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Hyperbolic functions obtained from k-Jacobsthal sequences

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In this paper, an expansion of the classical hyperbolic functions is presented and studied. Also, many features of the $k-$Jacobsthal hyperbolic functions are given. Finally, we introduced some graph and curved surfaces related to the $k-$Jacobsthal hyperbolic functions.

Keywords: k-Jacobsthal sequence; hyperbolic functions; k-Jacobsthal number.

Mathematics Subject Classification 2020: 11B37, 11B39, 11B83

1. Introduction

The simplest and most used sequences of numbers are Fibonacci sequences. It is a sequence obtained by adding the last two terms consecutively with the initial terms 0 and 1, usually denoted by F_n. That is, $\{F_n\} = \{0, 1, 1, 2, 3, 5, \dots\}$. Fibonacci numbers and their generalizations and polynomials have many exciting properties and applications in different branches of science, art, nature in [13, 23, 15, 11, 7, 10]. $\phi = (1 + \sqrt{5})/2$, the convergence of the ratio of two consecutive Fibonacci numbers, called the Golden Mean, or Golden Section appears in many studies in many fields [4, 21]. Jacobsthal sequences are also studied by many researchers [14, 9, 25, 1, 16, 2, 3]. Many generalizations of the hyperbolic functions are given [6, 22, 8, 24]. Also, Golden Section and its applications in many areas are given [20, 5, 12, 18]. One of the most recent works in this area is [6], in which the authors describe new k-Fibonacci hyperbolic functions and give some of their properties. In this study, we want to show that k-Jacobsthal sequences form a very close relationship between classical hyperbolic functions and k-Jacobsthal sequences by adapting many properties of k-Jacobsthal sequences to k-Jacobsthal hyperbolic

functions while this relationship is exactly the same in some cases, in some cases also the coefficients are different, but still, they are similar.

Definition 1.1. For $k \geq 1$, the $k-$Jacobsthal sequence is defined by:
$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}$ for with $J_{k,0} = 0$ and $J_{k,1} = 1$
The characteristic equation for k-Jacobsthal numbers is as follows:

$$r^2 = kr + 2 \tag{1.1}$$

Binet's formula for the k-Jacobsthal numbers is given by [9]:
$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$
where $r_1 = \frac{k+\sqrt{k^2+8}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+8}}{2}$ are the roots of Eq.1.1 and $r_1 > r_2$.
We know that, $r_1 + r_2 = k$, $r_1.r_2 = -2$ and $r_1 - r_2 = \sqrt{k^2+8}$.

2. New k-Jacobsthal hyperbolic functions

New Jacobsthal hyperbolic functions, like the known classical hyperbolic numbers, are defined as

$$sJhx = \frac{\alpha^{2x} - 2^{2x}\alpha^{-2x}}{3}$$

$$cJhx = \frac{\alpha^{2x+1} + 2^{2x+1}\alpha^{-(2x+1)}}{3}$$

where sJh and cJh are named, respectively, the Jacobsthal hyperbolic sine and cosine.

New k-Jacobsthal hyperbolic functions of the above functions, called sin k-Jacobsthal hyperbolic functions and cosine k-Jacobsthal hyperbolic functions, respectively, are defined as follows:

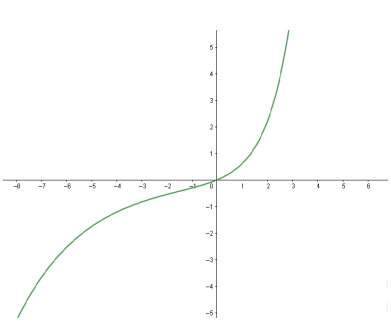
$$sJ_kh(x) = \frac{\alpha_k^{2x} - 2^{2x}\alpha_k^{-2x}}{\sqrt{k^2+8}}$$

$$cJ_kh(x) = \frac{\alpha_k^{2x+1} + 2^{2x+1}\alpha_k^{-(2x+1)}}{\sqrt{k^2+8}}$$

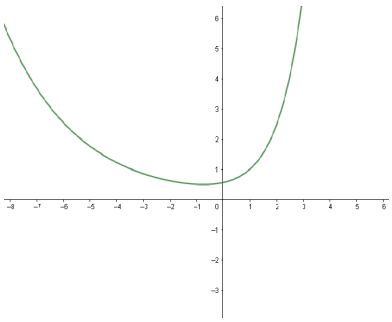
where $\alpha_k = \frac{k+\sqrt{k^2+8}}{2}$. If $x = 2n$ then $sJ_kh(x) = J_{k,4n}$, while if $x = 2n+1$ then $cJ_kh(x) = J_{k,2n+1}$. By doing a change of variable $2x + 1 = t$ in function cJ_khx new variable is t and similarly in sJ_khx for $2x = a$ new variable is a. k-Jacobsthal hyperbolic sine and cosine functions are defined as follows:

$$sJ_kh(x) = \frac{\alpha_k^{x} - 2^{x}\alpha_k^{-x}}{\alpha_k + 2\alpha_k^{-1}}$$

$$cJ_kh(x) = \frac{\alpha_k^{x} + 2^{x}\alpha_k^{-x}}{\alpha_k + 2\alpha_k^{-1}}$$



