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On the new Narayana polynomials, the Gauss Narayana numbers and their polynomials

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We give a new definition of Narayana polynomials and show that there is a relationship between the coefficient of the new Narayana polynomials and Pascal's triangle. We define the Gauss Narayana numbers and their polynomials. Then we show that there is a relationship between the Gauss Narayana polynomials and the new Narayana polynomials. Also, we show that there is a relationship between the derivatives of the new Narayana polynomials and Pascal's triangle. We also explain the relationship between the new Narayana polynomials and the known Pell numbers. Finally, we give the Hankel transform of the new Narayana polynomials.

Keywords: The new Narayana polynomials; Gauss Narayana numbers; Gauss Narayana polynomials.

AMS Subject Classification: 11B39, 11B83, 11C08

1. Introduction

Some number sequences have great importance in mathematics and they have many great applications such as calculus, applied mathematics and so on [1, 10, 12–14]. It is not well known whether Narayana numbers have similar applications. But if we establish a close relationship between the studies and the features of the Fibonacci sequences, we can confidently talk about the application area in many fields, just like the Fibonacci sequence.

The Narayana numbers come up as a result of solving the problem of Narayana's cow which was given by the Indian mathematician Narayana. This problem is that: *A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years*? [2].

Let n show the year. So, the Narayana problem can be given by the recurrence relation:

$$N_{n+1} = N_n + N_{n-2}, n \geq 2$$

where $N_0 = 0, N_1 = 1$and $N_2 = 1$ [17].

The first few terms of the Narayana numbers are $0, 1, 1, 1, 2, 3, 4, 6, 9, 13, \ldots$. This sequence is called Narayana sequence [17]. It is well known that the Narayan numbers are given by Binet's formula

$$N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.1}$$

where

$$\alpha = \frac{1}{3}\left(1 + \sqrt[3]{\frac{2}{29+3\sqrt{93}}} + \sqrt[3]{\frac{29+3\sqrt{93}}{2}}\right),$$

$$\beta = \frac{1}{3}\left(1 - w\sqrt[3]{\frac{2}{29+3\sqrt{93}}} + w^2\sqrt[3]{\frac{29+3\sqrt{93}}{2}}\right),$$

and

$$\gamma = \frac{1}{3}\left(1 + w^2\sqrt[3]{\frac{2}{29+3\sqrt{93}}} + w\sqrt[3]{\frac{29+3\sqrt{93}}{2}}\right),$$

where $w = \frac{1+i\sqrt{3}}{2}$ is the primitive cube root of unity [17].

The Fibonacci polynomials were studied in 1883 by Catalan. The Fibonacci polynomials $F_n(x)$ are defined by the recurrence relation:

$$F_n(x) = xF_{n+1}(x) + F_n(x), \quad n \geq 1,$$

with the initial condition $F_1(x) = 1$ and $F_2(x) = x$.

When $x = 1$, we get the Fibonacci numbers [9].

We can see some important properties of the Narayana sequence in [4, 11, 17]. Sergio Falcon [5–7] presented the definition of the k-Fibonacci sequence, the first and second derivatives of the polynomials corresponding to this sequence, and the Pascal 2-triangle. Bolat and Köse gave some features of the k-Fibonacci sequence [3]. Bolat and Köse gave some properties of the k-Fibonacci sequence [3].

The Pell polynomials are defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \quad n \geq 2,$$

where $P_0(x) = 0$ and $P_1(x) = 1$ [8, 16].

They are related to the Fibonacci polynomials by $P_n(x) = F_n(2x)$.

Özkan and Taştan [15] obtained some important properties of Gauss Fibonacci polynomials.

In Sec. 2, we give a definition of the new Narayana polynomials. Then we prove some important theorems and Cassini's Identity, related to the new Narayana polynomials.

In Sec. 3, we also define the Hankel Transform of them.

In Sec. 4, we define the Gauss Narayana numbers and their polynomials. We give some of their exciting properties via propositions and theorems.

