On Gaussian Fibonacci and Gaussian Lucas Hybrid Numbers

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Abstract

In this paper, we examine a new application of hybrid numbers. We take Gaussian Fibonacci and Gaussian Lucas numbers instead of the real components of hybrid numbers. We define the Gaussian Fibonacci and Lucas hybrid numbers. We analyze the properties of these numbers related to hybrid numbers.. We give recurrence relations, the Binet formulas, sum formulas and generating functions of these numbers. We give the relation between these numbers, Fibonacci hybrid and Lucas hybrid numbers. We gave the relation between these numbers, Fibonacci hybrid and Lucas hybrid numbers.We investigate the relations between Gaussian Fibonacci hybrid and Gaussian Lucas hybrid numbers. Finally, we have calculated the identities that are important for sequences of numbers. We give Cassini, Catalan, Vajda, d'Ocagne and Honsberger identities of these numbers.

Keywords: Hybrid numbers, the Gaussian Fibonacci number, the Gaussian Lucas number The Gaussian Fibonacci hybrid number, the Gaussian Lucas hybrid numbers

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1. Introduction

Number sequences have been researched by many scientists until today. In particular, Fibonacci numbers have found application in mathematics and other branches of science.

Many researchers have studied Fibonacci numbers and studied their different generalizations [2,3,6,8,10]. Some of the generalizations made are the Gaussian Fibonacci numbers and Gaussian Lucas numbers [4,7,11,12,13,19].

Definition 1.1. Gaussian Fibonacci numbers are defined by,

$$GF_n = GF_{n-1} + GF_{n-2}, \text{ for } n \geq 2$$

with $GF_0 = i$ and $GF_1 = 1$ [6].

Definition 1.2. Gaussian Fibonacci numbers are defined by,

$$GL_n = GL_{n-1} + GL_{n-2}, \text{ for } n \geq 2$$

with $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$ [6].

The second order characteristic equation of the recurrence relations given according to Definition 1.1 and Definition 1.2 is in the form below:

$$x^2 - x - 1 = 0.$$

The roots of the characteristic equation are a and b.

- $a + b = 1$,
- $ab = -1$.

Definition 1.3. (Binet formula)

$$\boldsymbol{a}.\, GF_n = \frac{a^n\bar{b} - b^n\bar{a}}{a - b},$$

$$\boldsymbol{b}.\, GL_n = a^n\bar{b} + b^n\bar{a}$$

where $\bar{a} = 1 - ai$ and $\bar{b} = 1 - bi$.

Theorem 1.4. We have [4].

$$\boldsymbol{a}.\, GF_n = F_n + F_{n-1}i,$$

$$\boldsymbol{b}.\, GL_n = L_n + L_{n-1}i,$$

$$\boldsymbol{c}.\, GF_{n+1} + GL_{n-1} = GL_n,$$

$$\boldsymbol{d}.\ 5GF_n + GL_n = 2GL_{n+1}.$$

Theorem 1.5. (Sum formula)

$$\boldsymbol{a}.\sum_{\boldsymbol{k=0}}^{\boldsymbol{n}} GF_k = GF_{n+2} - 1,$$

$$\boldsymbol{b}.\sum_{\boldsymbol{k=0}}^{\boldsymbol{n}} GL_k = GL_{n+2} - GL_1 \ \text{[4]}.$$

Later, a new set of numbers was introduced by Özdemir. Hybrid numbers are a new non-commutative number system. Hybrid numbers can be considered as a generalization of complex ($i^2 = -1$) , hyperbolic ($h^2 = 1$) and dual numbers ($\varepsilon^2 = 0$) [9].

A hybrid number $\hbar$ has form as follow

$$\hbar = a + bi + c\varepsilon + dh$$

where, i, ε, h are basis hybrid and a, b, c, d are real numbers [9].
The product rule for hybrid units is shown in Table 2.

.	i	ε	h
i	-1	$1-h$	$\varepsilon+i$
ε	$1+h$	0	$-\varepsilon$
h	$-\varepsilon-i$	ε	1

Table 1. The product rule of the hybrid units

Hybrid numbers have been examined by many researchers and their relations with other numbers has been studied [5,9,14,15,16,17,18,19].

Definition 1.7. Let $z=a+bi+c\varepsilon+dh$ be any hybrid number. There are definitions given below.

1. The conjugate of z is $\bar{z}=a-bi-c\varepsilon-dh$.
2. The character of z *is* $C(z)=z\bar{z}$.
3. The norm of z is $\|z\|=N(z)=\sqrt{C(z)}$.
4. If $C(z)<0$, then z is a spacelike.