(a) sine k-Jacobsthal hyperbolic functions $sJh(x)$



(b) cosine k-Jacobsthal hyperbolic functions $cJh(x)$

Figure 1: sine and cosine k-Jacobsthal hyperbolic functions

since $\alpha_k + 2(\alpha_k)^{-1} = \sqrt{k^2+8}$.

The graphics for new k-Jacobsthal hyperbolic functions are shown in Fig. 1a and Fig. 1b .From last definitions, it holds that $sJ_kh(2n) = J_{k,2n}$ and $cJ_kh(2n+1) = J_{k,2n+1}$, as it can be easily confirmed by using $\alpha_k^{-2n} = (\frac{-1}{\alpha_k})^{2n}$ and $\alpha_k^{-(2n+1)} = -(\frac{-1}{\alpha_k})^{2n+1}$.

After that, we will substitute α for α_k to simplify.

2.1. *Properties of the sJhx and cJhx functions*

Proposition 2.1.1. *The basic feature of these functions, which can be called a version of the Pythagorean trigonometric identity, is*

$$(cJ_kh(x))^2 - (sJ_kh(x))^2 = \frac{2^x 4}{k^2+8} \tag{2.1}$$

Proposition 2.1.2. *We have the following sum and difference identity*

$$cJ_kh(x+y) = \tfrac{\sqrt{k^2+8}}{2}(cJ_kh(x).cJ_kh(y) + sJ_kh(x).sJ_kh(y)$$
$$cJ_kh(x-y) = \tfrac{\sqrt{k^2+8}}{2}(cJ_kh(x).cJ_kh(y) - sJ_kh(x).sJ_kh(y)$$
$$sJ_kh(x+y) = \tfrac{\sqrt{k^2+8}}{2}(sJ_kh(x).cJ_kh(y) + cJ_kh(x).sJ_kh(y)$$
$$sJ_kh(x-y) = \tfrac{\sqrt{k^2+8}}{2}(sJ_kh(x).cJ_kh(y) - cJ_kh(x).sJ_kh(y)$$

Proof. We will now prove the first equality. The proof of the others is similar to

the proof of the first equality.

$$(cJ_kh(x).cJ_kh(y) + sJ_kh(x).sJ_kh(y) = \frac{(\alpha_k^x + 2^x\alpha_k^{-x})(\alpha_k^y + 2^y\alpha_k^{-y}) + (\alpha_k^x - 2^x\alpha_k^{-x})(\alpha_k^y - 2^y\alpha_k^{-y})}{\alpha_k + 2\alpha_k^{-1}}$$
$$= \frac{2}{\sqrt{k^2+8}}cJ_kh(x+y) \qquad \square$$

By making $y = x$ in the previous first and third formulas, we get the following result.

Corollary 2.1.1. *Double variable formulas*

$$cJ_kh(2x) = \frac{\sqrt{k^2+8}}{2}\Big[(cJ_kh(x))^2 + (sJ_kh(x))^2\Big] \tag{2.2}$$

$$sJ_kh(2x) = \sqrt{k^2+8}sJ_kh(x)cJ_kh(x) \tag{2.3}$$

Corollary 2.1.2. *Half variable formulas*

$(cJ_kh(x))^2 = \frac{2^{x+1}}{\sqrt{k^2+8}}\Big[cJ_kh(2x) + \frac{2}{\sqrt{k^2+8}}\Big]$

$(sJ_kh(x))^2 = \frac{2^{x+1}}{\sqrt{k^2+8}}\Big[cJ_kh(2x) - \frac{2}{\sqrt{k^2+8}}\Big]$

For $k = 2i$, the classical expression of hyperbolic functions is seen in the previous formula.