2. The New Narayana Polynomials, Their Derivatives

Definition 2.1. The new Narayana polynomials are defined recurrently by

$$N_{n+1}(x) = xN_n(x) + N_{n-2}(x), \quad n \geq 2,$$

with initial conditions $N_0(x) = 0, N_1(x) = 1$ and $N_2(x) = 1$.

It is well observed that there is a relation between the new Narayana polynomials and the Narayana numbers. If $x = 1$ then $N_n(1) = N_n$ is the nth Narayana number.

The first few Narayana polynomials and their coefficients are displayed in Table 1.

After this, we will use the Narayana Polynomials instead of the new Narayana polynomials.

Let us consider the rule of formation of the Narayana polynomials, then we have the polynomials in descending order. The array of formed coefficients has the same

Table 1. The first few new Narayana polynomials and their coefficients.

The new Narayana polynomials	Coefficients
$N_1(x) = 1$	1
$N_2(x) = 1$	1
$N_3(x) = 1$	1
$N_4(x) = x + 1$	1 1
$N_5(x) = x^2 + x + 1$	1 1 **1**
$N_6(x) = x^3 + x^2 + x + 1$	1 **1** 1 **1**
$N_7(x) = x^4 + x^3 + x^2 + 2x + 1$	**1** 2 **1** 1 **1**
$N_8(x) = x^5 + x^4 + x^3 + 3x^2 + 2x + 1$	1 **2** 3 **1** 1 **1**
$N_9(x) = x^6 + x^5 + x^4 + 4x^3 + 3x^2 + 2x + 1$	**1** 2 **3** 4 **1** 1 **1**
$N_{10}(x) = x^7 + x^6 + x^5 + 5x^4 + 4x^3 + 3x^2 + 3x + 1$	1 **3** 3 **4** 5 **1** 1 **1**
$N_{11}(x) = x^8 + x^7 + x^6 + 6x^5 + 5x^4 + 4x^3 + 6x^2 + 3x + 1$	**1** 3 **6** 4 **5** 6 **1** 1 **1**
$N_{12}(x) = x^9 + x^8 + x^7 + 7x^6 + 6x^5 + 5x^4 + 10x^3 + 6x^2 + 3x + 1$	1 3 6 **10** 5 **6** 7 **1** 1 1
$N_{13}(x) = x^{10} + x^9 + x^8 + 8x^7 + 7x^6 + 6x^5 + 15x^4 + 10x^3 + 6x^2 + 4x + 1$	1 4 6 10 **15** 6 **7** 8 1 1 1
$N_{14}(x) = x^{11} + x^{10} + x^9 + 9x^8 + 8x^7 + 7x^6 + 21x^5 + 15x^4 + 10x^3 + 10x^2 + 4x + 1$	1 4 10 10 15 **21** 7 8 9 1 1 1
$N_{15}(x) = x^{12} + x^{11} + x^{10} + 10x^9 + 9x^8 + 8x^7 + 28x^6 + 21x^5 + 15x^4 + 20x^3 + 10x^2 + 4x + 1$	1 4 10 20 15 21 28 8 9 10 1 1 1

rule of formation as Pascal's triangle when it is written in a left-justified form. But in the first three terms of Pascal's triangle, respectively, the first two terms appear, then with sliding the box a term appears crosswise after that we have two terms and we sum them. After this process, the same process is repeated by taking three terms in the boxes. So, all terms of Pascal's triangle are obtained.

If we take $x = 1, 2$ then it is obtained the Narayana polynomials whose first elements are $\{N_n(1)\} = \{1, 1, 1, 2, 3, 4, 6, 9, 13, \ldots\}$ and $\{N_n(2)\} = \{1, 1, 1, 3, 7, 15, 33, 73, 161, 355, \ldots\}$, respectively.

$\{N_n(1)\}$ and $\{N_n(2)\}$ are numbered as sequences A000930 and A193641 in *The On-Line Encyclopedia of Integer Sequences* [18].