Definition 1.8. The Fibonacci and Lucas hiybrid numbers are defined by respectively

$$FH_n=F_n+F_{n+1}i+F_{n+2}\varepsilon+F_{n+3}h$$

and

$$FL_n=L_n+L_{n+1}i+L_{n+2}\varepsilon+L_{n+3}h \text{ [15].}$$

In this work, we define Gaussian Fibonacci and Gaussian Lucas Hybrid Numbers. We investigate of properties of these numbers. We give Binet formula, generating functions and sum formula. We give the relationship between these numbers and the relationship between Fibonacci hybrid and Lucas hybrid numbers. Finally, we calculated Cassini, Catalan, Vajda, d'Ocagne and Honsberger identities of these numbers.

2. Gaussian Fibonacci and Gaussian Lucas Hybrid Numbers

In this chapter, we introduce the Gaussian Fibonacci hybrid numbers and give some properties of them.

Definition 2.1. The Gaussian Fibonacci hybrid numbers are defined by

$$GFH_n=GF_n+GF_{n+1}i+GF_{n+2}\varepsilon+GF_{n+3}h$$

where i,ε,h are hybrid units and GF_n is nth the Gaussian Fibonacci number.

Let's give some terms of the Gaussian Fibonacci hybrid numbers in Table 2.

n	GFH_n
0	$1+3i+2\varepsilon+h$
1	$1+3i+4\varepsilon+2h$
2	$2+6i+6\varepsilon+3h$

3	$3+9i+10\varepsilon+5h$

Table 2. Some values of the Gaussian Fibonacci hybrid numbers

Definition 2.2. The Gaussian Lucas hybrid numbers are defined by

$$GLH_n = GL_n + GL_{n+1}i + GL_{n+2}\varepsilon + GL_{n+3}h$$

where i, ε, h are hybrid units and GL_n is nth Gaussian Lucas number.

Let us give a few terms of Gaussian Lucas hybrid numbers in Table 2.

n	GLH_n
0	$1+3i+6\varepsilon+3h$
1	$3+9i+8\varepsilon+4h$
2	$4+12i+14\varepsilon+7h$
3	$7+21i+22\varepsilon+11h$

Table 3. Some values of Gaussian Fibonacci hybrid numbers

Definition 2.3. The scaler sections of the Gaussian Fibonacci and Lucas hybrid numbers are

$$\boldsymbol{a}.\, S(GFH_n) = GF_n$$

and

$$\boldsymbol{b}.\, S(GLH_n) = GL_n.$$

Definition 2.4. The vector sections of the Gaussian Fibonacci and Lucas hybrid numbers are

$$\boldsymbol{a}.\, V(GFH_n) = GF_{n+1}i + GF_{n+2}\varepsilon + GF_{n+3}h$$

and

$$\boldsymbol{b}.\, V(GLH_n) = GL_{n+1}i + GL_{n+2}\varepsilon + GL_{n+3}h.$$

Definition 2.5. For negative indices, the Gaussian Fibonacci hybrid numbers are defined by

$$GFH_{-n} = (-1)^n i(GF_{n+1} - GF_n i + GF_{n-1}\varepsilon - GF_{n-2}h)$$

where i, ε, h are hybrid units and $\overline{GF_n}$ is the conjugate of nth Gaussian Fibonacci number.

Definition 2.6. For negative indices, the Gaussian Lucas hybrid numbers are defined by

$$GLH_{-n} = (-1)^n i(-GL_{n+1} + GL_n i - GL_{n-1}\varepsilon + GL_{n-2}h)$$

where i, ε, h are hybrid units and GL_n is nth Gaussian Fibonacci number.

Theorem 2.7. The Gaussian Fibonacci and Lucas hybrid numbers provide the recurrence relations given below for $n \geq 2$.

$$\boldsymbol{a}.\ GFH_n = GFH_{n-1} + GFH_{n-2},$$

$$\boldsymbol{b}.\ GLH_n = GLH_{n-1} + GLH_{n-2}.$$

Proof. We use Definition 1.1. for the proof.

$$GFH_n = GF_n + GF_{n+1}i + GF_{n+2}\varepsilon + GF_{n+3}h$$
$$= (GF_{n-1} + GF_{n-2}) + (GF_n + GF_{n-1})i + (GF_{n+1} + GF_n)\varepsilon + (GF_{n+2} + GF_{n+1})h$$
$$= (GF_{n-1} + GF_n i + GF_{n+1}\varepsilon + GF_{n+2}h) + (GF_{n-2} + GF_{n-1}i + GF_n\varepsilon + GF_{n+1}h)$$
$$= GFH_{n-1} + GFH_{n-2}.$$

Thus, the proof is obtained.

