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A New Approach to Dickson k-Lucas Polynomials

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Abstract: In this study, we define a Dickson *k*-Lucas polynomial inspired by Dickson polynomials and give some terms of these polynomials. Then we examine the relations between the terms of these polynomials. Also, we find Binet formulas, special summation formulas, generating functions for these polynomials. In addition, we obtain the relationships between the roots of the characteristic equation of these sequences and general terms of these sequences. What is interesting here is that the relations obtained from the relationship between the roots of the characteristic equation of these new sequences and the terms of the sequences are satisfied in both roots. Moreover, we calculate some important identities like Cassini identity, Halton identity, Gelin Cesaro identity. Furthermore, Catalan transformation is applied to these polynomials, and their terms are found. Finally, the Hankel transform is applied to the Catalan transform of these polynomials, and the results obtained are associated with Dickson *k*-Lucas polynomials.

Keywords: Lucas number; Generating function; Dickson polynomials; Cassini Identity; Catalan number.

Mathematics Subject Classification: 11B39, 11B83, 05A15

1 Introduction

Fibonacci and Lucas sequences are famous number sequences. These sequences have intrigued scientists for a long time. Fibonacci and Lucas sequences have been applied to various fields such as Algebraic Coding Theory, Physics, Phylotaxis, Biomathematics, Computer Science, Chemistry, etc. New sequences are obtained by changing the recurrence relation and initial conditions of the generalized Fibonacci sequence. The known examples of such sequences are the Horadam, *k*-Pell, *k*-Chebsyhev sequence, Gaussian Fibonacci, Oresme numbers, *k*-Jacobsthal-Lucas, Perrin, Narayana sequences, etc. (see for details in [1-8]).

For $n \in \mathbb{N}$, the Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ and } L_{n+2} = L_{n+1} + L_n,$$

with the initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$.

For F_n and L_n, the Binet formulas are given by the following relations, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n.$$

Here $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation, $r^2 - r - 1 = 0$. The number α is the known golden ratio.

With the help of the recurrence relation of the Fibonacci sequence, *k*-sequences have been introduced, and these sequences have an important place in number theory [9].

For $n \in \mathbb{N}$ and $k \in \mathbb{R}$, the k-Fibonacci sequence f_n and k-Lucas sequence l_n are defined by the recurrence relations, respectively,

$$f_{n+2} = kf_{n+1} + f_n, \text{ and } l_{n+2} = kl_{n+1} + l_n,$$

with the initial conditions $f_0 = 0, f_1 = 1$ and $l_0 = 2, l_1 = k$.

For f_n and l_n, the Binet formulas are given by the following relations, respectively,

$$f_n = \frac{\delta^n - \theta^n}{\delta - \theta} \text{ and } l_n = \delta^n + \theta^n.$$

Here $\delta = \frac{k+\sqrt{k^2+4}}{2}$ and $\theta = \frac{k-\sqrt{k^2+4}}{2}$ are the roots of the characteristic equation, $s^2 - ks - 1 = 0$.

El-Mikkawy and Sogabe obtained many features related to the *k*-Fibonacci sequence [10]. Also, Godase and Dhakne showed new properties of *k*-Fibonacci and *k*-Lucas sequences, and they found correlations

between these sequences [11]. In [12], Falcon and Plaza obtained new properties by applying different transformations to the k-Fibonacci sequence. In addition, Falcon worked on the k-Fibonacci difference sequence [13]. Moreover, Falcon in another study, found many new formulas on *k*-Fibonacci and *k*-Lucas sequences [14].

Dickson polynomials were introduced by L. E. Dickson (1897). For integer $n \geq 2$ and α with identity in a commutative ring R, Dickson polynomials first kind $D_n(x,\alpha)$ and Dickson polynomials second kind $E_n(x,\alpha)$ are defined by recurrence relations, respectively,

$$D_n(x,\alpha) = xD_{n-1}(x,\alpha) - \alpha D_{n-2}(x,\alpha) \quad \text{and} \quad E_n(x,\alpha) = xE_{n-1}(x,\alpha) - \alpha E_{n-2}(x,\alpha)$$

with the initial conditions $D_0 = 2$, $D_1 = x$ and $E_0 = 1$, $E_1 = x$.

In addition, Dickson polynomials attracted the attention of scientists, and they did a lot of work on these polynomials (see for details in [15-17]).