2.2. *The sJhx and cJhx functions and the k-Jacobsthal numbers*

Proposition 2.2.1. *We have the recursive relation*

$sJ_kh(x+1) = kcJ_kh(x) + 2sJ_kh(x-1)$

Proof. $kcJ_kh(x) + 2sJ_kh(x-1) = k\frac{(\alpha^x+2^x\alpha^{-x})}{\alpha+2\alpha^{-1}} + 2\frac{(\alpha^{x-1}-2^{x-1}\alpha^{-x+1})}{\alpha+2\alpha^{-1}}$

$= \frac{1}{\alpha+2\alpha^{-1}}(\alpha^x(k+2/\alpha) + 2^x\alpha^{-x}(k-\alpha)) = \frac{1}{\alpha+2\alpha^{-1}}(\alpha^{x+1} - 2^{x+1}\alpha^{-x-1})$

Since $\alpha^2 = k\alpha + 2$ and $\alpha - k = \frac{2}{\alpha}$, we have $\frac{\alpha^{x+1}-2^{x+1}\alpha^{-x-1}}{\alpha+2\alpha^{-1}} = sJ_kh(x+1)$

□

Proposition 2.2.2. *Catalan's identity for the kJacobsthal hyperbolic functions*

$$cJ_kh(x-s).cJ_kh(x+s) - (cJ_kh(x))^2 = 2^{x-s}(sJ_kh(s))^2$$

Proof. By definition, (LHS) is

$$\text{(LHS)}=\frac{(\alpha^{x-s}+2^{x-s}\alpha^{-x+s})(\alpha^{x+s}+2^{x+s}\alpha^{-x-s})-(\alpha^x+2^x\alpha^{-x})^2}{(\alpha+2\alpha^{-1})^2} = \frac{2^{x+s}\alpha^{-2s}+2^{x-s}\alpha^{2s}-2^{x+1}}{(\alpha+2\alpha^{-1})^2}$$

$$= \frac{2^{x-s}\alpha^{2s}+\alpha^{-2s}2^{2s}-2^{s+1}}{(\alpha+2\alpha^{-1})^2} = 2^{x-s}(sJ_kh(s))^2$$ □

Corollary 2.2.1.

$$cJ_kh(x-s)cJ_kh(x+s) - (sJ_kh(x))^2 = 2^{x-s}(cJ_kh(s))^2$$

$$sJ_kh(x-s)sJ_kh(x+s) - (sJ_kh(x))^2 = -2^{x-s}(sJ_kh(s))^2$$

$$sJ_kh(x-s)sJ_kh(x+s) - (cJ_kh(x))^2 = -2^{x-s}(cJ_kh(s))^2$$

If we put $s = 1$ into classical Catalan's identity, then we obtain the Cassini's identity for the $k-$Jacobsthal numbers [9]:

$$J_{k,n-1}J_{k,n+1} - J_{k,n}^2 = (-1)^n 2^{n-1}$$

The suitable identity for the $k-$Jacobsthal hyperbolic functions are given below.

Proposition 2.2.3. *For the $k-$Jacobsthal hyperbolic functions, Cassini's or identity are given as follows*

$$cJ_kh(x-1)cJ_kh(x+1) - (sJ_kh(x))^2 = 2^{x-1}$$

$$sJ_kh(x-1)sJ_kh(x+1) - (sJ_kh(x))^2 = -2^{x-1}$$

Proposition 2.2.4. *d'Ocagne's identity for the $k-$Jacobsthal numbers* [9] *is*

$J_{k,m}J_{k,n+1} - J_{k,m+1}J_{k,n} = (-2)^n J_{k,m-n}$ *if* $m > n$.

For the $k-$Jacobsthal hyperbolic functions, d'Ocagne's identity is

$$cJ_kh(x)cJ_kh(y+s) - sJ_kh(x+s)sJ_kh(y) = 2^y cJ_kh(s)cJ_kh(x-y) \tag{2.4}$$

$$cJ_kh(x)sJ_kh(y+s) - cJ_kh(x+s)sJ_kh(y) = 2^y sJ_kh(s)cJ_kh(x-y) \tag{2.5}$$

Proof. From Eq. 2.4 we have:

$$\begin{aligned}(LHS) &= \frac{(\alpha^x + 2^x\alpha^{-x})(\alpha^{y+s} + 2^{y+s}\alpha^{-y-s}) - (\alpha^{x+s} - 2x + s\alpha^{-x-s})(\alpha^y - 2^y\alpha^{-y})}{(\alpha + 2\alpha^{-1})^2} \\ &= \frac{2^{y+s}\alpha^{x-y-s} + 2^x\alpha^{-x+y+s} + 2^y\alpha^{x-y+s} + 2^{x+s}\alpha^{-x+y-s}}{(\alpha + 2\alpha^{-1})^2} \\ &= \frac{\alpha^s + 2^s\alpha^{-s}}{\alpha + 2\alpha^{-1}} . \frac{2^y\alpha^{x-y} + 2^x\alpha^{-x+y}}{\alpha + 2\alpha^{-1}} = 2^y cJ_kh(s)cJ_kh(x-y)\end{aligned}$$