The proof of part ***b*** can be shown in a similar way.

Theorem 2.8. The vector sections of the Gaussian Fibonacci and Lucas hybrid numbers provides the following relations.

$$\boldsymbol{a}.\ V(GFH_n) = V(GFH_{n-1}) + V(GFH_{n-2}) \text{ for } n \geq 2,$$
$$\boldsymbol{b}.V(GLH_n) = V(GLH_{n-1}) + V(GLH_{n-2}) \text{ for } n \geq 2.$$

Proof. ***a***. Let's start the proof from the right side of the equation.

$$V(GFH_{n-1}) + V(GFH_{n-2})$$
$$= GF_n i + GF_{n+1}\varepsilon + GF_{n+2}h + GF_{n-1}i + GF_n\varepsilon + GF_{n+1}h$$
$$= (GF_n + GF_{n-1})i + (GF_{n+1} + GF_n)\varepsilon + (GF_{n+2} + GF_{n+1})h, \text{ from Definition 1.1., we have}$$
$$= GF_{n+1}i + GF_{n+2}\varepsilon + GF_{n+3}h$$
$$= V(GFH_n).$$

Thus, the proof is obtained.

The proof of part ***b*** can be shown in a similar way.

Definition 2.9. The conjugate of the of the Gaussian Fibonacci and Gaussian Lucas hybrid numbers are defined by

$$\boldsymbol{a}.\ \overline{GFH_n} = GF_n - GF_{n+1}i - GF_{n+2}\varepsilon - GF_{n+3}h,$$
$$\boldsymbol{b}.\ \overline{GLH_n} = GL_n - GL_{n+1}i - GL_{n+2}\varepsilon - GL_{n+3}h.$$

Theorem 2.10. We have,

$$\boldsymbol{a}.GFH_n + \overline{GFH_n} = 2GF_n,$$
$$\boldsymbol{b}.GLH_n + \overline{GLH_n} = 2GL_n.$$

Proof. ***a***. From Definition 2.6., we have

$$GFH_n + \overline{GFH_n} = GF_n + GF_{n+1}i + GF_{n+2}\varepsilon + GF_{n+3}h + GF_n - GF_{n+1}i - GF_{n+2}\varepsilon - GF_{n+3}h$$
$$= 2GF_n.$$

Thus, the proof is obtained.

The proof is of ***b*** can be shown similarly.

Theorem 2.11. The character of the GFH_n and GLH_n is

$$\boldsymbol{a}.\, C(\, GFH_n) = -5{GF_{n+1}}^2 - 6GF_nGF_{n+1},$$

$$\boldsymbol{b}.\, C(\, GLH_n) = -5{GL_{n+1}}^2 - 6GL_nGL_{n+1}.$$

where GF_n is nth Gaussian Fibonacci numbers and GL_n is nth Gaussian Lucas numbers.

Proof. $\boldsymbol{a}.\, C(\, GFH_n) = GFH_n\, \overline{GFH_n}$

$$= (GF_n + GF_{n+1}i + GF_{n+2}\varepsilon + GF_{n+3}h)(GF_n - GF_{n+1}i - GF_{n+2}\varepsilon - GF_{n+3}h)$$

$$= {GF_n}^2 + {GF_{n+1}}^2 - 2GF_{n+1}GF_{n+2} - {GF_{n+3}}^2$$

From Definition 1.1., we have

$$= {GF_n}^2 + {GF_{n+1}}^2 - 2{GF_{n+1}}^2 - 2GF_{n+1}GF_n - 4{GF_{n+1}}^2 - 4GF_{n+1}GF_n - {GF_n}^2$$

$$= -5{GF_{n+1}}^2 - 6GF_nGF_{n+1}.$$

Thus, the proof is obtained.

Since $C(\, GFH_n) < 0$ for any $n \geq 0$, Gaussian Fibonacci hybrid numbers is spacelike.

The proof is of ***b*** can be shown similarly.

Definition 2.12. The norm of the GFH_n and GLH_n is

$$\boldsymbol{a}.\, N(\, GFH_n) = \sqrt{-5{GF_{n+1}}^2 - 6GF_nGF_{n+1}},$$

$$\boldsymbol{b}.\, N(\, GLH_n) = \sqrt{-5{GL_{n+1}}^2 - 6GL_nGL_{n+1}}.$$

where GF_n is nth Gaussian Fibonacci numbers and GL_n is nth Gaussian Lucas numbers.