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the Dickson polynomial. We call these polynomials the Dickson *k*-Lucas polynomial and denote them as $D\mathcal{L}_{k,n}(x)$.

We separate the article into three parts.

In chapter 2, we define the Dickson *k*-Lucas polynomials. Also, we give the characteristic equation, the Binet formulas, and some properties for these polynomials. Then, we examine the relationship between these polynomials. In addition, we calculate the special identities of these sequences. Moreover, we obtain the special generating functions and special summation formulas for these polynomials.

In chapter 3, we define the Catalan transformation of the Dickson *k*-Lucas polynomials and some properties are given. In addition, we obtain the Catalan generating functions of these polynomials. Then Hankel transformations are applied to the Catalan transformations of the Dickson k-Lucas polynomials and the results are associated with Dickson k-Lucas polynomials.

2 Dickson *k*-Lucas Polynomials

In this chapter, we define the Dickson *k*-Lucas polynomials. Also, we give the characteristic equation, the

Binet formulas, and some properties for these polynomials. Then, we examine the relationship between these polynomials. In addition, we calculate the special identities of these sequences. Moreover, we obtain the special generating functions and special summation formulas for these polynomials.

Definition 2.1. For $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the Dickson *k*-Lucas $D\mathcal{L}_{k,n}(x)$ is defined by the recurrence relation:

$$D\mathcal{L}_{k,n+2}(x) = kxD\mathcal{L}_{k,n+1}(x) + D\mathcal{L}_{k,n}(x),$$

with $D\mathcal{L}_{k,0}(x) = 2$ and $D\mathcal{L}_{k,1}(x) = kx$.

The characteristic equation of these polynomials is $r^2 - kxr - 1 = 0$. The roots of this equation are as follows;

$$r_1 = \frac{kx+\sqrt{k^2x^2+4}}{2} \text{ and } r_2 = \frac{kx-\sqrt{k^2x^2+4}}{2}.$$

The relationship between these roots is given with

$$r_1 + r_2 = kx,\ r_1 - r_2 = \sqrt{k^2x^2+4},\ r_1^2 + r_2^2 = k^2x^2 + 2 \text{ and } r_1r_2 = -1.$$

Some values for $D\mathcal{L}_{k,n}(x)$ are given below,

$D\mathcal{L}_{k,0}(x) = 2,\ D\mathcal{L}_{k,1}(x) = kx,\ D\mathcal{L}_{k,2}(x) = k^2x^2 + 2,\ D\mathcal{L}_{k,3}(x) = k^3x^3 + 3kx,$
$D\mathcal{L}_{k,4}(x) = k^4x^4 + 4k^2x^2 + 2,\ D\mathcal{L}_{k,5}(x) = k^5x^5 + 5k^3x^3 + 5kx.$

Also, the terms of these polynomials can be found with the help of the following relation. Let $n \in \mathbb{N}^+$:

$$D\mathcal{L}_{k,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \frac{n}{n-i} (kx)^{n-2i}.$$

In the following theorem, we express the Binet formula of the Dickson *k*-Lucas polynomials $D\mathcal{L}_{k,n}$.

Theorem 2.1. Let $n \in \mathbb{N}$. We obtain

$$D\mathcal{L}_{k,n}(x) = r_1^n + r_2^n.$$

Proof. The Binet form of a sequence is as follows

$$D\mathcal{L}_{k,n}(x) = c{r_1}^n + d{r_2}^n.$$

Here, the scalars c and d can be obtained by substituting the initial conditions. Now let's solve the system of equations. For $n = 0$, $D\mathcal{L}_{k,0} = 2$ and for $n = 1$, $D\mathcal{L}_{k,1} = kx$. Thus, we obtain $c = 1$ and $d = 1$. Thus,

$$D\mathcal{L}_{k,n}(x) = r_1^n + r_2^n. \qquad \square$$

In the following theorem, it is seen that $D\mathcal{L}_{k,n}(x)$ has some important relations with roots of the characteristic equation for these polynomials. Please note that these relations are independent of the choice of roots.

Theorem 2.2. Let $n \in \mathbb{N}$ and $y = r_1$ or $y = r_2$. We obtain

i. $y^{2n} = y^n D\mathcal{L}_{k,n}(x) - (-1)^n$,

ii.. $kxy + y^{2(2^{n+1}+1)} + 1 = D\mathcal{L}_{k,2^{n+1}}(x) y^{2(2^n+1)}$.