Eq.2.5 can be obtained as in Eq.2.4. □

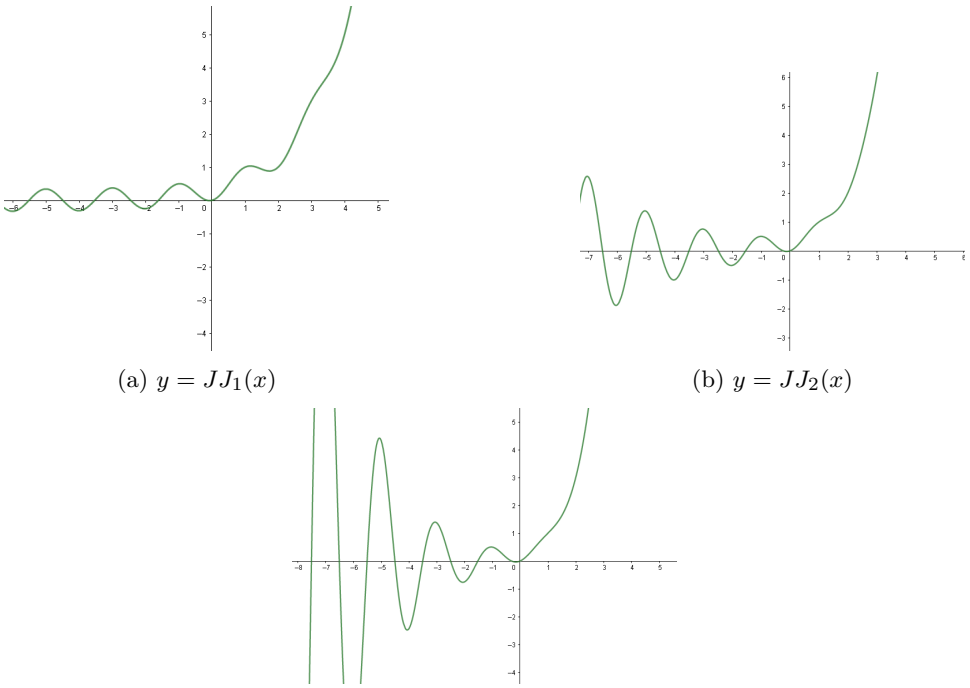
3. The quasi-sine k-Jacobsthal functions and the k-Jacobsthal numbers

Binet's formula for the $k-$Jacobsthal sequence has the form $J_{k,n} = \frac{\alpha^n - (-2)^n\alpha^{-n}}{\alpha + 2\alpha^{-1}}$. Also, we have given the $k-$jacobsthal hyperbolic sine function as $sJ_kh(x) = \frac{\alpha^x - 2^x\alpha^{-x}}{\alpha + 2\alpha^{-1}}$. So, if x is an even integer number we have $sJ_kh(x) = J_{k,n}$. Therefore, we give the following definition.

Definition 3.1. The quasi-sine $k-$Jacobsthal function is defined as

$$JJ_k(x) = \frac{\alpha^x - 2^x cos(\pi x)\alpha^{-x}}{\alpha + 2\alpha^{-1}}$$

Notice that $JJ_k(n) = J_{k,n}$ for $n \in \mathbf{Z}$. So, the points of the form $(n, J_{k,n})$ concern to the graphics of these functions. For $k = 1, 2, 3$, the graphics of $JJ_k(x)$ are given in Fig 2. while functions $JJ_k(n)$ and $J_{k,n}$ are similar in many properties, they differ considerably in the values taken by the relative extremum points and the number of these extremes. For instance, for $k = 1$ if x takes positive values, $JJ_1(x)$ has only maximum and minimum values. These values are $x = 1.1646981\ldots$ and



(a) $y = JJ_1(x)$ (b) $y = JJ_2(x)$

(c) $y = JJ_3(x)$

Figure 2: The quasi-sine $k-$Jacobsthal functions, $JJ_k(x)$ for $k = 1, 2, 3$.

$x = 1.7369690\ldots$, respectively. But if $k > 1$, the $JJ_k(x)$ function does not have maximum and minimum values. In addition, if k takes values in the range of 0 and 1, for $x > 0$ the number of maximum and minimum values that $JJ_k(x)$ will take will increase. For give an example, there will be a maximum and a minimum values at $k = 0.9$, two values for $k = 0.5$ and seven values for $k = 0.2$. In other words, as k, gets closer to 0, the number of maximum and minimum values will increase. In Fig. 3 for $k = 0.01$ and $x < 0$ the maximum and minimum values that $JJ_k(x)$ will take will be infinite. For $x < 0$, the infinite relative extreme points of these functions are given at different values in x . For instance, for $x < 0$ and $k = 1$ the first minimum is at $x = -0.0675244\ldots$ and for $k = 3$ is at $x = -0.1544596\ldots$.