Theorem 2.13. For $n \geq 0$, we have

$$\boldsymbol{a}.\; GFH_{n+1} + GFH_{n-1} = GLH_n,$$

$$\boldsymbol{b}.\; GFH_n = FH_n + iFH_{n-1},$$

$$\boldsymbol{c}.\; GLH_n = LH_n + iLH_{n-1},$$

$$\boldsymbol{d}.\; GFH_{n+2} - GFH_{n-2} = GLH_n,$$

$$\boldsymbol{e}.\; 5\, GFH_n + GLH_n = 2\, GLH_{n+1}$$

where FH_n is nth Fibonacci hybrid numbers and LH_n is nth Lucas hybrid numbers.

Proof.

$\boldsymbol{a}.\; GFH_{n-1} + GFH_{n-1}$

$= GF_{n+1} + GF_{n+2}i + GF_{n+2}\varepsilon + GF_{n+3}h + GL_{n-1} + GL_n i + GL_{n+1}\varepsilon + GL_{n+2}h$

$= GF_{n+1} + GL_{n-1} + (GF_{n+2} + GL_n)i + (GF_{n+3} + GL_{n+1})\varepsilon + (GF_{n+4} + GL_{n+2})h$

from Theorem 1.4., we have

$= GL_n + GL_{n+1}i + GL_{n+2}\varepsilon + GL_{n+3}h = GLH_n.$

Thus, the proof is obtained.

The proof of the other parts can be shown by using Theorem 1.4.

Theorem 2.14. (Binet's Formula) For $n \geq 0$, we have

$$\boldsymbol{a}.\ GFH_n = \frac{a^n\bar{b}\hat{a} - b^n\bar{a}\hat{b}}{a-b},$$

$$\boldsymbol{b}.\ GLH_n = a^n\bar{b}\hat{a} + b^n\bar{a}\hat{b},$$

$$\boldsymbol{c}.\ GFH_{-n} = (-1)^{-n}\left(\frac{b^n\bar{b}\hat{a} - a^n\bar{a}\hat{b}}{a-b}\right),$$

$$\boldsymbol{d}.\ GLH_{-n} = (-1)^{-n}\left(b^n\bar{b}\hat{a} + a^n\bar{a}\hat{b}\right).$$

where $\bar{a} = 1 - ai,\quad \bar{b} = 1 - bi,\quad \hat{a} = 1 + ai + a^2\varepsilon + a^3h$ and $\hat{b} = 1 + bi + b^2\varepsilon + b^3h.$

Proof.

$\boldsymbol{a}.\ GFH_n = GF_n + GF_{n+1}i + GF_{n+2}\varepsilon + GF_{n+3}h$ from (), we obtain

$$= \left(\frac{a^n\bar{b} - b^n\bar{a}}{a-b}\right) + \left(\frac{a^{n+1}\bar{b} - b^{n+1}\bar{a}}{a-b}\right)i + \left(\frac{a^{n+2}\bar{b} - b^{n+2}\bar{a}}{a-b}\right)\varepsilon + \left(\frac{a^{n+3}\bar{b} - b^{n+3}\bar{a}}{a-b}\right)h$$

$$= \frac{a^n\bar{b}}{a-b}(1 + ai + a^2\varepsilon + a^3h) - \frac{b^n\bar{a}}{a-b}(1 + bi + b^2\varepsilon + b^3h)$$

$$= \frac{a^n\bar{b}\hat{a} - b^n\bar{a}\hat{b}}{a-b}.$$

Thus, the proof is obtained.

The proof of the others can be shown similarly to $\boldsymbol{a}$.

Theorem 2.15. (Sum formula) We have,

$$\boldsymbol{a}.\ \sum_{k=0}^{n} GFH_k = GFH_{n+2} - GFH_1,$$

$$\boldsymbol{b}.\ \sum_{k=0}^{n} GLH_k = GLH_{n+2} - GLH_1.$$

Proof.

$$\boldsymbol{a}.\ \sum_{k=0}^{n} GFH_k = \sum_{k=0}^{n} (GF_k + GF_{k+1}i + GF_{k+2}\varepsilon + GF_{k+3}h)$$

$$= \sum_{k=0}^{n} GF_k + \sum_{k=0}^{n} GF_{k+1}\, i + \sum_{k=0}^{n} GF_{k+2}\, \varepsilon + \sum_{k=0}^{n} GF_{k+3}\, h$$

from Theorem 1.5., we have

$$= (GF_{n+2} - 1) + (GF_{n+3} - 1 - GF_0)i + (GF_{n+4} - 1 - GF_0 - GF_1)\varepsilon$$
$$+(GF_{n+5} - 1 - GF_0 - GF_1 - GF_2)h$$
$$= GF_{n+2} + GF_{n+3}i + GF_{n+4}\varepsilon + GF_{n+5}h - 1 - (1 + GF_0)i$$
$$-(1 + GF_0 + GF_1)\varepsilon - (1 + GF_0 + GF_1 + GF_2)h$$
$$= GFH_{n+2} - GFH_1.$$

Thus, the proof is obtained.

