_{s=0}^{n}\mathcal{S}_{k,s}$$

$$=4\left(-\mathcal{S}_{k,n}-\mathcal{S}_{k,0}+\sum_{s=0}^{n}\mathcal{S}_{k,s}\right)-4\left(-\mathcal{S}_{k,n}-\mathcal{S}_{k,n-1}+\sum_{s=0}^{n}\mathcal{S}_{k,s}\right)-(n-1)k.$$

Thus, we obtain

$$\sum_{s=0}^{n}\mathcal{S}_{k,s}=4\mathcal{S}_{k,n-1}-(n-1)\,k+1.$$

The proofs of the others are shown similarly. □

Theorem 2.4. *Let $k\in\mathbb{R}^{+}$ and $n\in\mathbb{N}$. We obtain*

i. $\sum_{s=0}^{n}\mathcal{O}_{k,2s}=\frac{16\mathcal{O}_{k,2n}+4\mathcal{O}_{k,2n+1}-9nk+21}{9}$,
ii. $\sum_{s=0}^{n}\mathcal{O}_{k,2s+1}=\frac{20\mathcal{O}_{k,2n}-4\mathcal{O}_{k,2n+1}-9nk+15}{9}$,
iii. $\sum_{s=0}^{n}\mathcal{S}_{k,2s}=\frac{20\mathcal{S}_{k,2n}-4\mathcal{S}_{k,2n+1}-9nk+5}{9}$,
iv. $\sum_{s=0}^{n}\mathcal{S}_{k,2s+1}=\frac{20\mathcal{S}_{k,2n}+4\mathcal{S}_{k,2n+1}-9nk+4}{9}$,
v. $\sum_{s=0}^{n}\mathcal{B}_{k,2s}=\frac{20\mathcal{B}_{k,2n}-4\mathcal{B}_{k,2n+1}+9nk+4}{9}$,
vi. $\sum_{s=0}^{n}\mathcal{B}_{k,2s+1}=\frac{16\mathcal{B}_{k,2n}+4\mathcal{B}_{k,2n+1}+9nk+5}{9}$.

Proof. The proofs are shown similarly to Theorem 2.3. □

3 Relations among Special Sequences

In this chapter, we examine the relationships among the k-Woodall, k-Woodall-Lucas, Modified k-Woodall, Woodall, Woodall-Lucas, Modified Woodall, Cullen, Cullen-Lucas, and Modified Cullen sequences. In addition, we calculate the specific identities of the k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences. Finally, we associate the terms of these sequences with matrices.

In the following theorems, we examine the relations among the k-Woodall $\mathcal{O}_{k,n}$, k-Woodall-Lucas $\mathcal{S}_{k,n}$, Modified k-Woodall $\mathcal{B}_{k,n}$, Woodall R_n, Woodall-Lucas D_n, and Modified Woodall W_n sequences.

Theorem 3.1. *Let $k\in\mathbb{R}^{+}$, and $n\in\mathbb{N}$. The following equations are true:*

i. $\mathcal{O}_{k,n}=\frac{2k-7}{k^2-4k+4}\mathcal{S}_{k,n+2}+\frac{-7k+23}{k^2-4k+4}\mathcal{S}_{k,n+1}+\frac{k^2+k-12}{k^2-4k+4}\mathcal{S}_{k,n}$,
ii. $\mathcal{O}_{k,n}=-\frac{k+5}{k^2+2k+1}\mathcal{B}_{k,n+2}+\frac{7k+19}{k^2+2k+1}\mathcal{B}_{k,n+1}-\frac{k^2+8k+15}{k^2+2k+1}\mathcal{B}_{k,n}$,
iii. $\mathcal{O}_{k,n}=(2k-2)R_{n+2}+\frac{-15k+15}{2}R_{n+1}+\frac{13k-11}{2}R_n$,
iv. $\mathcal{O}_{k,n}=(5k-10)D_{n+2}+(-17k+33)D_{n+1}+(13k-23)D_n$,
v. $\mathcal{O}_{k,n}=-\frac{k+5}{4}W_{n+2}+\frac{5k+21}{4}W_{n+1}-(2k+4)W_n$.

Proof. i. The following relation is used for proofs;

$$\mathcal{O}_{k,n}=a\,x\,\mathcal{S}_{k,n+2}+b\,x\,\mathcal{S}_{k,n+1}+c\,x\,\mathcal{S}_{k,n}.$$

For these n values, we obtain;

$$\mathcal{O}_{k,0}=a\,x\,\mathcal{S}_{k,2}+b\,x\,\mathcal{S}_{k,1}+c\,x\,\mathcal{S}_{k,0},$$

$$\mathcal{O}_{k,1}=a\,x\,\mathcal{S}_{k,3}+b\,x\,\mathcal{S}_{k,2}+c\,x\,\mathcal{S}_{k,1},$$

$$\mathcal{O}_{k,2}=a\,x\,\mathcal{S}_{k,4}+b\,x\,\mathcal{S}_{k,3}+c\,x\,\mathcal{S}_{k,2}.$$

We find

$a=\frac{2k-7}{k^2-4k+4}$, $b=\frac{-7k+23}{k^2-4k+4}$, and $c=\frac{k^2+k-12}{k^2-4k+4}$.