Characteristic equation of the Narayana polynomials is that

$$
\begin{aligned}
&k^3 - k^2 x - 1 = 0,\\
&r = \frac{1}{3}\left(\frac{\sqrt[3]{2x^3 + 3\sqrt{3}\sqrt{4x^3+27}+27}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}x^2}{\sqrt[3]{2x^3 + 3\sqrt{3}\sqrt{4x^3+27}+27}} + x\right)\\
&s = -\frac{(1 - i\sqrt{3})\sqrt[3]{2x^3 + 3\sqrt{3}\sqrt{4x^3+27}+27}}{6\sqrt[3]{2}}\\
&-\frac{(1 + i\sqrt{3})x^2}{3.2^{2/3}}\sqrt[3]{2x^3 + 3\sqrt{3}\sqrt{4x^3+27}+27} + \frac{x}{3},\\
&t = -\frac{(1 + i\sqrt{3})\sqrt[3]{2x^3 + 3\sqrt{3}\sqrt{4x^3+27}+27}}{6\sqrt[3]{2}}\\
&-\frac{(1 - i\sqrt{3})x^2}{3.2^{2/3}}\sqrt[3]{2x^3 + 3\sqrt{3}\sqrt{4x^3+27}+27} + \frac{x}{3}.
\end{aligned}
\tag{2.1}
$$

Binet's formula of the Narayana polynomials is

$$N_n(x) = \frac{r^{n+1}}{(r-s)(r-t)} + \frac{s^{n+1}}{(s-r)(s-t)} + \frac{t^{n+1}}{(t-r)(t-s)} \tag{2.2}$$

where rs and t are the zeros of the characteristic equation of the Narayana polynomials.

Proposition 2.2 (Cassini's Identity).

$$N_{\mathbf{n-1}}(\mathbf{x})N_{\mathbf{n+1}}(\mathbf{x}) - N_{\mathbf{n}}^{\mathbf{2}}(\mathbf{x}) = -N_{\mathbf{-n-1}}(\mathbf{x}).$$

Proof. By using Binet's formula, we get

$$
\begin{aligned}
&N_{n-1}(x)N_{n+1}(x) - N_n^2(x)\\
&\quad = \left(\frac{r^n}{(r-s)(r-t)} + \frac{s^n}{(s-r)(s-t)} + \frac{t^n}{(t-r)(t-s)}\right)\\
&\qquad \times \left(\frac{r^{n+2}}{(r-s)(r-t)} + \frac{s^{n+2}}{(s-r)(s-t)} + \frac{t^{n+2}}{(t-r)(t-s)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{r^{n+1}}{(r-s)(r-t)} + \frac{s^{n+1}}{(s-r)(s-t)} + \frac{t^{n+1}}{(t-r)(t-s)} \right)^2 \\
&= \frac{r^n s^{n+2} - 2r^{n+1}s^{n+1} + s^n r^{n+2}}{(r-s)(r-t)(s-r)(s-t)} + \frac{r^n t^{n+2} + r^{n+2}t^n - 2r^{n+1}t^{n+1}}{(r-s)(r-t)(t-r)(t-s)} \\
&\quad + \frac{s^n t^{n+2} - 2s^{n+1}t^{n+1} + s^{n+2}t^n}{(s-r)(s-t)(t-r)(t-s)} \\
&= \frac{r^n s^n (r-s)^2}{(r-s)(r-t)(s-r)(s-t)} + \frac{r^n t^n (r-t)^2}{(r-s)(r-t)(t-r)(t-s)} \\
&\quad + \frac{t^n s^n (t-s)^2}{(s-r)(s-t)(t-r)(t-s)} \\
&= - \left(\frac{r^n s^n}{(r-t)(s-t)} + \frac{r^n t^n}{(r-s)(t-s)} + \frac{s^n t^n}{(s-r)(t-r)} \right) \\
&\quad - \left(\frac{t^{-n}}{(r-t)(s-t)} + \frac{s^{-n}}{(r-s)(t-s)} + \frac{r^{-n}}{(s-r)(t-r)} \right) \\
&= -N_{-n-1}(x). \qquad \square
\end{aligned}
$$

Definition 2.3. The Pell polynomials generated by the recurrence relation by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x)$$

where $P_0(x) = 0$ and $P_1(x) = 1$ [16].