The proof $\boldsymbol{b}$
can be shown similarly to $\boldsymbol{a}$.

Theorem 2.16. The generating functions for the Gaussian Fibonacci and Gaussian Lucas hybrid numbers are

$$\boldsymbol{a}.\ \sum_{n=0}^{\infty} GFH_n x^n = \frac{GFH_0 + (GFH_1 - GFH_0)x}{1 - x - x^2},$$

$$\boldsymbol{b}.\ \sum_{n=0}^{\infty} GLH_n x^n = \frac{GLH_0 + (GLH_1 - GLH_0)x}{1 - x - x^2}.$$

Proof.

$\boldsymbol{a}$. Let's

$$g(x) = \sum_{n=0}^{\infty} GFH_n x^n.$$

$$g(x) = GFH_0 + GFH_1 x + GFH_2 x^2 + GFH_3 x^3 + \cdots + GFH_n x^n + \cdots\ ,$$

If we multiply $g(x)$ by and x, x^2 respectively, we get

$$xg(x) = GFH_0 x + GFH_1 x^2 + GFH_2 x^3 + GFH_3 x^4 + \cdots + \cdots\ ,$$

$$x^2 g(x) = GFH_0 x^2 + GFH_1 x^3 + GFH_2 x^4 + GFH_3 x^5 + \cdots + \cdots\ .$$

So,

$$g(x)(1 - x - x^2) = GFH_0 + GFH_1 x - GFH_0 x$$

$$g(x) = \frac{GFH_0 + (GFH_1 - GFH_0)x}{1 - x - x^2}.$$

Thus, the proof is obtained.

The proof of part ***b*** can be shown in a similar way.

Theorem 2.17. The exponential generating functions for the Gaussian Fibonacci and Gaussian Lucas hybrid numbers are

$$\boldsymbol{a}.\ \sum_{n=0}^{\infty} GFH_n \frac{x^n}{n!} = \frac{\bar{b}\hat{a}e^{ax} - \bar{a}\hat{b}e^{bx}}{a-b},$$

$$\boldsymbol{b}.\ \sum_{n=0}^{\infty} GLH_n \frac{x^n}{n!} = \bar{b}\hat{a}e^{ax} + \bar{a}\hat{b}e^{bx}.$$

Proof.

$$\boldsymbol{a}.\ \sum_{n=0}^{\infty} GFH_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n\bar{b}\hat{a} - b^n\bar{a}\hat{b}}{a-b}\left(\frac{x^n}{n!}\right)$$

$$= \frac{\bar{b}\hat{a}}{a-b}\sum_{n=0}^{\infty}\frac{(ax)^n}{n!} - \frac{\bar{a}\hat{b}}{a-b}\sum_{n=0}^{\infty}\frac{(bx)^n}{n!}$$

$$= \frac{\bar{b}\hat{a}e^{ax} - \bar{a}\hat{b}e^{bx}}{a-b}.$$

Thus, the proof is completed.

The proof of part ***b*** can be shown in a similar way.

3. Some Identities of the Gaussian Fibonacci and Lucas Hybrid Numbers

In this chapter, we introduce some identities that are important for number sequences.

Theorem 3.1. (Cassini identiy) For $n \geq 1$, we have

$$\boldsymbol{a}.\ GFH_{n-1}GFH_{n+1} - (GFH_n)^{\mathbf{2}} = \frac{(-1)^{n-1}}{\delta}\left(b\bar{b}\hat{a}\bar{a}\hat{b} - a\bar{a}\hat{b}\bar{b}\hat{a}\right),$$

$$\boldsymbol{b}.\ GLH_{n-1}GLH_{n+1} - (GLH_n)^{\mathbf{2}} = \delta(-1)^{n-1}\left(a\bar{b}\hat{a}\bar{a}\hat{b} - b\bar{a}\hat{b}\bar{b}\hat{a}\right),$$

where $\delta = a - b$.