Thus, we have

$$\mathcal{O}_{k,n} = \frac{2k-7}{k^2-4k+4}\mathcal{S}_{k,n+2} + \frac{-7k+23}{k^2-4k+4}\mathcal{S}_{k,n+1} + \frac{k^2+k-12}{k^2-4k+4}\mathcal{S}_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 3.2. *Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:*

i. $\mathcal{S}_{k,n} = \frac{-2k+8}{k^2-6k+9}\mathcal{O}_{k,n+2} + \frac{7k-27}{k^2-6k+9}\mathcal{O}_{k,n+1} + \frac{k^2-11k+28}{k^2-6k+9}\mathcal{O}_{k,n}$,

ii. $\mathcal{S}_{k,n} = \frac{k-2}{k^2+2k+1}\mathcal{B}_{k,n+2} + \frac{9}{k^2+2k+1}\mathcal{B}_{k,n+1} - \frac{k^2+3k+8}{k^2+2k+1}\mathcal{B}_{k,n}$,

iii. $\mathcal{S}_{k,n} = \frac{4k-1}{2}R_{n+2} + \frac{-15k+5}{2}R_{n+1} + \frac{13k-4}{2}R_n$,

iv. $\mathcal{S}_{k,n} = (5k-5)D_{n+2} + (-17k+17)D_{n+1} + (13k-12)D_n$,

v. $\mathcal{S}_{k,n} = -\frac{k}{4}W_{n+2} + \frac{5k+4}{4}W_{n+1} - (2k+1)W_n$.

Proof. v. If Binet formulas are used for proofs, we get

$$-\frac{k}{4}W_{n+2} + \frac{5k+4}{4}W_{n+1} - (2k+1)W_n$$

$$= -\frac{k}{4}\left((n+1)2^{n+1}+1\right) + \frac{5k+4}{4}\left(n2^n+1\right) - (2k+1)\left((n-1)2^n+1\right)$$

$$= \left(k+1+\frac{2-k}{2}n\right)2^n - k = \mathcal{S}_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 3.3. *Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:*

i. $\mathcal{B}_{k,n} = \frac{k-11}{k^2-6k+9}\mathcal{O}_{k,n+2} + \frac{-7k+45}{k^2-6k+9}\mathcal{O}_{k,n+1} + \frac{-k^2+12k-43}{k^2-6k+9}\mathcal{O}_{k,n}$,

ii. $\mathcal{B}_{k,n} = -\frac{k+7}{k^2-4k+4}\mathcal{S}_{k,n+2} + \frac{27}{k^2-4k+4}\mathcal{S}_{k,n+1} + \frac{-k^2+5k-24}{k^2-4k+4}\mathcal{S}_{k,n}$,

iii. $\mathcal{B}_{k,n} = -\frac{4k+1}{2}R_{n+2} + \frac{15k+4}{2}R_{n+1} - \frac{13k+3}{2}R_n$,

iv. $\mathcal{B}_{k,n} = -(5k+3)D_{n+2} + (17k+10)D_{n+1} - (13k+7)D_n$,

v. $\mathcal{B}_{k,n} = \frac{k-1}{4}W_{n+2} + \frac{-5k+5}{4}W_{n+1} + (2k-1)W_n$.

Proof. iv. If Binet formulas are used for proofs, we get

$$-(5k+3)D_{n+2} + (17k+10)D_{n+1} - (13k+7)D_n = -(5k+3)\left((6+n)2^{n+1}-1\right)$$

$$+(17k+10)\left((5+n)2^n-1\right) - (13k+7)\left((4+n)2^{n-1}-1\right)$$

$$= \left(-k+\frac{k+1}{2}n\right)2^n + k = \mathcal{B}_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 3.4. *Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:*

(i) $R_n = \frac{k^2-10k+4}{k^3-4k^2+4k}\mathcal{S}_{k,n+2} + \frac{-6k^2+38k-16}{k^3-4k^2+4k}\mathcal{S}_{k,n+1} + \frac{6k^2-32k+16}{k^3-4k^2+4k}\mathcal{S}_{k,n}$,

$$R_n = \frac{k^2-10k+9}{k^3-6k^2+9k}\mathcal{O}_{k,n+2} + \frac{-6k^2+42k-36}{k^3-6k^2+9k}\mathcal{O}_{k,n+1} + \frac{6k^2-38k+36}{k^3-6k^2+9k}\mathcal{O}_{k,n},$$

$$R_n = -\frac{k^2+4k+1}{k^3+2k^2+k}\mathcal{B}_{k,n+2} + \frac{6k^2+16k+4}{k^3+2k^2+k}\mathcal{B}_{k,n+1} - \frac{6k^2+14k+4}{k^3+2k^2+k}\mathcal{B}_{k,n},$$