In Fig.4, it is possible to see the tangent curves obtained along the values taken

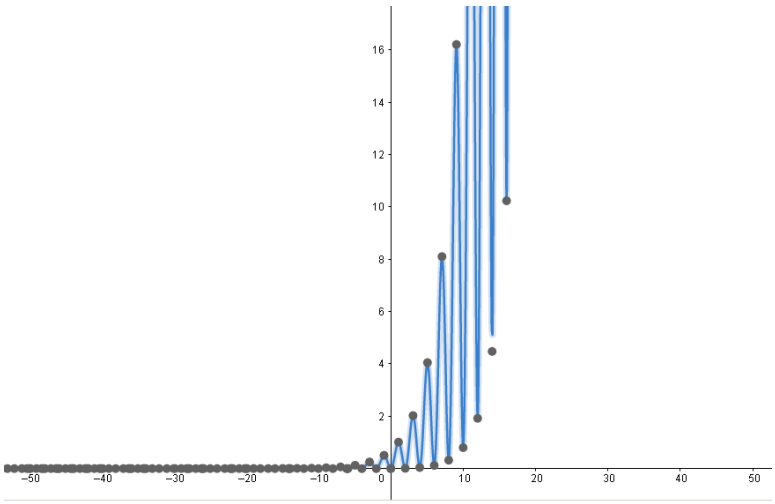
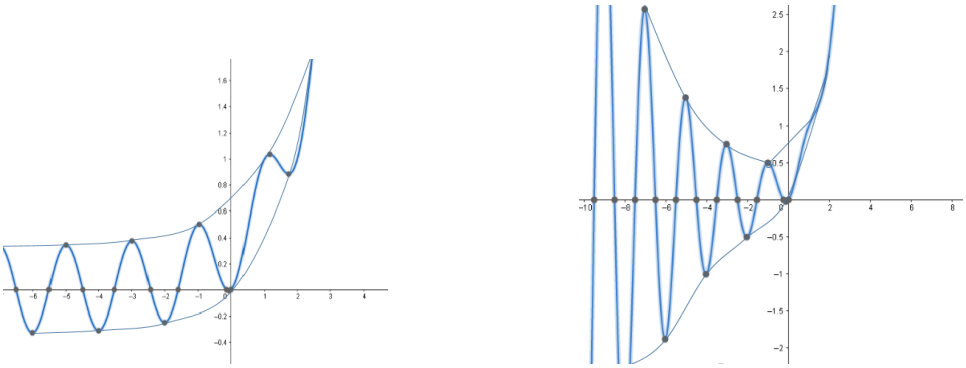


Figure 3: The quasi-sine 0.01-Jacobsthal function.



(a) $y = JJ_1(x)$ (b) $y = JJ_2(x)$

Figure 4: The quasi-sine k-Jacobsthal functions for $k = 1, 2$ with their improving tangent curves.

by the $JJ_k(x)$ function for k= 1,2. Notice that the tangent points have the form $(n, J_{k,n})$ for $n \in \mathbf{Z}$. For these quasi-sine k - Jacobsthal functions, there are several identities that are versions of the analogues identities of k - Jacobsthal numbers [9],[25],[1]. Some of them are revealed here.

Theorem 3.1 (Recursive relation). $JJ_k(x+2) = kJJ_k(x+1) + 2JJ_k(x)$

Proof.

$$\begin{aligned} JJ_k(x+1) + 2JJ_k(x) &= k\frac{\alpha^{x+1} - 2^{x+1}cos(\pi(x+1))\alpha^{-x-1}}{\alpha + 2\alpha^{-1}} + 2\frac{\alpha^x - 2^x cos(\pi x)\alpha^{-x}}{\alpha + 2\alpha^{-1}} \\ &= \frac{\alpha^x(k\alpha + 2) + cos(\pi x)\alpha^{-x-2}2^{x+1}(k\alpha - \alpha^2)}{\alpha + 2\alpha^{-1}} = \frac{\alpha^{x+2} - cos(\pi x)\alpha^{-x-2}2^{x+2}}{\alpha + 2\alpha^{-1}} \\ &= JJ_k(x+2) \end{aligned}$$

Because $k/\alpha = 1 = -2/\alpha^2$. □

Theorem 3.2 (Catalan's identity). *If $s \in \mathbf{Z}$, then:*

$$JJ_k(x+s).JJ_k(x-s) - (JJ_k(x))^2 = 2^{x-s}(-1)^{s+1}cos(\pi x)(JJ_k(s))^2$$

Proof.