Proof. We use the Binet formula for the proof.

a. $GFH_{n-1}GFH_{n+1} - (GFH_n)^{\mathbf{2}}$

$$= \left(\frac{a^{n-1}\bar{b}\hat{a} - b^{n-1}\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{n+1}\bar{b}\hat{a} - b^{n+1}\bar{a}\hat{b}}{a-b}\right) - \left(\frac{a^{n}\bar{b}\hat{a} - b^{n}\bar{a}\hat{b}}{a-b}\right)^{2}$$

$$= \frac{\mathbf{1}}{(a-b)^{\mathbf{2}}}\left(-a^{n-1}b^{n+1}\bar{b}\hat{a}\bar{a}\hat{b} - b^{n-1}a^{n+1}\bar{a}\hat{b}\bar{b}\hat{a} + a^{n}b^{n}\bar{b}\hat{a}\bar{a}\hat{b} + b^{n}a^{n}\bar{a}\hat{b}\bar{b}\hat{a}\right)$$

$$= \frac{1}{(a-b)^{\mathbf{2}}}\left[a^{n}b^{n}\bar{b}\hat{a}\bar{a}\hat{b}\left(1-\frac{b}{a}\right) + b^{n}a^{n}\bar{a}\hat{b}\bar{b}\hat{a}\left(1-\frac{a}{b}\right)\right]$$

$= \frac{(ab)^{\boldsymbol{n-1}}(a-b)}{(a-b)^{\mathbf{2}}}\left(b\bar{b}\hat{a}\bar{a}\hat{b} - a\bar{b}\hat{a}\bar{a}\hat{b}\right)$ because of $ab = -1$, we have

$$\frac{(-1)^{n-1}}{\delta}\left(b\bar{b}\hat{a}\bar{a}\hat{b} - a\bar{a}\hat{b}\bar{b}\hat{a}\right).$$

Thus, the proof is obtained.

The proof of part ***b*** can be shown in a similar way.

Theorem 3.2. (Catalan identity) We have

$$\boldsymbol{a}.\ GFH_{n-t}GFH_{n+t} - (GFH_n)^{\mathbf{2}} = \frac{(-1)^{n-t}\left(b^{t}\bar{b}\hat{a}\bar{a}\hat{b} - a^{t}\bar{a}\hat{b}\bar{b}\hat{a}\right)F_t}{\delta},$$

$$\boldsymbol{b}.\ GLH_{n-t}GLH_{n+t} - (GLH_n)^{\mathbf{2}} = (-1)^{n-t}\delta\left(a^{t}\bar{a}\hat{b}\bar{b}\hat{a} - b^{t}\bar{b}\hat{a}\bar{a}\hat{b}\right)F_t.$$

where $\delta = a - b$.

Proof. We use the Binet formula for the proof.

$\boldsymbol{a}.\ GFH_{n-t}GFH_{n+t} - (GFH_n)^{\mathbf{2}}$

$$= \left(\frac{a^{n-t}\bar{b}\hat{a} - b^{n-1}\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{n+t}\bar{b}\hat{a} - b^{n+1}\bar{a}\hat{b}}{a-b}\right) - \left(\frac{a^{n}\bar{b}\hat{a} - b^{n}\bar{a}\hat{b}}{a-b}\right)^{2}$$

$$= \frac{\mathbf{1}}{(a-b)^{\mathbf{2}}}\left(-a^{n-t}b^{n+t}\bar{b}\hat{a}\bar{a}\hat{b} - b^{n-t}a^{n+t}\bar{a}\hat{b}\bar{b}\hat{a} + a^{n}b^{n}\bar{b}\hat{a}\bar{a}\hat{b} + b^{n}a^{n}\bar{a}\hat{b}\bar{b}\hat{a}\right)$$

$$= \frac{1}{(a-b)^{\mathbf{2}}}\left[a^{n}b^{n}\bar{b}\hat{a}\bar{a}\hat{b}\left(1-\left(\frac{b}{a}\right)^{t}\right) + b^{n}a^{n}\bar{a}\hat{b}\bar{b}\hat{a}\left(1-\left(\frac{a}{b}\right)^{t}\right)\right]$$

$= \frac{(ab)^{\boldsymbol{n-t}}(a^{t}-b^{t})}{(a-b)^{\mathbf{2}}}\left(b^{t}\bar{b}\hat{a}\bar{a}\hat{b} - a^{t}\bar{a}\hat{b}\bar{b}\hat{a}\right)$ because of $ab = -1$, we have

$$= \frac{(-1)^{n-1}(a^{t}-b^{t})}{\delta^{2}}\left(b\bar{b}\hat{a}\bar{a}\hat{b} - a\bar{a}\hat{b}\bar{b}\hat{a}\right)$$

$$= \frac{(-1)^{n-t}\left(b^{t}\bar{b}\hat{a}\bar{a}\hat{b} - a^{t}\bar{a}\hat{b}\bar{b}\hat{a}\right)F_t}{\delta}.$$

Thus, the proof is obtained.