(ii) $D_n = \frac{-k^2-3k+4}{k^3-4k^2+4k}\mathcal{S}_{k,n+2} + \frac{k^2+15k-16}{k^3-4k^2+4k}\mathcal{S}_{k,n+1} + \frac{k^2-16k+16}{k^3-4k^2+4k}\mathcal{S}_{k,n}$,

$$D_n = \frac{-k^2-2k+9}{k^3-6k^2+9k}\mathcal{O}_{k,n+2} + \frac{k^2+15k-36}{k^3-6k^2+9k}\mathcal{O}_{k,n+1} + \frac{k^2-19k+36}{k^3-6k^2+9k}\mathcal{O}_{k,n},$$

$$D_n = \frac{k^2-k-1}{k^3+2k^2+k}\mathcal{B}_{k,n+2} + \frac{-k^2+6k+4}{k^3+2k^2+k}\mathcal{B}_{k,n+1} - \frac{k^2+7k+4}{k^3+2k^2+k}\mathcal{B}_{k,n}.$$

(iii) $W_n = -\frac{4k+4}{k^3-4k^2+4k}\mathcal{S}_{k,n+2} + \frac{-k^2+12k+16}{k^3-4k^2+4k}\mathcal{S}_{k,n+1} - \frac{4k+16}{k^3-4k^2+4k}\mathcal{S}_{k,n}$,

$$W_n = -\frac{k+9}{k^3-6k^2+9k}\mathcal{O}_{k,n+2} + \frac{-k^2+3k+36}{k^3-6k^2+9k}\mathcal{O}_{k,n+1} + \frac{4k-36}{k^3-6k^2+9k}\mathcal{O}_{k,n},$$

$$W_n = \frac{-k+1}{k^3+2k^2+k}\mathcal{B}_{k,n+2} + \frac{k^2+3k-4}{k^3+2k^2+k}\mathcal{B}_{k,n+1} + \frac{4}{k^3+2k^2+k}\mathcal{B}_{k,n}.$$

Proof. The proofs are shown similarly to Theorem 3.1. □

In the following theorem, we examine the relationships between the k-Woodall, k-Woodall-Lucas, Modified k-Woodall sequences, and special numbers sequences.

Theorem 3.5. *Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The relationships of the k-Woodall $\mathcal{O}_{k,n}$, k-Woodall-Lucas $\mathcal{S}_{k,n}$, and Modified k-Woodall $\mathcal{B}_{k,n}$, sequences with the Cullen sequence C_n, Cullen-Lucas sequence E_n, and Modified Cullen sequence H_n are as follows:*

i.

(i) $\mathcal{O}_{k,n} = -2C_{n+2} + \frac{k+15}{2}C_{n+1} - \frac{3k+11}{2}C_n$,

(ii) $\mathcal{O}_{k,n} = -\frac{k+4}{3}E_{n+2} + \frac{5k+15}{3}E_{n+1} - \frac{7k+11}{3}E_n$,

(iii) $\mathcal{O}_{k,n} = \frac{5k-4}{3}H_{n+2} + \frac{-19k+15}{3}H_{n+1} + \frac{17k-11}{3}H_n$,

(iv) $C_n = \frac{1}{k^3-6k^2+9k}\left(\left(-k^2+2k-9\right)\mathcal{O}_{k,n+2} + \left(2k^2-6k+36\right)\mathcal{O}_{k,n+1} + (-2k^2+10k-36)\mathcal{O}_{k,n}\right)$,

(v) $E_n = \frac{1}{k^3-6k^2+9k}\left(\left(-k^2-9\right)\mathcal{O}_{k,n+2} + \left(k^2+3k+36\right)\mathcal{O}_{k,n+1} + (-k^2+3k-36)\mathcal{O}_{k,n}\right)$,

(vi) $H_n = \frac{1}{k^3-6k^2+9k}\left(\left(k^2-12k+9\right)\mathcal{O}_{k,n+2} + \left(-7k^2+51k-36\right)\mathcal{O}_{k,n+1} + (7k^2-45k+36)\mathcal{O}_{k,n}\right)$.

ii.

(i) $\mathcal{S}_{k,n} = -\frac{1}{2}C_{n+2} + \frac{k+5}{2}C_{n+1} - \frac{3k+4}{2}C_n$,

(ii) $\mathcal{S}_{k,n} = -\frac{k+1}{3}E_{n+2} + \frac{5k+5}{3}E_{n+1} - \frac{7k+4}{3}E_n$,

(iii) $\mathcal{S}_{k,n} = \frac{5k-1}{3}H_{n+2} + \frac{-19k+5}{3}H_{n+1} + \frac{17k-4}{3}H_n$,

(iv) $C_n = \frac{1}{k^3-4k^2+4k}\left(-\left(k^2+2k+4\right)\mathcal{S}_{k,n+2} + \left(2k^2+6k+16\right)\mathcal{S}_{k,n+1} - (2k^2+16)\mathcal{S}_{k,n}\right)$,

(v) $E_n = \frac{1}{k^3-4k^2+4k}\left(-\left(k^2+5k+4\right)\mathcal{S}_{k,n+2} + \left(k^2+17k+16\right)\mathcal{S}_{k,n+1} - (k^2+8k+16)\mathcal{S}_{k,n}\right)$,

(vi) $H_n = \frac{1}{k^3-4k^2+4k}\left(\left(k^2-13k+4\right)\mathcal{S}_{k,n+2} + \left(-7k^2+49k-16\right)\mathcal{S}_{k,n+1} + (7k^2-40k+16)\mathcal{S}_{k,n}\right)$.

iii.