$$\begin{aligned} JJ_k(x+s).JJ_k(x-s) - (JJ_k(x))^2 &= \frac{\alpha^{x+s} - 2^{x+s}cos(\pi(x+s))\alpha^{-x-s}}{\alpha + 2\alpha^{-1}} \frac{\alpha^{x-s} - 2^{x-s}cos(\pi(x-s))\alpha^{-x+s}}{\alpha + 2\alpha^{-1}} \\ &\quad - (\frac{\alpha^x - 2^x cos(\pi x)\alpha^{-x}}{\alpha + 2\alpha^{-1}})^2 \\ &= 2^{x-s}\frac{(-1)^{s+1}cos(\pi x)\alpha^{2s} + 2^{2s}\alpha^{-2s}(-1)^{s+1}cos(\pi x) + 2^{s+1}cos(\pi x)}{(\alpha + 2\alpha^{-1})^2} \\ &= (-1)^{s+1}cos(\pi x)2^{x-s}\frac{\alpha^{2s} + 2^{2s}\alpha^{-2s} - 2^{s+1}cos(\pi s)}{(\alpha + 2\alpha^{-1})^2} \\ &= (-1)^{s+1}cos(\pi x)2^{x-s}(\frac{\alpha^s - 2^s cos(\pi s)\alpha^{-s}}{\alpha + 2\alpha^{-1}})^2 \\ &= (-1)^{s+1}cos(\pi x)2^{x-s}(JJ_k(s))^2 \end{aligned}$$

□

If x is an integer n in the previous result then Catalan's identity for the kJacobsthal numbers appears. Moreover, if $s = 1$, then we get:

$$JJ_k(x+1).JJ_k(x-1) - (JJ_k(x))^2 = cos(\pi x)2^{x-1}.$$

If we take $x = n$, then we find Cassini's identity [9] for the $k-$Jacobsthal numbers:

$$J_{k,n+1}.J_{k,n-1} - (J_{k,n})^2 = (-1)^n 2^{n-1}$$

Theorem 3.3 (Asymptotic quotient). *We have*

$\lim_{x\to\infty} \frac{JJ_k(x+s)}{JJ_k(x)} = \alpha^s$ *where* s *is any intger.*

Proof. For $\alpha > 1$, then:

$lim_{x\to\infty} \frac{JJ_k(x+s)}{JJ_k(x)} = lim_{x\to\infty} \frac{\alpha^s - 2^{x+s}cos(\pi(x+s))\alpha^{-2x-s}}{1-2^x cos(\pi(x))\alpha^{-2x}} = \alpha^s$

Specialy, if $k = 1$, then $\alpha = r_1$ and $\lim_{x\to\infty} \frac{JJ_k(x+1)}{JJ_k(x)} = r_1$ □

4. 3D curved surfaces for the $sJhx$ and $cJhx$ functions

Definition 4.1. The complex valued function given as follows is denoted by $CJJ_k(x)$ and named the three-dimensional $k-$Jacobsthal spirals:

$$CJJ_k(x) = \frac{\alpha^x - 2^x cos(\pi x)\alpha^{-x}}{\alpha + 2\alpha^{-1}} + i\frac{2^x sin(\pi x)\alpha^{-x}}{\alpha + 2\alpha^{-1}}.$$

$\Re(CJJ_k(x)) = JJ_k(x)$ is the quasi-sine $k-$ Jacobsthal function. The shape of the three-$k-$ Jacobsthal spiral recalls a three-dimensional spiral. Function $CJJ_k(x)$ can be given as:

$$CJJ_k(x) = \frac{\alpha^x + 2^x ie^{i\pi(1/2-x)}\alpha^{-x}}{\alpha + 2\alpha^{-1}}$$

We give some of the properties for $CJJ_k(x)$ as follow:

- If $k = 1$, then we obtain the three-dimensional Jacobsthal spiral. $CJJ_1(x) = CJJ_k(x) = \frac{r^x + 2^x ie^{i\pi(1/2-x)}r^{-x}}{3}$.
- If x is an integer then, the imaginery part of the three-dimensional $k-$Jacobsthal spiral is zero, and hence $CJJ_k(n)$ results in Binet's formula for the general term of the $k-$Jacobsthal sequence [9].
- Lastly, for every k, it is $CJJ_k(0) = 0$ and $CJJ_k(1) = 1$.