They are related to the Narayana polynomials. Namely, if we write $2x$ instead of x in the Narayana polynomials, we get

$$
\begin{aligned}
N_6(2x) &= 8x^3 + 4x^2 + 2x + 1 \\
N_7(2x) &= 16x^4 + 8x^3 + 4x^2 + 4x + 1 \\
N_8(2x) &= 32x^5 + 16x^4 + 8x^3 + 12x^2 + 4x + 1 \\
N_9(2x) &= 64x^6 + 32x^5 + 16x^4 + 32x^3 + 12x^2 + 4x + 1.
\end{aligned}
$$

For example, the coefficients of $N_7(2x)$ $N_8(2x)$ $N_9(2x)$ are, respectively, 16,8,4,4,1 and 32,16,8,12,4,1 and 64,32,16,32,12,4,1.

We show these numbers on the Pell polynomials and we re-express the Pell polynomials:

$$
\begin{aligned}
P_1(x) &= 1 \\
P_2(x) &= 2x \\
P_3(x) &= 4x^2 + 1 \\
P_4(x) &= 8x^3 + 4x
\end{aligned}
$$

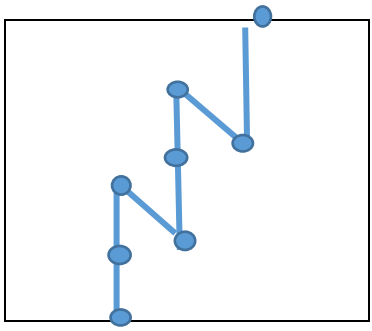


Fig. 1. Linear representation of the Pell number.

$$P_5(x) = 16x^4 + 12x^2 + 1$$
$$P_6(x) = 32x^5 + 32x^3 + 6x$$
$$P_7(x) = 64x^6 + 80x^4 + 24x^2 + 1.$$

If we follow these lines, then we can get the Narayana polynomials, for example; for $N_9(2x)$ we can follow the line starting with $P_7(x)$.

It would be more explanatory if we show this in Fig. 1.

In that case, we give the following theorem in light of the above explanations.

Let $P_{i,j}(x)$ be the jth term of the Pell polynomial $P_i(x)$.

Theorem 2.4. *There is a relationship between the Narayana polynomials and the Pell polynomials*

(1) *If* $n \equiv 0(\text{mod}3)$ *then we have*

$$N_n(2x) = \sum_{i=n-2}^{n} P_{i-2,1}(x) + \sum_{i=n-2}^{n} P_{i-3,2}(x) + \cdots + \sum_{i=n-2}^{n} P_{i-\frac{n}{3},\frac{n-3}{3}}(x) + P_{n-(\frac{n}{3}+1),\frac{n}{3}}(x).$$

(2) *If* $n \equiv 1(\text{mod}3)$ *then we have*

$$N_n(2x) = \sum_{i=n-2}^{n} P_{i-2,1}(x) + \sum_{i=n-2}^{n} P_{i-3,2}(x) + \cdots + \sum_{i=n-2}^{n} P_{i-(\frac{n-1}{3}),\frac{n-4}{3}}(x) + P_{n-(\frac{n-1}{3}+1),\frac{n-1}{3}}(x) + P_{n-(\frac{n-1}{3}+2),\frac{n-1}{3}}(x).$$

(3) *If* $n \equiv 2(\text{mod}3)$ *then we have*

$$N_n(2x) = \sum_{i=n-2}^{n} P_{i-2,1}(x) + \sum_{i=n-2}^{n} P_{i-3,2}(x) + \cdots + \sum_{i=n-2}^{n} P_{i-(\frac{n+1}{3}),\frac{n-2}{3}}(x).$$

Proof. If $n \equiv 2(\mathrm{mod}3)$

$$\underbrace{P_{n-2,1}(x) + P_{n-3,1}(x) + P_{n-4,1}(x)} + \underbrace{P_{n-3,2}(x)+P_{n-4,2}(x) + P_{n-5,2}(x)} + \cdots$$
$$+ \underbrace{P_{n-(\frac{n+1}{3}),\frac{n-2}{3}}(x)+P_{n-(\frac{n+1}{3}+1),\frac{n-2}{3}}(x) + P_{n-(\frac{n+1}{3}+2),\frac{n-2}{3}}(x)}$$

According to the Definition 2.3, the first three terms of the last equation give the first three terms of $N_n(2x)$, the next three terms of the equation give the next three terms of $N_n(2x)$. If we continue in this way, then the last three terms of the equation give the last three terms of $N_n(2x)$. So, all the terms of $N_n(2x)$ are obtained. □

Other proofs of the theorem can be made in a similar way.