(i) $\mathcal{B}_{k,n} = -\frac{1}{2}C_{n+2} + \frac{-k+4}{2}C_{n+1} + \frac{3k-3}{2}C_n$,

(ii) $\mathcal{B}_{k,n} = \frac{k-1}{3}E_{n+2} + \frac{-5k+4}{3}E_{n+1} + \frac{7k-3}{3}E_n$,

(iii) $\mathcal{B}_{k,n} = -\frac{5k+1}{3}H_{n+2} + \frac{19k+4}{3}H_{n+1} - \frac{17k+3}{3}H_n$,

(iv) $C_n = \frac{1}{k^3+2k^2+k}\left(\left(k^2+1\right)\mathcal{B}_{k,n+2} - \left(2k^2+4\right)\mathcal{B}_{k,n+1} + (2k^2+2k+4)\mathcal{B}_{k,n}\right)$,

(v) $E_n = \frac{1}{k^3+2k^2+k}\left(\left(k^2-k+1\right)\mathcal{B}_{k,n+2} + \left(-k^2+4k-4\right)\mathcal{B}_{k,n+1} + (k^2-k+4)\mathcal{B}_{k,n}\right)$,

(vi) $H_n = \frac{1}{k^3+2k^2+k}\left(-\left(k^2+5k+1\right)\mathcal{B}_{k,n+2} + \left(7k^2+20k+4\right)\mathcal{B}_{k,n+1}\right.$

$$\left. -(7k^2+17k+4)\mathcal{B}_{k,n}\right).$$

Proof. i. If Binet formulas are used for proofs, we get

$$-2C_{n+2} + \frac{k+15}{2}C_{n+1} - \frac{3k+11}{2}C_n$$

$$= -2((n+2)2^{n+2}+1) + \frac{k+15}{2}((n+1)2^{n+1}+1) - \frac{3k+11}{2}(n2^n+1)$$

$$= -2k((n+2)\,2^{n+2}-1) + \frac{15k+1}{2}((n+1)\,2^{n+1}-1) - \frac{13k+1}{2}(n2^n-1)$$

$$= \left(k-1+\frac{3-k}{2}n\right)2^n - k = \mathcal{O}_{k,n}.$$

The proofs of the others are shown similarly. □

In the following theorems, we calculate the specific identities of k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences.

Theorem 3.6. ***(Cassini Identity)*** *Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. We obtain*

i. $\mathcal{O}_{k,n+1}\mathcal{O}_{k,n-1} - \mathcal{O}_{k,n}^2 = \left(-2C_{n+3} + \frac{k+15}{2}C_{n+2} - \frac{3k+11}{2}C_{n+1}\right)$

$$\left(-2C_{n+1} + \frac{k+15}{2}C_n - \frac{3k+11}{2}C_{n-1}\right) - \left(-2C_{n+2} + \frac{k+15}{2}C_{n+1} - \frac{3k+11}{2}C_n\right)^2,$$

ii. $\mathcal{S}_{k,n+1}\mathcal{S}_{k,n-1} - \mathcal{S}_{k,n}^2 = \left(-\frac{k+1}{3}E_{n+3} + \frac{5k+5}{3}E_{n+2} - \frac{7k+4}{3}E_{n+1}\right)$

$$\left(-\frac{k+1}{3}E_{n+1} + \frac{5k+5}{3}E_n - \frac{7k+4}{3}E_{n-1}\right) - \left(-\frac{k+1}{3}E_{n+2} + \frac{5k+5}{3}E_{n+1} - \frac{7k+4}{3}E_n\right)^2,$$

iii. $\mathcal{B}_{k,n+1}\mathcal{B}_{k,n-1} - \mathcal{B}_{k,n}^2 = \left(-\frac{5k+1}{3}H_{n+3} + \frac{19k+4}{3}H_{n+2} - \frac{17k+3}{3}H_{n+1}\right)$

$$\left(-\frac{5k+1}{3}H_{n+1} + \frac{19k+4}{3}H_n - \frac{17k+3}{3}H_{n-1}\right) - \left(-\frac{5k+1}{3}H_{n+2} + \frac{19k+4}{3}H_{n+1} - \frac{17k+3}{3}H_n\right)^2.$$