Proof. i. For $y = r_1$, we have

$$y^a D\mathcal{L}_{k,a}(x) - (-1)^a = {r_1}^a (r_1^a + r_2^a) - (r_1 r_2)^a = {r_1}^{2a}.$$

For $y = r_2$, we have

$$y^a D\mathcal{L}_{k,a}(x) - (-1)^a = {r_2}^a (r_1^a + r_2^a) - (r_1 r_2)^a = {r_2}^{2a}.$$

The proof of the other may be found similarly. □

In the following theorem, we examine the relationship between of the Dickson *k*-Lucas polynomials.

Theorem 2.3. Let $m, n \in \mathbb{N}$, $m > n$ and $k \in \mathbb{R}^+$. We obtain

i. $D\mathcal{L}_{k,m}(x)D\mathcal{L}_{k,n}(x) = D\mathcal{L}_{k,m+n}(x) + (-1)^n D\mathcal{L}_{k,m-n}(x)$,

ii. $(-1)^{2m+1}D\mathcal{L}_{k,2n-2m}(x) + D\mathcal{L}_{k,2m+2n+2}(x) = D\mathcal{L}_{k,2m+1}(x)D\mathcal{L}_{k,2n+1}(x)$,

iii. $D\mathcal{L}_{k,n+3}(x)D\mathcal{L}_{k,n}^2(x) - D\mathcal{L}_{k,n+1}^3(x)$

$$= (-1)^{n+3}D\mathcal{L}_{k,n-3}(x) - 3(-1)^{n+1}D\mathcal{L}_{k,n+1}(x) + D\mathcal{L}_{k,n+3}(x)2(-1)^n,$$

iv. $(-1)^n D\mathcal{L}_{k,-n}(x) = D\mathcal{L}_{k,n}(x)$.

Proof. i. If the Binet formula is used, we obtain

$$D\mathcal{L}_{k,m}(x)D\mathcal{L}_{k,n}(x) = (r_1^m + r_2^m)(r_1^n + r_2^n) = (r_1^{m+n} + r_2^{m+n}) + r_1^n r_2^n (r_1^{m-n} + r_2^{m-n})$$

$$= D\mathcal{L}_{k,m+n}(x) + (-1)^n D\mathcal{L}_{k,m-n}(x).$$

The proof of the other may be found similarly. □

In the following theorems, we calculate the some identities of the Dickson *k*-Lucas polynomials.

Theorem 2.4. ***(Cassini Identity)*** Let $n \in \mathbb{N}$. We obtain

$$D\mathcal{L}_{k,n-1}(x)D\mathcal{L}_{k,n+1}(x) - D\mathcal{L}_{k,n}^2(x) = (-1)^{n-1}(k^2x^2 + 4).$$

Proof. If the Binet formula is used, we obtain

$$D\mathcal{L}_{k,n-1}(x)D\mathcal{L}_{k,n+1}(x) - D\mathcal{L}_{k,n}^2(x) = (r_1^{n-1} + r_2^{n-1})(r_1^{n+1} + r_2^{n+1}) - (r_1^n + r_2^n)^2$$

$$= r_1^{2n} + r_1^{n-1}r_2^{n+1} + r_1^{n+1}r_2^{n-1} + r_2^{2n} - r_1^{2n} - 2r_1^n r_2^n - r_2^{2n}$$

$$= r_1^n r_2^n \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2\right)$$

$$= (-1)^{n-1}(k^2x^2 + 4).$$ □

Theorem 2.5. ***(Halton Identity)*** Let $n, i, j \in \mathbb{N}$, and $i, j < n$. We obtain

$$D\mathcal{L}_{k,n-i}(x)D\mathcal{L}_{k,n+i}(x) - D\mathcal{L}_{k,n+j}(x)D\mathcal{L}_{k,n-j}(x)$$

$$= (-1)^{n-i-j}[(-1)^j D\mathcal{L}_{k,2i}(x) + (-1)^i D\mathcal{L}_{k,2j}(x)].$$