Theorem 4.1. *The three-dimensional* $k-$*Jacobsthal spiral confirm the following Jacobsthal-type recurrence relation:*

$$CJJ_k(x+2) = kCJJ_k(x+1) + 2CJJ_k(x)$$

Proof. Since $\alpha > 1$ and $k\alpha + 2 = \alpha^2$, so $\alpha - k = 2/\alpha$, we have:

$$\begin{aligned}(RHS) &= k\frac{\alpha^{x+1} + 2^{x+1}ie^{i\pi(1/2-x-1)}\alpha^{-x-1}}{\alpha + 2\alpha^{-1}} + 2\frac{\alpha^x + 2^x ie^{i\pi(1/2-x)}\alpha^{-x}}{\alpha + 2\alpha^{-1}} \\ &= \frac{\alpha^{x+2} + ie^{-i\pi x}\alpha^{-x-1}2^{x+1}(-k+\alpha)i}{\alpha + 2\alpha^{-1}} \\ &= \frac{\alpha^{x+2} + 2^{x+2}ie^{i\pi(1/2-x-2)}\alpha^{-x-2}}{\alpha + 2\alpha^{-1}} = CJJ_k(x)\end{aligned}$$ □

Since we have showed, we can give Catalan's identity for the three-dimensional $k-$ Jacobsthal spiral in the next theorem.

Theorem 4.2. *Catalan's identity for the three-dimensional $k-$Jacobsthal spiral is as follows:*

$$CJJ_k(x-s).CJJ_k(x+s) - (CJJ_k(x))^2 = (-1)^{x+s+1}2^{x-s}(CJJ_k(s))^2$$

Proof. Since $e^{i\pi(1/2-x)} = ie^{-i\pi x}$ and $e^{i\pi s} = (-1)^s$, $\quad e^{i(\pi/2\pm\pi s)} = i(-1)^s$ and s is an integer. Therefore,

$$\begin{aligned}(LHS) &= \frac{\alpha^{x+s} + 2^{x+s}ie^{i\pi(1/2-x-s)}\alpha^{-x-s}}{\alpha + 2\alpha^{-1}}.\frac{\alpha^{x-s} + 2^{x-s}ie^{i\pi(1/2-x+s)}\alpha^{-x+s}}{\alpha + 2\alpha^{-1}} - \Big(\frac{\alpha^x + 2^x ie^{i\pi(1/2-x)}\alpha^{-x}}{\alpha + 2\alpha^{-1}}\Big)^2 \\ &= \frac{\alpha^{2s}2^{x-s}ie^{i\pi(1/2-x+s)} + \alpha^{-2s}2^{x+s}ie^{i\pi(1/2-x-s)} - 2^{s+1}ie^{i\pi(1/2-x)}}{(\alpha + 2\alpha^{-1})^2} \\ &= \frac{2^{x-s}ie^{i\pi(1/2-x+s)}(\alpha^{2s} + \alpha^{-2s}2^{2s}e^{-2is\pi} - 2^{s+1}e^{-is\pi})}{(\alpha + 2\alpha^{-1})^2} \\ &= 2^{x-s}ie^{i\pi(1/2-x+s)}CJJ_k(s))^2 = (-1)^{x+s+1}2^{x-s}(CJJ_k(s))^2\end{aligned}$$ □

where if $s = 1$, then we obtain Cassini's identity:

$$CJJ_k(x-1).CJJ_k(x+1) - (CJJ_k(x))^2 = (-2)^{x-1}$$

5. The Incomplete Metallic Shofars

Jacobsthal spirals, which are examined in three dimensions, are discussed from a geometric point of view in connection with the Jacobsthal sequences and the hy-

perbolic functions of these sequences. We focus on the geometry features (curvature and torsion) of the three-dimensional $k-$Jacobsthal spirals. The three-dimensional $k-$Jacobsthal spiral has real and imaginary parts as follow,

$\mathfrak{R}(CJJ_k(x)) = \frac{\alpha^x - 2^x cos(\pi x)\alpha^{-x}}{\alpha+2\alpha^{-1}} = y(x)$

$\mathfrak{I}(CJJ_k(x)) = \frac{2^x sin(\pi x)\alpha^{-x}}{\alpha+2\alpha^{-1}} = z(x)$

If we accept Y axis as real axis and Z axis as imaginary axis then we can give the following equations:

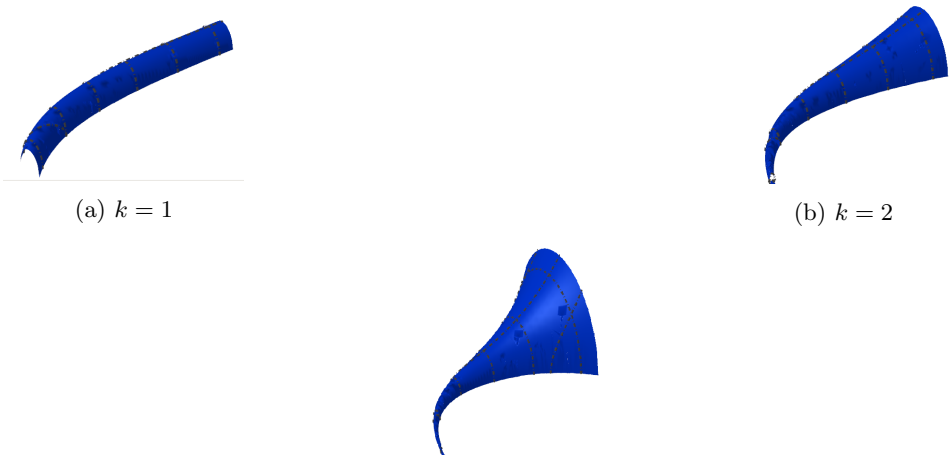
$y(x) - \frac{\alpha^x}{\alpha+2\alpha^{-1}} = \frac{-2^x cos(\pi x)\alpha^{-x}}{\alpha+2\alpha^{-1}}$