Theorem 3.7. ***(Catalan Identity)*** *Let $k \in \mathbb{R}^+$ and $n, r \in \mathbb{N}$. We obtain*

i. $\mathcal{O}_{k,n+r}\mathcal{O}_{k,n-r} - \mathcal{O}_{k,n}^2 = \left(-2C_{n+r+2} + \frac{k+15}{2}C_{n+r+1} - \frac{3k+11}{2}C_{n+r}\right)$

$$\left(-2C_{n-r+2} + \frac{k+15}{2}C_{n-r+1} - \frac{3k+11}{2}C_{n-r}\right) - \left(-2C_{n+2} + \frac{k+15}{2}C_{n+1} - \frac{3k+11}{2}C_n\right)^2,$$

ii. $\mathcal{S}_{k,n+r}\mathcal{S}_{k,n-r} - \mathcal{S}_{k,n}^2 = \left(-\frac{k+1}{3}E_{n+r+2} + \frac{5k+5}{3}E_{n+r+1} - \frac{7k+4}{3}E_{n+r}\right)$

$$\left(-\frac{k+1}{3}E_{n-r+2} + \frac{5k+5}{3}E_{n-r+1} - \frac{7k+4}{3}E_{n-r}\right) - \left(-\frac{k+1}{3}E_{n+2} + \frac{5k+5}{3}E_{n+1} - \frac{7k+4}{3}E_n\right)^2,$$

iii. $\mathcal{B}_{k,n+r}\mathcal{B}_{k,n-r} - \mathcal{B}_{k,n}^2 = \left(-\frac{5k+1}{3}H_{n+r+2} + \frac{19k+4}{3}H_{n+r+1} - \frac{17k+3}{3}H_{n+r}\right)$

$$\left(-\frac{5k+1}{3}H_{n-r+2} + \frac{19k+4}{3}H_{n-r+1} - \frac{17k+3}{3}H_{n-r}\right) - \left(-\frac{5k+1}{3}H_{n+2} + \frac{19k+4}{3}H_{n+1} - \frac{17k+3}{3}H_n\right)$$

Theorem 3.8. ***(D'ocagne's Identity)*** *Let $k \in \mathbb{R}^+$ and $n, r \in \mathbb{N}$. We have*

i. $\mathcal{O}_{k,n+1}\mathcal{O}_{k,r} - \mathcal{O}_{k,n}\mathcal{O}_{k,r+1}$

$$= \left(-2C_{n+3} + \frac{k+15}{2}C_{n+2} - \frac{3k+11}{2}C_{n+1}\right)\left(-2C_{r+2} + \frac{k+15}{2}C_{r+1} - \frac{3k+11}{2}C_r\right)$$

$$- \left(-2C_{n+2} + \frac{k+15}{2}C_{n+1} - \frac{3k+11}{2}C_n\right)\left(-2C_{r+3} + \frac{k+15}{2}C_{r+2} - \frac{3k+11}{2}C_{r+1}\right),$$

ii. $\mathcal{S}_{k,n+1}\mathcal{S}_{k,r} - \mathcal{S}_{k,n}\mathcal{S}_{k,r+1}$

$$= \left(-\frac{k+1}{3}E_{n+3} + \frac{5k+5}{3}E_{n+2} - \frac{7k+4}{3}E_{n+1}\right)\left(-\frac{k+1}{3}E_{r+2} + \frac{5k+5}{3}E_{r+1} - \frac{7k+4}{3}E_r\right)$$

$$- \left(-\frac{k+1}{3}E_{n+2} + \frac{5k+5}{3}E_{n+1} - \frac{7k+4}{3}E_n\right)\left(-\frac{k+1}{3}E_{r+3} + \frac{5k+5}{3}E_{r+2} - \frac{7k+4}{3}E_{r+1}\right),$$

iii. $\mathcal{B}_{k,n+1}\mathcal{B}_{k,r} - \mathcal{B}_{k,n}\mathcal{B}_{k,r+1}$

$$= \left(-\frac{5k+1}{3}H_{n+3} + \frac{19k+4}{3}H_{n+2} - \frac{17k+3}{3}H_{n+1}\right)\left(-\frac{5k+1}{3}H_{r+2} + \frac{19k+4}{3}H_{r+1} - \frac{17k+3}{3}H_r\right)$$

$$- \left(-\frac{5k+1}{3}H_{n+2} + \frac{19k+4}{3}H_{n+1} - \frac{17k+3}{3}H_n\right)\left(-\frac{5k+1}{3}H_{r+3} + \frac{19k+4}{3}H_{r+2} - \frac{17k+3}{3}H_{r+1}\right).$$

Theorem 3.9. ***(Vajda's Identity)*** *Let $k \in \mathbb{R}^+$ and $n, i, j \in \mathbb{N}$. We have*

i. $\mathcal{O}_{k,n+i}\mathcal{O}_{k,n+j} - \mathcal{O}_{k,n}\mathcal{O}_{k,n+i+j}$

$$= \left(-2C_{n+i+2} + \frac{k+15}{2}C_{n+i+1} - \frac{3k+11}{2}C_{n+i}\right)\left(-2C_{n+j+2} + \frac{k+15}{2}C_{n+j+1} - \frac{3k+11}{2}C_{n+j}\right)$$

$$-(-2C_{n+2} + \frac{k+15}{2}C_{n+1} - \frac{3k+11}{2}C_n)(-2C_{n+i+j+2} + \frac{k+15}{2}C_{n+i+j+1} - \frac{3k+11}{2}C_{n+i+j}),$$

ii. $\mathcal{S}_{k,n+i}\mathcal{S}_{k,n+j} - \mathcal{S}_{k,n}\mathcal{S}_{k,n+i+j}$