Theorem 2.6. ***(Gelin-Cesaro Identity)*** Let $n \in \mathbb{N}$. We obtain

$$D\mathcal{L}_{k,n-2}(x)D\mathcal{L}_{k,n-1}(x)D\mathcal{L}_{k,n+2}(x)D\mathcal{L}_{k,n+1}(x) - D\mathcal{L}_{k,n}^4(x) = (-1)^{n-2}D\mathcal{L}_{k,2n+4}(x)$$

$$+(-1)^{n-1}D\mathcal{L}_{k,2n+2}(x) + (-1)^{2n-3}D\mathcal{L}_{k,6}(x) + (-1)^{n+1}D\mathcal{L}_{k,2n-2}(x) + (-1)^{2n-1}D\mathcal{L}_{k,2}(x)$$

$$+(-1)^{n+2}D\mathcal{L}_{k,2n-4}(x) - 4(-1)^n D\mathcal{L}_{k,2n}(x) - 5.$$

The proofs of Theorem 2.5., and 2.6., are shown using the Binet formulas in a similar way to Theorem 2.4.

In the following theorems, we obtain special sum formulas of the Dickson *k*-Lucas polynomials. In addition, we find binomial sum formulas of these polynomials.

Theorem 2.7. Let $n \in \mathbb{N}$ and $k \in \mathbb{R}^+$. We obtain

$$\sum_{s=0}^{n} D\mathcal{L}_{k,s} = \frac{D\mathcal{L}_{k,n+1} + D\mathcal{L}_{k,n} + kx - 2}{kx}.$$

Proof. If the definition 2.1., is used, we obtain

$$D\mathcal{L}_{k,2}(x) = kxD\mathcal{L}_{k,1}(x) + D\mathcal{L}_{k,0}(x),$$

$$D\mathcal{L}_{k,3}(x) = kxD\mathcal{L}_{k,2}(x) + D\mathcal{L}_{k,1}(x),$$

$$\vdots$$

$$D\mathcal{L}_{k,n}(x) = kxD\mathcal{L}_{k,n-1}(x) + D\mathcal{L}_{k,n-2}(x).$$

So, we have

$$-2 - kx + \sum_{s=0}^{n} D\mathcal{L}_{k,s}(x) = kx \sum_{s=1}^{n-1} D\mathcal{L}_{k,s}(x) + \sum_{s=0}^{n-2} D\mathcal{L}_{k,s}(x),$$

$$-2 - kx + \sum_{s=0}^{n} D\mathcal{L}_{k,s}$$

$$= kx\left(-D\mathcal{L}_{k,n} - D\mathcal{L}_{k,0} + \sum_{s=0}^{n} D\mathcal{L}_{k,s}\right) - \left(-D\mathcal{L}_{k,n} - D\mathcal{L}_{k,n-1} + \sum_{s=0}^{n} D\mathcal{L}_{k,s}\right).$$

Thus, we obtain

$$\sum_{s=0}^{n} D\mathcal{L}_{k,s} = \frac{D\mathcal{L}_{k,n+1} + D\mathcal{L}_{k,n} + kx - 2}{kx}.$$ □

Theorem 2.8. Let $k \in \mathbb{R}^{+}$, $c, p, s, t \in \mathbb{N}$ and $c > s$. We obtain

i. $\sum_{j=0}^{n} \binom{n}{j} (kx)^{j} D\mathcal{L}_{k,j}(x) = D\mathcal{L}_{k,2n}(x),$

ii. $\sum_{j=0}^{n} \binom{n}{j} (kx)^{j} D\mathcal{L}_{k,cn+s+j}(x) = D\mathcal{L}_{k,cn+2n+s}(x),$

iii. $\sum_{j=0}^{n} \binom{n}{j} (-1)^{j} \frac{D\mathcal{L}_{k,cn+s+j}(x)}{(kx)^{j}} = (-1)^{n} \frac{D\mathcal{L}_{k,cn-n+s}(x)}{(kx)^{n}},$

iv. $\sum_{j=0}^{n}(\frac{t}{k})^{j}\left(tD\mathcal{L}_{k,j-s+2}(x)-kxD\mathcal{L}_{k,j-s+1}(x)\right)=(\frac{t}{k})^{n}tD\mathcal{L}_{k,n-s+2}(x)-kD\mathcal{L}_{k,-s+1}(x)$.