Proposition 2.5. *For $n = 3$, we have*

$$\begin{vmatrix} N_{n+1}(x) & N_{n-1}(x) & N_n(x) \\ N_n(x) & N_{n-2}(x) & N_{n-1}(x) \\ N_{n-1}(x) & N_{n-3}(x) & N_{n-2}(x) \end{vmatrix} = x$$

and for $n > 3$, we have

$$\begin{vmatrix} N_{n+1}(x) & N_{n-1}(x) & N_n(x) \\ N_n(x) & N_{n-2}(x) & N_{n-1}(x) \\ N_{n-1}(x) & N_{n-3}(x) & N_{n-2}(x) \end{vmatrix} = x^2.$$

Proof. For $n = 3$, the proof is clear.

For $n > 3$, we give a proof by induction on n.

For $n = 4$, we get

$$\begin{vmatrix} x^2+x+1 & 1 & x+1 \\ x+1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = x^2.$$

Assume it is true for $n = k$, that is

$$\begin{vmatrix} N_{k+1}(x) & N_{k-1}(x) & N_k(x) \\ N_k(x) & N_{k-2}(x) & N_{k-1}(x) \\ N_{k-1}(x) & N_{k-3}(x) & N_{k-2}(x) \end{vmatrix} = x^2. \tag{2.3}$$

Now, prove that it is true for $n = k + 1$.

If we use elementary row, column operations and properties of the determinates, we have

$$\begin{vmatrix} xN_{k+1}(x) + N_{k-1}(x) & N_{k+1}(x) & N_k(x) \\ xN_k(x) + N_{k-2}(x) & N_k(x) & N_{k-1}(x) \\ xN_{k-1}(x) + N_{k-3}(x) & N_{k-1}(x) & N_{k-2}(x) \end{vmatrix}$$

$$= \begin{vmatrix} N_{k+2}(x) & N_{k+1}(x) & N_k(x) \\ N_{k+1}(x) & N_k(x) & N_{k-1}(x) \\ N_k(x) & N_{k-1}(x) & N_{k-2}(x) \end{vmatrix} = -x^2$$

$$= \begin{vmatrix} N_{k+2}(x) & N_k(x) & N_{k+1}(x) \\ N_{k+1}(x) & N_{k-1}(x) & N_k(x) \\ N_k(x) & N_{k-2}(x) & N_{k-1}(x) \end{vmatrix} = x^2.$$

As desired. □

Now, we study on derivations of the Narayana polynomials. Then we give a relation between the coefficients corresponding to the polynomials and Pascal's triangle.

If we take the derivative of the Narayana polynomials, we get

$$N_3'(x) = 0$$

$$N_4'(x) = 1$$

$$N_5'(x) = 2x + 1$$

$$N_6'(x) = 3x^2 + 2x + 1$$

$$N_7'(x) = 4x^3 + 3x^2 + 2x + 2$$

$$N_8'(x) = 5x^4 + 4x^3 + 3x^2 + 6x + 2$$

$$N_9'(x) = 6x^5 + 5x^4 + 4x^3 + 12x^2 + 6x + 2$$

$$N_{10}'(x) = 7x^6 + 6x^5 + 5x^4 + 20x^3 + 12x^2 + 6x + 3$$

$$N_{11}'(x) = 8x^7 + 7x^6 + 6x^5 + 30x^4 + 20x^3 + 12x^2 + 12x + 3$$

$$N_{12}'(x) = 9x^8 + 8x^7 + 7x^6 + 42x^5 + 30x^4 + 20x^3 + 30x^2 + 12x + 3.$$

Let us consider only the coefficients of the derivative polynomials. Then we may write them in a triangle like shown in Fig. 2.