$$= \left(-\frac{k+1}{3}E_{n+i+2} + \frac{5k+5}{3}E_{n+i+1} - \frac{7k+4}{3}E_{n+i}\right)\left(-\frac{k+1}{3}E_{n+j+2} + \frac{5k+5}{3}E_{n+j+1} - \frac{7k+4}{3}E_{n+j}\right)$$

$$-(-\frac{k+1}{3}E_{n+2} + \frac{5k+5}{3}E_{n+1} - \frac{7k+4}{3}E_n)(-\frac{k+1}{3}E_{n+i+j+2} + \frac{5k+5}{3}E_{n+i+j+1} - \frac{7k+4}{3}E_{n+i+j}),$$

iii. $\mathcal{B}_{k,n+i}\mathcal{B}_{k,n+j} - \mathcal{B}_{k,n}\mathcal{B}_{k,n+i+j} = \left(-\frac{5k+1}{3}H_{n+i+2} + \frac{19k+4}{3}H_{n+i+1} - \frac{17k+3}{3}H_{n+i}\right)$

$$\left(-\frac{5k+1}{3}H_{n+j+2} + \frac{19k+4}{3}H_{n+j+1} - \frac{17k+3}{3}H_{n+j}\right) - (-\frac{5k+1}{3}H_{n+2} + \frac{19k+4}{3}H_{n+1} - \frac{17k+3}{3}H_n)$$

$$(-\frac{5k+1}{3}H_{n+i+j+2} + \frac{19k+4}{3}H_{n+i+j+1} - \frac{17k+3}{3}H_{n+i+j}).$$

The proofs of Theorems 3.6-3.9 are shown using Theorem 3.5.

In the following theorem, we associate the terms of the k-Woodall $\mathcal{O}_{k,n}$, k-Woodall-Lucas $\mathcal{S}_{k,n}$, and Modified k-Woodall $\mathcal{B}_{k,n}$ sequences with matrices.

Theorem 3.10. *Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:*

i. For the k-Woodall sequence,

$$\begin{bmatrix} \mathcal{O}_{k,n+2} \\ \mathcal{O}_{k,n+1} \\ \mathcal{O}_{k,n} \end{bmatrix} = \begin{bmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 8-k \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \mathcal{O}_{k,n+2} \\ \mathcal{O}_{k,n+1} \\ \mathcal{O}_{k,n} \end{bmatrix} = \begin{bmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{O}_{k,n+1} \\ \mathcal{O}_{k,n} \\ \mathcal{O}_{k,n-1} \end{bmatrix},$$

and

$$\begin{bmatrix} \mathcal{O}_{k,n} \\ \mathcal{O}_{k,n+1} \\ \mathcal{O}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -8 & 4 \end{bmatrix}^n \begin{bmatrix} -1 \\ 1 \\ 8-k \end{bmatrix}.$$

ii. For the k-Woodall-Lucas sequence,

$$\begin{bmatrix} \mathcal{S}_{k,n+2} \\ \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n} \end{bmatrix} = \begin{bmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 12-k \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{k,n+2} \\ \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n} \end{bmatrix} = \begin{bmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n} \\ \mathcal{S}_{k,n-1} \end{bmatrix},$$

and

$$\begin{bmatrix} \mathcal{S}_{k,n} \\ \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -8 & 4 \end{bmatrix}^n \begin{bmatrix} 1 \\ 4 \\ 12-k \end{bmatrix}.$$

iii. For the Modified k-Woodall sequence,

$$\begin{bmatrix} \mathcal{B}_{k,n+2} \\ \mathcal{B}_{k,n+1} \\ \mathcal{B}_{k,n} \end{bmatrix} = \begin{bmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathcal{B}_{k,n+2} \\ \mathcal{B}_{k,n+1} \\ \mathcal{B}_{k,n} \end{bmatrix} = \begin{bmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{B}_{k,n+1} \\ \mathcal{B}_{k,n} \\ \mathcal{B}_{k,n-1} \end{bmatrix},$$

and

$$\begin{bmatrix} \mathcal{B}_{k,n} \\ \mathcal{B}_{k,n+1} \\ \mathcal{B}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -8 & 4 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ k+4 \end{bmatrix}.$$

Proof. The proofs are shown by the induction method using definitions of sequences and properties of multiplication in matrices. □

Theorem 3.11. ***(Simpson Formulas)*** *Let $k \in \mathbb{R}^+$ and $t \in \mathbb{N}$. The following equations are satisfied.*

$$i.\ det \begin{bmatrix} \mathcal{O}_{k,t+2} & \mathcal{O}_{k,t+1} & \mathcal{O}_{k,t} \\ \mathcal{O}_{k,t+1} & \mathcal{O}_{k,t} & \mathcal{O}_{k,t-1} \\ \mathcal{O}_{k,t} & \mathcal{O}_{k,t-1} & \mathcal{O}_{k,t-2} \end{bmatrix} = 4^{t-2}k(k-3)^2,$$