Proof. The following equations are obtained with the help of the characteristic equation of the Dickson *k*-Lucas polynomials:

$$r_1^2=kxr_1+1 \text{ and } r_2^2=kxr_2+1.$$

i. By employing the binomial theorem, we obtain

$$\sum_{j=0}^{n}\binom{n}{j}(kx)^{j}D\mathcal{L}_{k,j}(x)=\sum_{j=0}^{n}\binom{n}{j}(kx)^{j}(r_1^{j}+r_2^{j}),$$

$$=\sum_{j=0}^{n}\binom{n}{j}(kxr_1)^{j}+(kxr_2)^{j},$$

$$=(1+kxr_1)^{n}+(1+kxr_2)^{n},$$

$$=(r_1^2)^{n}+(r_2^2)^{n},$$

$$=r_1^{2n}+r_2^{2n},$$

$$=D\mathcal{L}_{k,2n}(x).$$

The proof of the other may be found similarly. □

In the following theorem, we give special generating function of the Dickson *k*-Lucas polynomials.

Theorem 2.9. Let $n\in\mathbb{N}$ and $k\in\mathbb{R}^{+}$. We obtain

$$l(t)=\sum_{n=0}^{\infty}D\mathcal{L}_{k,n}(x)t^{n}=\frac{2-txk}{1-tkx-t^2}.$$

Proof. If the definition 2.1., is used, we obtain

$$l(t)=\sum_{n=0}^{\infty}D\mathcal{L}_{k,n}(x)t^{n}=2+kxt+\sum_{n=2}^{\infty}D\mathcal{L}_{k,n}(x)t^{n}$$

$$= 2 + kxt + \sum_{n=2}^{\infty}[kxD\mathcal{L}_{k,n-1}(x) + D\mathcal{L}_{k,n-2}(x)]t^n$$

$$= 2 + kxt + kx\sum_{n=1}^{\infty} D\mathcal{L}_{k,n}(x)t^n + \sum_{n=0}^{\infty} D\mathcal{L}_{k,n}(x)t^n$$

$$= \frac{2-txk}{1-tkx-t^2}.$$ □

3 Catalan Transformation of the Dickson *k*-Lucas Polynomials

In this chapter, we apply Catalan transformation to these polynomials, and their terms are found. Then we find the generating function of the Catalan transformation of these polynomials. In addition, the Hankel transform is applied to the Catalan transform of these polynomials. Finally, we associate the terms of the Hankel transformation of the Dickson *k*-Lucas polynomials with the terms of the Dickson *k*-Lucas polynomials.

Definition 3.1. (Catalan Number) For $n \in \mathbb{Z}^+$ the n^{th} Catalan numbers are as follows

$$C_n = \frac{C(2n,n)}{n+1}$$

and the generating function is obtained as

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

For some terms natural numbers n. Catalan numbers are $1, 1, 2, 5, 14, 132, 429, 1430, \ldots$ (see for details in [18]).

Using the Catalan transform, we define the Catalan transform of the Dickson *k*-Lucas polynomials, as follows.

$$CD\mathcal{L}_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} D\mathcal{L}_{k,i},\ n \geq 1 \text{ with } CD\mathcal{L}_{k,0} = 0.$$

Now we can give the Catalan transformation of the first elements of the Dickson *k*-Lucas polynomials. The $CD\mathcal{L}_{k,n}$ values for the first four n natural numbers are given below:

$CD\mathcal{L}_{k,0} = 0, CD\mathcal{L}_{k,1} = kx, CD\mathcal{L}_{k,2} = k^2x^2 + kx + 2, CD\mathcal{L}_{k,3} = k^3x^3 + 2k^2x^2 + 5kx + 4,$

$CD\mathcal{L}_{k,4} = k^4x^4 + 3k^3x^3 + 9k^2x^2 + 14kx + 12,$
$CD\mathcal{L}_{k,5} = k^5x^5 + 4k^4x^4 + 14k^3x^3 + 30k^2x^2 + 46kx + 36.$

We can write $CD\mathcal{L}_{k,n}$ as the product of $nx1$ type $D\mathcal{L}_{k,n}$'s of the lower triangular Catalan matrix C:

$$\begin{bmatrix} CD\mathcal{L}_{k,1} \\ CD\mathcal{L}_{k,2} \\ CD\mathcal{L}_{k,3} \\ CD\mathcal{L}_{k,4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} D\mathcal{L}_{k,1} \\ D\mathcal{L}_{k,2} \\ D\mathcal{L}_{k,3} \\ D\mathcal{L}_{k,4} \\ \vdots \end{bmatrix}.$$