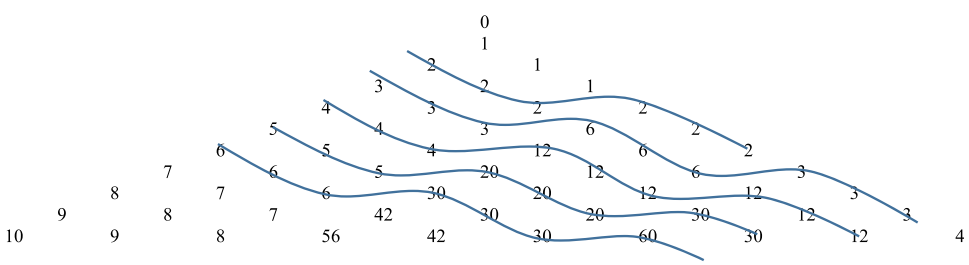


Fig. 2. The coefficients of the derivative polynomials in a triangle.

An exciting property of the triangle is that if we continue by following each line and dividing each line by the starting number, that is, dividing by $2, 3, 4, 5, \ldots$. Respectively, we get Pascal's triangle.

If we again take the derivative of them, we obtain the second derivatives of the Narayana polynomials:

$$\begin{aligned}
N_4''(x) &= 0\\
N_5''(x) &= 2\\
N_6''(x) &= 6x + 2\\
N_7''(x) &= 12x^2 + 6x + 2\\
N_8''(x) &= 20x^3 + 12x^2 + 6x + 6\\
N_9''(x) &= 30x^4 + 20x^3 + 12x^2 + 24x + 6\\
N_{10}''(x) &= 42x^5 + 30x^4 + 20x^3 + 60x^2 + 24x + 6\\
N_{11}''(x) &= 56x^6 + 42x^5 + 30x^4 + 120x^3 + 60x^2 + 24x + 12.
\end{aligned}$$

Let us consider only the coefficients of the second derivative polynomials. So, we may write in a triangle, as shown in Fig. 3.

So, if we continue by following each line and dividing each line by the starting number, that is, dividing by 2,3,4,5, ... respectively, we will get Pascal's triangle.

Different numerical sequences are obtained by simple substitution of variable x for an integer into the derivative of Narayana polynomials. For example, for the first and second derivatives of them, we get

$$\begin{aligned}
\{N_n'(1)\} &= \{0, 1, 3, 6, 10, 20, 35, \ldots\}\\
\{N_n'(2)\} &= \{0, 1, 5, 17, 50, 138, 366, \ldots\}\\
\{N_n'(3)\} &= \{0, 1, 7, 34, 143, 560, \ldots\},\\
\{N_n''(1)\} &= \{0, 2, 8, 20, 44, 92, 182, \ldots\}\\
\{N_n''(2)\} &= \{0, 2, 14, 62, 226, 742, \ldots\}\\
\{N_n''(3)\} &= \{0, 2, 20, 128, 672, \ldots\}
\end{aligned}$$

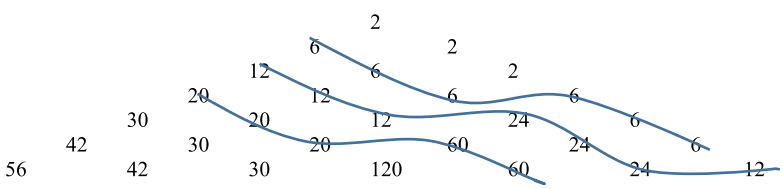


Fig. 3. The coefficients of the second derivative polynomials in a triangle.