The proof of part ***b*** can be shown in a similar way.

Note that, for $t = 1$ we obtain the Cassini identity.

Theorem 3.3. (d' Ocagne identity) We have,

$$\boldsymbol{a}.\ GFH_m GFH_{n+1} - GFH_n GFH_{m+1} = \frac{(-1)^n\left(a\bar{a}\hat{b}\bar{b}\hat{a} - b\bar{b}\hat{a}\bar{a}\hat{b}\right)F_{m-n}}{\delta},$$

$$\boldsymbol{b}.\ GLH_m GLH_{n+1} - GLH_n GLH_{m+1} = (-1)^{n+1}\delta\left(a\bar{a}\hat{b}\bar{b}\hat{a} - b\bar{b}\hat{a}\bar{a}\hat{b}\right)F_{m-n}.$$

where F_n is nth Fibonacci numbers.

Proof. We use the Binet formula for the proof.

$\boldsymbol{a}.\ GFH_m GFH_{n+1} - GFH_n GFH_{m+1}$

$$= \left(\frac{a^m\bar{b}\hat{a} - b^m\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{n+1}\bar{b}\hat{a} - b^{n+1}\bar{a}\hat{b}}{a-b}\right) - \left(\frac{a^n\bar{b}\hat{a} - b^n\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{m+1}\bar{b}\hat{a} - b^{m+1}\bar{a}\hat{b}}{a-b}\right)$$

$$= \frac{1}{(a-b)^2}\left(-a^m b^{n+1}\bar{b}\hat{a}\bar{a}\hat{b} - b^m a^{n+1}\bar{a}\hat{b}\bar{b}\hat{a} + a^n b^{m+1}\bar{b}\hat{a}\bar{a}\hat{b} + b^n a^{m+1}\bar{a}\hat{b}\bar{b}\hat{a}\right)$$

$$= \frac{a^m b^n\left(a\bar{a}\hat{b}\bar{b}\hat{a} - b\bar{b}\hat{a}\bar{a}\hat{b}\right) + a^n b^m\left(b\bar{b}\hat{a}\bar{a}\hat{b} - a\bar{a}\hat{b}\bar{b}\hat{a}\right)}{(a-b)^2}$$

$$= \frac{\left(a\bar{a}\hat{b}\bar{b}\hat{a} - b\bar{b}\hat{a}\bar{a}\hat{b}\right)(a^m b^n - a^n b^m)}{(a-b)^2}$$

$$= \frac{\left(a\bar{a}\hat{b}\bar{b}\hat{a} - b\bar{b}\hat{a}\bar{a}\hat{b}\right)a^n b^n(a^{m-n} - b^{m-n})}{\delta}$$

$$\frac{(-1)^n\left(a\bar{a}\hat{b}\bar{b}\hat{a} - b\bar{b}\hat{a}\bar{a}\hat{b}\right)F_{m-n}}{\delta}.$$

Thus, the proof is completed.

The proof of part ***b*** can be shown in a similar way.

Theorem 3.4. (Vajda identity) We have,

$$\boldsymbol{a}.\ GFH_{n+i} GFH_{n+j} - GFH_n GFH_{n+i+j} = \frac{(-1)^n\left(a^j\bar{a}\hat{b}\bar{b}\hat{a} - b^j\bar{b}\hat{a}\bar{a}\hat{b}\right)F_i}{\delta},$$

$$\boldsymbol{b}.\ GLH_{n+i} GLH_{n+j} - GLH_n GLH_{n+i+j} = (-1)^n\delta\left(b^j\bar{b}\hat{a}\bar{a}\hat{b} - a^j\bar{a}\hat{b}\bar{b}\hat{a}\right)F_i.$$

where F_i is ith Fibonacci numbers.

Proof. We use the Binet formula for the proof.