So, we obtain

$$\begin{bmatrix} kx \\ k^2x^2 + kx + 2 \\ k^3x^3 + 2k^2x^2 + 5kx + 4 \\ k^4x^4 + 3k^3x^3 + 9k^2x^2 + 14kx + 12 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} kx \\ k^2x^2 + 2 \\ k^3x^3 + 3kx \\ k^4x^4 + 4k^2x^2 + 2 \\ \vdots \end{bmatrix}.$$

3.1 The Generating Functions of the Catalan Transform Dickson *k*-Lucas Polynomials

Let $l(t)$ and $C(t)$ be generating functions of Dickson *k*-Lucas polynomials and Catalan sequence, respectively. $D_{k,n}\big(t * C(t)\big)$, shown the generating function of the Catalan Dickson *k*-Lucas polynomials. The following equations are written for the generating function of the Catalan transform of the Dickson *k*-Lucas polynomials,

$$\ell(t) = CD\mathcal{L}_{k,n}(t) = D\mathcal{L}_{k,n}\big(t * C(t)\big) = \frac{4 - 2kx + 2kx\sqrt{1-4t}}{4 - 2kx - 4t + (2kx+2)\sqrt{1-4t}}.$$

3.2 Hankel Transform of the Catalan Dickson *k*-Lucas Polynomials

Let the terms of a sequence be $A = \{v_1, v_2, v_3, \ldots\}$. In [19], the Hankel transform H_n of the terms of this sequence was defined as follows:

$$H_n = \begin{vmatrix} v_1 & v_2 & v_3 & v_4 & \dots \\ v_2 & v_3 & v_4 & v_5 & \dots \\ v_3 & v_4 & v_5 & v_6 & \dots \\ v_4 & v_5 & v_6 & v_7 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}.$$

Let's apply Hankel's work to the Catalan Dickson *k*-Lucas polynomials, respectively. Its results are associated with the terms of the sequences. We get;

$$HCD\mathcal{L}_1 = det\left[CD\mathcal{L}_{k,1}\right] = det[kx] = kx = D\mathcal{L}_{k,1}$$

$$HCD\mathcal{L}_2 = det\begin{bmatrix} CD\mathcal{L}_{k,1} & CD\mathcal{L}_{k,2} \\ CD\mathcal{L}_{k,2} & CD\mathcal{L}_{k,3} \end{bmatrix} = det\begin{bmatrix} kx & k^2x^2+kx+2 \\ k^2x^2+kx+2 & k^3x^3+2k^2x^2+5kx+4 \end{bmatrix}$$

$$= -4 = -2D\mathcal{L}_{k,0},$$

$$HCD\mathcal{L}_3 = det\begin{bmatrix} CD\mathcal{L}_{k,1} & CD\mathcal{L}_{k,2} & CD\mathcal{L}_{k,3} \\ CD\mathcal{L}_{k,2} & CD\mathcal{L}_{k,3} & CD\mathcal{L}_{k,4} \\ CD\mathcal{L}_{k,3} & CD\mathcal{L}_{k,4} & CD\mathcal{L}_{k,5} \end{bmatrix} = -k^3x^3 - 4k^2x^2 - 8kx - 16$$

$$= -D\mathcal{L}_{k,3} - 4D\mathcal{L}_{k,2} - 5D\mathcal{L}_{k,1} - 7D\mathcal{L}_{k,0}$$

4 Conclusions

In this study, we defined a Dickson *k*-Lucas polynomial inspired by Dickson polynomials and given some terms of these polynomials. Then we presented the relations between the terms of Dickson *k*-Lucas polynomials. Also, we found Binet formulas and generating functions for these polynomials. In addition, we calculated some important identities like Cassini identity, and Halton identity. Moreover, we applied Catalan transformation to these polynomials, and found their terms. Finally, we applied the Hankel transform to the Catalan transform of these polynomials, and the results obtained associated with Dickson *k*-Lucas polynomials. If this study is examined, it is possible to come across such features in the Fermat, Narayana and Mersenne polynomials.

Conflicts of Interest The authors have no conflict of interest.

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