3. Hankel Transform of Narayana Polynomials

Let $C = \{c_1\, c_2\, c_3 \ldots$ be a sequence of real numbers. The Hankel tranform of C is the sequence of determinants $H_n = \mathrm{Det}[c_{i+j-2}]$ [11]. That is,

$$H_n = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & \ldots \\ c_1 & c_2 & c_3 & c_4 & c_5 & \ldots \\ c_2 & c_3 & c_4 & c_5 & c_6 & \ldots \\ c_3 & c_4 & c_5 & c_6 & c_7 & \ldots \\ c_4 & c_5 & c_6 & c_7 & c_8 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}.$$

The Hankel determinates of order n of C are the upper-left $n \times n$ subdeterminant of H_n [7].

$$H_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$H_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & x+1 \\ 1 & x+1 & x^2+x+1 \end{vmatrix} = -x^2$$

$$H_4 = \begin{vmatrix} 1 & 1 & 1 & x+1 \\ 1 & 1 & x+1 & x^2+x+1 \\ 1 & x+1 & x^2+x+1 & x^3+x^2+x+1 \\ x+1 & x^2+x+1 & x^3+x^2+x+1 & x^4+x^3+x^2+x+1 \end{vmatrix} = -x^2.$$

If we can continue in this way, we find that Hankel determinants of Narayana polynomials are $-x^2$.

4. The Gauss Narayana Numbers and Their Polynomials

Now, we define the Gauss Narayana number GN_n, the Gauss Narayana polynomials $GN_n(x)$ and give some interesting properties.

Definition 4.1. Gauss Narayana numbers are defined $GN_{n+1} = GN_n + GN_{n-2}$, where $GN_0 = i, GN_1 = 1$ and $GN_2 = 1$.

The first six terms of the Gauss Narayana numbers are $i, 1, 1, 1+i, 2+i, 3+i, 4+2i$.

$GN_n = N_n + iN_{n-2}, n \geq 2$ consequently, $\|GN_n\| = N_n^2 + N_{n-2}^2 = N_{2n-2} + N_{n-3}^2$.

Theorem 4.2.

$$\sum_{n=0}^{n} GN_i = GN_{n-3} - 1.$$

Proof.

$$GN_o = GN_3 - GN_2,$$

$$GN_1 = GN_4 - GN_3$$

$$GN_2 = GN_5 - GN_4$$

$$\vdots$$

$$GN_n = GN_{n-3} - GN_{n-4}$$

By the above equations, we have

$$\sum_{n=0}^{n} GN_i = GN_{n-3} - GN_2.$$

We know that $GN_2 = 1$, we get

$$\sum_{n=0}^{n} GN_i = GN_{n-3} - 1.$$

As desired. □

Theorem 4.3.

$$\sum_{i=1}^{n} GN_{3i-2} = GN_{3n-1}.$$

Proof. By using the definition of the Gauss Narayana numbers, we get

$$GN_4 = GN_5 - GN_2$$

$$GN_7 = GN_8 - GN_{5,}$$

$$\vdots$$

$$GN_{3n-2} = GN_{3n-1} - GN_{3n-4}.$$

From the last equations, we have

$$GN_4 + GN_7 + GN_{10} + \cdots + GN_{3n-2} = GN_{3n-1} - GN_{2.}$$

If we add GN_1 to both sides of the last equation, it is desired. □

We have the following interesting properties related to these numbers.
Let $a + ib$ be the Gauss Narayana number.

(1) $\sum_{i=1}^{n} GN_{3i-1} = GN_{3n}$
(2) $\sum_{i=1}^{n} GN_{3i} = GN_{3n+1} - 1$ For $k = 0, 1, 2, 3, \ldots$,
(3) If $n = 8k, n = 8k - 1$ and $n = 8k - 3$ then $3 \mid a$,
(4) If $n = 7k + 4, n = 7k + 6$ and $n = 7k$ then a is an even number.

Proposition 4.4. *For $n+k \geq 5$, we have*

$$GN_{n+k+1} = GN_{n+k-1} + N_4 GN_{n+k-3} + GN_{n+k-5}.$$

Proof. We give this proof by induction on k. For $k = 1$, we have

$$\begin{aligned} GN_{n+2} &= GN_{n+1} + GN_{n-1} \\ &= GN_{n+1} + GN_{n-2} + GN_{n-4} \\ &= GN_n + GN_{n-2} + GN_{n-2} + GN_{n-4} \\ &= GN_n + 2GN_{n-2} + GN_{n-4} \end{aligned}$$