$z(x) = \frac{2^x sin(\pi x)\alpha^{-x}}{\alpha+2\alpha^{-1}}$

By adding the squares of previous expressions and taking y and z as independent variables, the following equation is obtained:

$$\left(y - \frac{\alpha^x}{\alpha + 2\alpha^{-1}}\right)^2 + z^2 = \left(\frac{2^x\alpha^{-x}}{\alpha + 2\alpha^{-1}}\right)^2 \tag{5.1}$$

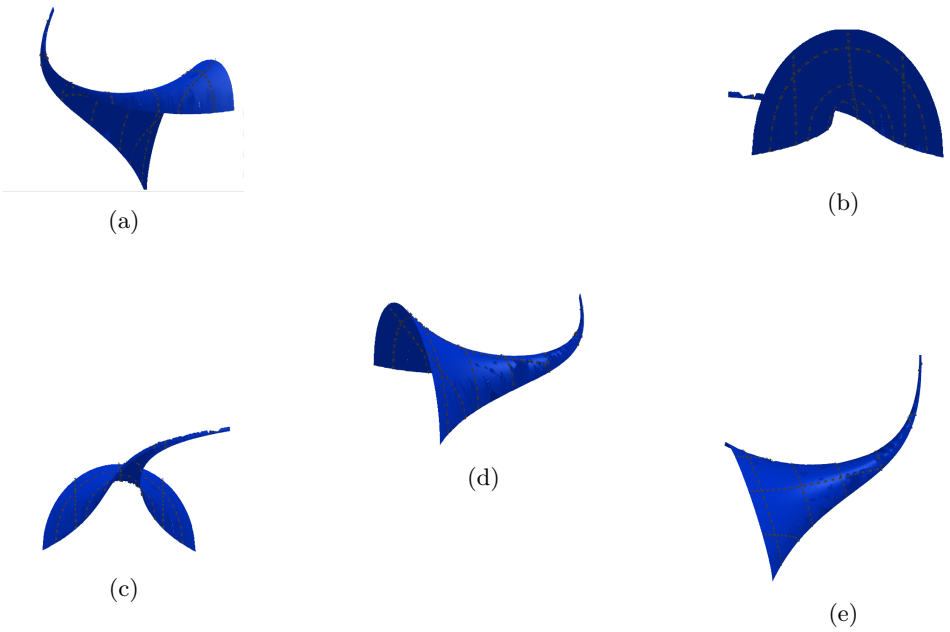
For $k = 1$ Eq5.1 corresponds to a surface called the *Incomplete Golden Shofar* with equation $\left(y - \frac{\alpha^x}{3}\right)^2 + z^2 = \left(\frac{2^{2x}}{9\alpha^{2x}}\right)$. For $k = 2$, we find the *Incomplete Silver Shofar* with equation $\left(y - \frac{\alpha^x}{\sqrt{12}}\right)^2 + z^2 = \left(\frac{2^{2x}}{12\alpha^{2x}}\right)$. Finally, for $k = 3$, we get the *Incomplete Bronze Shofar* with equation $\left(y - \frac{\alpha^x}{\sqrt{17}}\right)^2 + z^2 = \left(\frac{2^{2x}}{17\alpha^{2x}}\right)$.We show the Incomplete Metallic Shofar for $k = 1, 2, 3$ in Fig.5 and the incomplete Bronze Shofar and its reflections in the coordinate planes in Fig.6.



(a) $k = 1$ (b) $k = 2$

(c) $k = 3$

Figure 5: The Incomplete Metallic shofars for $k = 1, 2, 3$.



(a) (b)

(c) (d) (e)

Figure 6: The Incomplete Bronze Shofar and its reflections on the coordinate planes

6. Conclusions

In this paper, we have introduced the new $k-$Jacobsthal hyperbolic functions connected with the $k-$Jacobsthal sequences. Various properties for this functions have been derived and associated with similar identities for $k-$Jacobsthal numbers. Moreover, the $k-$Jacobsthal hyperbolic functions are used to present the alleged Incomplete Metallic Shofars which expand the Incomplete Bronze Shofar.

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