$\boldsymbol{a}.\ GFH_{n+i} GFH_{n+j} - GFH_n GFH_{n+i+j}$

$$= \left(\frac{a^{n+i}\bar{b}\hat{a} - b^{n+i}\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{n+j}\bar{b}\hat{a} - b^{n+j}\bar{a}\hat{b}}{a-b}\right) - \left(\frac{a^{n}\bar{b}\hat{a} - b^{n}\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{n+i+j}\bar{b}\hat{a} - b^{n+i+j}\bar{a}\hat{b}}{a-b}\right)$$

$$= \frac{-a^{n+i}b^{n+j}\bar{b}\hat{a}\bar{a}\hat{b} - b^{n+i}a^{n+j}\bar{a}\hat{b}\bar{b}\hat{a} + a^{n}b^{n+i+j}\bar{b}\hat{a}\bar{a}\hat{b} + b^{n}a^{n+i+j}\bar{a}\hat{b}\bar{b}\hat{a}}{(a-b)^{\mathbf{2}}}$$

$$= \frac{a^{n}b^{n+j}\bar{b}\hat{a}\bar{a}\hat{b}\left(b^{i} - a^{i}\right) + a^{n+j}b^{n}\bar{a}\hat{b}\bar{b}\hat{a}\left(a^{i} - b^{i}\right)}{(a-b)^{\mathbf{2}}}$$

$$= \frac{\left(a^{i} - b^{i}\right)a^{n}b^{n}\left(a^{j}\bar{a}\hat{b}\bar{b}\hat{a} - b^{j}\bar{b}\hat{a}\bar{a}\hat{b}\right)}{(a-b)^{\mathbf{2}}}$$

$$= \frac{(-1)^{n}\left(a^{j}\bar{a}\hat{b}\bar{b}\hat{a} - b^{j}\bar{b}\hat{a}\bar{a}\hat{b}\right)F_{i}}{\delta}.$$

Theorem 3.4. (Honsberger identity) We have,

$$\boldsymbol{a}.\ GFH_{n}GFH_{m} + GFH_{n+1}GFH_{m+1} = \frac{a^{n+m}\left(\bar{b}\hat{a}\right)^{2}(1+a^{2}) + b^{n+m}\left(\bar{a}\hat{b}\right)^{2}(1+b^{2})}{(a-b)^{\mathbf{2}}},$$

$$\boldsymbol{b}.\ GLH_{n}GLH_{m} + GLH_{n+1}GLH_{m+1} = a^{n+m}\left(\bar{b}\hat{a}\right)^{2}(1+a^{2}) + b^{n+m}\left(\bar{a}\hat{b}\right)^{2}(1+b^{2}).$$

Proof. $\boldsymbol{a}.\ GFH_{n}GFH_{m} + GFH_{n+1}GFH_{m+1}$

$$= \left(\frac{a^{n}\bar{b}\hat{a} - b^{n}\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{m}\bar{b}\hat{a} - b^{m}\bar{a}\hat{b}}{a-b}\right) + \left(\frac{a^{n+1}\bar{b}\hat{a} - b^{n+1}\bar{a}\hat{b}}{a-b}\right)\left(\frac{a^{m+1}\bar{b}\hat{a} - b^{m+1}\bar{a}\hat{b}}{a-b}\right)$$

$$= \frac{a^{n+m}(\bar{b}\hat{a})^{2} - a^{n}b^{m}\bar{b}\hat{a}\bar{a}\hat{b} - b^{n}a^{m}\bar{a}\hat{b}\bar{b}\hat{a} + b^{n+m}\left(\bar{a}\hat{b}\right)^{2}}{(a-b)^{\mathbf{2}}}$$

$$+ \frac{a^{n+m+2}(\bar{b}\hat{a})^{2} - a^{n+1}b^{m+1}\bar{b}\hat{a}\bar{a}\hat{b} - b^{n+1}a^{m+1}\bar{a}\hat{b}\bar{b}\hat{a} + b^{n+m+2}\left(\bar{a}\hat{b}\right)^{2}}{(a-b)^{\mathbf{2}}}$$

$$= \frac{a^{n+m}\left(\bar{b}\hat{a}\right)^{2}(1+a^{2}) + b^{n+m}\left(\bar{a}\hat{b}\right)^{2}(1+b^{2}) - a^{n}b^{m}\bar{b}\hat{a}\bar{a}\hat{b}(1+ab) - b^{n}a^{m}\bar{a}\hat{b}\bar{b}\hat{a}(1+ab)}{(a-b)^{\mathbf{2}}}$$

since $ab = -1$, we have

$$= \frac{a^{n+m}\left(\bar{b}\hat{a}\right)^{2}(1+a^{2}) + b^{n+m}\left(\bar{a}\hat{b}\right)^{2}(1+b^{2})}{(a-b)^{\mathbf{2}}}.$$

The proof of part $\boldsymbol{b}$ can be shown in a similar way.

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