$$ii.\ det \begin{bmatrix} \mathcal{S}_{k,t+2} & \mathcal{S}_{k,t+1} & \mathcal{S}_{k,t} \\ \mathcal{S}_{k,t+1} & \mathcal{S}_{k,t} & \mathcal{S}_{k,t-1} \\ \mathcal{S}_{k,t} & \mathcal{S}_{k,t-1} & \mathcal{S}_{k,t-2} \end{bmatrix} = 4^{t-2}k(k-2)^2,$$

$$iii.\ det \begin{bmatrix} \mathcal{B}_{k,t+2} & \mathcal{B}_{k,t+1} & \mathcal{B}_{k,t} \\ \mathcal{B}_{k,t+1} & \mathcal{B}_{k,t} & \mathcal{B}_{k,t-1} \\ \mathcal{B}_{k,t} & \mathcal{B}_{k,t-1} & \mathcal{B}_{k,t-2} \end{bmatrix} = -4^{t-2}k(k+1)^2.$$

Proof. The proofs are shown by the induction method using definition and determinant properties. □

4 Conclusions

In this study, we defined the k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences. Then, we showed the features of these sequences. In addition, we examined the relationships between the terms of these sequences. Moreover, we associated k-Woodall, k-Woodall-Lucas, and Modified k-Woodall sequences with Cullen, Cullen-Lucas, and Modified Cullen numbers. Finally, we associate the terms of these sequences with matrices. If this study is examined, such features can be found in other sequences such as Fermat, and Mersenne sequences.

References

[1] de Oliveira, R. R., Alves, F. R. V., An investigation of the bivariate complex fibonacci polynomials supported in didactic engineering: an application of Theory of Didactics Situations (TSD)*, Acta Scientiae,* **21***, 170-195, (2019).*

[2] Aydınyüz, S., Aşcı, M., The Moore-Penrose Inverse of the Rectangular Fibonacci Matrix and Applications to the Cryptology*, Advances and Applications in Discrete Mathematics, **40**, 195-211, (2023). DOI: 10.17654/0974165823066.*

[3] Avazzadeh, Z., Hassani, H., Agarwal, P., Mehrabi, S., Javad Ebadi, M., Hosseini Asl, M. K., Optimal study on fractional fascioliasis disease model based on generalized Fibonacci polynomials, *Mathematical Methods in the Applied Sciences, **46**, 9332-9350, (2023). DOI:10.1002/mma.9057.*

[4] Turner, H. A., Humpage, M., Kerp, H., Hetherington, A. J., Leaves and sporangia developed in rare non-Fibonacci spirals in early leafy plants, *Science, **380(6650)**, 1188-1192, (2023). DOI: 10.1126/science.adg4014.*

[5] Otto, H. H., Fibonacci Stoichiometry and Superb Performance of Nb16W5O55 and Related Super-Battery Materials, *Journal of Applied Mathematics and Physics, **10**, 1936-1950, (2022). DOI: 10.4236/jamp.2022.106133.*

[6] Akkuş, H., Özkan, E., A New Approach to Olivier Sequences, *Mathematica Montisnigri, **61**, 14-24, (2024). DOI: 10.20948/mathmontis-2024-61-2*

[7] Özkan, E., Akkuş, H., Generalized Bronze Leonardo sequence, *Notes on Number Theory and Discrete Mathematics, **30**, 811-824, (2024). DOI: 10.7546/nntdm.2024.30.4.811-824*

[8] Özkan, E., Akkuş, H., A New Approach to k-Oresme and k-Oresme-Lucas Sequences, *Symmetry, **16**, 1-12 (2024). DOI: 10.3390/sym16111407*

[9] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, *Earthline Journal of Mathematical Sciences, **11**, 23-114, (2023) DOI: https://doi.org/10.34198/ejms.11123.23114*

[10] Özkan, E., Şen, B., Akkuş, H., Uysal, M., A Study on the k-Mersenne and k-Mersenne-Lucas Sequences, *Universal Journal of Mathematics and Applications, **8**, 1-7, (2025). DOI: 10.32323/ujma.1566270*

[11] Özkan, E., Akkuş, H., Hyperbolic k-Oresme and k-Oresme-Lucas Quaternions, *Proof, **4**, 141-149, (2024). DOI: 10.37394/232020.2024.4.14*

[12] Akkuş, H., Üregen, R. N., Özkan E., A New Approach to Jacobsthal Lucas Sequences, *Sakarya University Journal of Science, **25**, 969-973, (2021). DOI: https://doi.org/10.16984/saufenbilder.842489*

[13] Sparavigna, A. C., The groupoids of Mersenne, Fermat, Cullen, Woodall and other Numbers and their representations by means of integer sequences, *Hal Science, 1-10, (2019)*

[14] Özkan, E., Akkuş, H., Özkan, A., Properties of Generalized Bronze Fibonacci Sequences and their Hyperbolic Quaternions, *Axioms, **14**, 14, (2025). DOI: 10.3390/axioms14010014*

[15] Çelik, S., Durukan, İ., Özkan, E., New Recurrences on Pell Numbers, Pell Lucas Numbers, Jacobsthal Numbers, and Jacobsthal Lucas Numbers, *Chaos, Solitons and Fractals, **150**, 111173 (2021). DOI:10.1016/j.chaos.2021.111173*

[16] Akkuş, H., Özkan, E., Hyperbolic (s,t)-Fibonacci and (s,t)-Lucas Quaternions, *Proof, **4**, 97-105, (2024). DOI: 10.37394/232020.2024.4.9*

[17] Özkan, E., Akkus, H., On K-Chebsyhev Sequence, *WSEAS Transactions on Mathematics, **22**, 503-507, (2023). DOI: 10.37394/23206.2023.22.56*

[18] Özkan, E., Akkus, H., On Generalizations of Dickson k-Fibonacci Polynomials, *WSEAS Transactions on Mathematics, **23**, 856-862, (2024). DOI: 10.37394/23206.2024.23.88*

[19] Shanmuganandham, P., Deepika, T., Sum of Squares of 'm'Consecutive Woodall Numbers, *Baghdad Science Journal, **20**, 0345-0345 (2023). DOI: https://doi.org/10.1155/2021/4249509*

[20] Shannon, A. G., Akkuş, H., Aküzüm, Y., Deveci, Ö., Özkan, E., A partial recurrence Fibonacci link, *Notes on Number Theory and Discrete Mathematics, **30**, 530-537, (2024). DOI: 10.7546/nntdm.2024.30.3.530-537*

[21] Tastan, M., Özkan, E., Shannon, A. G., The generalized k-Fibonacci polynomials and generalized k-Lucas polynomials, *Notes on Number Theory and Discrete Mathematics, **27** 148-158 (2021). DOI: 10.7546/nntdm.2021.27.2.148-158*

[22] Daoui, A., Boussayoud, A., Boubellouta, K., Complex bivariate Fibonacci and Lucas polynomials and new generating functions with some special numbers, *Palestine Journal of Mathematics, **13**, 593-610, (2024).*

[23] Marques, D., Chaves, A. P., Fibonacci s-Cullen and s-Woodall Numbers, *Journal of Integer Sequences, **18**, 1-4, (2015).*

[24] Belaggoun, N., Belbachir, H., Sums of the Powers of Bi-Periodic Fibonacci and Lucas Numbers, *Palestine Journal of Mathematics, **13**, 415-427, (2024).*

[25] Kapoor, S., Kumar, P., On Narayana-Lucas Hybrinomial Sequence, *Palestine Journal of Mathematics, **13**, 865-872, (2024).*

[26] Marques, D., On Generalized Cullen and Woodall Numbers That are Also Fibonacci Numbers, *Journal of Integer Sequences, **17**, 14-9, (2014).*

[27] Falcon, S., On the k-Lucas numbers, *International Journal of Contemporary Mathematical Sciences, **6**, 1039-1050, (2011).*

[28] Bilu, Y., Marques, D., Togbé, A., Generalized Cullen numbers in linear recurrence sequences*, Journal of Number Theory, **202**, 412-425, (2019).*

[29] Soykan, Y., İrge, V., Generalized Woodall Numbers: An Investigation of Properties of Woodall and Cullen Numbers via Their Third Order Linear Recurrence Relations*, Universal Journal of Mathematics and Applications, **5**, 69-81, (2022).*

[30] Soykan, Y., Summing formulas for generalized Tribonacci numbers*, Universal Journal of Mathematics and Applications, **3**, 1-11, (2020). DOI: https://doi.org/10.32323/ujma.637876*

[31] Soykan, Y., Generalized tribonacci polynomials*, Earthline Journal of Mathematical Sciences, **13**, 1-120, (2023).*

[32] Soykan, Y., A study on generalized (r, s, t)-numbers*, MathLAB Journal, **7**, 101-129, (2020).*

[33] Falcon, S., Plaza, A., On the Fibonacci k-numbers*, Chaos Solitons Fractals, **32**, 1615–1624 (2007). DOI: https://doi.org/10.1016/j.chaos.2006.09.022*

[34] Akkuş, H., Özkan, E., Generalization of the k-Leonardo Sequence and their Hyperbolic Quaternions*, Mathematica Montisnigri, **60**, 14-31, (2024). DOI: 10.20948/mathmontis-2024-60-2*

[35] Özkan, E., Akkuş, H., Copper ratio obtained by generalizing the Fibonacci sequence*, AIP Advances, **14**, 1-11, (2024). DOI: 10.1063/5.0207147*

[36] Falcon, S., On the k-Jacobsthal numbers*, American Review of Mathematics and Statistics, **2**, 67-77, (2014).*

[37] Faye, M. N., Rihane, S. E., Togbé, A., On Repdigits Which are Sums or Differences of Two k-Pell Numbers*, Mathematica Slovaca, **73**, 1409-1422, (2023). DOI: https://doi.org/10.1515/ms-2023-0102*

Author information

Hakan Akkuş, Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, Türkiye.
E-mail: hakan.akkus@ogr.ebyu.edu.tr

Engin Özkan, Department of Mathematics, Faculty of Sciences, Marmara University, Göztepe Campus, 34722, İstanbul, Türkiye.
E-mail: engin.ozkan@marmara.edu.tr

Received: 2024-11-03
Accepted: 2025-03-20