We know $N_4 = 2$ so we have

$$GN_{n+2} = GN_n + N_4 GN_{n-2} + GN_{n-4}$$

Now, supposing that the proposition is true for $k = t$. That is,

$$GN_{n+t+1} = GN_{n+t-1} + N_4 GN_{n+t-3} + GN_{n+t-5}$$

Now, prove it for $k = t + 1$. That is,

$$\begin{aligned} GN_{n+t+2} &= GN_{n+t+1} + GN_{n+t-1} \\ GN_{n+t+1} + GN_{n+t-1} &= GN_{n+t-1} + 2GN_{n+t-3} + GN_{n+t-5} + GN_{n+t-1} \\ &= GN_{n+t-1} + GN_{n+t-3} + GN_{n+t-3} + GN_{n+t-5} + GN_{n+t-1} \\ &= GN_{n+t} + GN_{n+t-2} + GN_{n+t-2} + GN_{n+t-4} \\ &= GN_{n+t} + N_4 GN_{n+t-2} + GN_{n+t-4} \end{aligned}$$

which are the desired results. So, by induction, we have proven our initial formula and thus for all values of k. □

In above proposition, if we take $k = n$

$$GN_{2n} = GN_{2n-2} + 2GN_{2n-4} + GN_{2n-6}$$

and $k = n + 1$

$$GN_{2n+1} = GN_{2n-1} + 2GN_{2n-3} + GN_{2n-5}.$$

Definition 4.5. The Gauss Narayana polynomials are defined by

$$GN_{n+1}(x) = xGN_n(x) + GN_{n-2}(x),$$

with initial conditions $GN_0(x) = i$, $GN_1(x) = 1$ and $GN_2 = 1$.

Some Gauss Narayana polynomials are

$$GN_0(x) = i$$

$$GN_1(x) = 1$$

$$GN_2 = 1$$
$$GN_3(x) = x + i$$
$$GN_4(x) = x^2 + xi + 1$$
$$GN_5(x) = x^3 + x^2 i + x + 1$$
$$GN_6(x) = x^4 + x^3 i + x^2 + 2x + i$$
$$GN_7(x) = x^5 + x^4 i + x^3 + 3x^2 + 2xi + 1$$

If we take $x = 1$. Then we get the Gauss Narayana numbers.

Theorem 4.6. *There is a relationship between the Narayana polynomials and the Gauss Narayana polynomials. For $n \geq 3$ and $a \in Z$*

(1) *If $n \neq 3a$, then we have*

$$\begin{aligned} GN_n(x) &= (N_{n+1}(x) - N_n(x)) + i(N_n(x) - N_{n-1}(x)) \\ &\quad + (N_{n-1}(x) - N_{n-3}(x)) + i(N_{n-3}(x) - N_{n-4}(x)) + (N_{n-4}(x) \\ &\quad - N_{n-6}(x)) + \cdots + i(N_4(x) - N_3(x)) + N_3(x) \end{aligned}$$

(2) *If $n = 3a$, then we have*

$$\begin{aligned} GN_n(x) &= (N_{n+1}(x) - N_n(x)) + i(N_n(x) - N_{n-1}(x)) \\ &\quad + (N_{n-1}(x) - N_{n-3}(x)) + i(N_{n-3}(x) + N_{n-4}(x)) + \ldots \\ &\quad + (N_5(x) - N_3(x)) + iN_3(x) \end{aligned}$$

Proof. (2) We give a proof by induction on n.

For $n = 3$, we have

$$\begin{aligned} GN_3(x) &= (N_4(x) - N_3(x)) + iN_3(x) \\ &= (x + 1 - 1) + i = x + i \end{aligned}$$

Assume it is true for $n = k$, and $k = n = 3a$, that is,

$$\begin{aligned} GN_k(x) &= (N_{k+1}(x) - N_k(x)) + i(N_k(x) \\ &\quad - N_{k-1}(x)) + \cdots + (N_5(x) - N_3(x)) + iN_3(x) \end{aligned}$$

Now, prove that it is true for $n = k + 3$. That is,

$$GN_{k+3}(x) = xGN_{k+2}(x) + GN_k(x)$$

$$\begin{aligned} xGN_{k+2}(x) + GN_k(x) &= x[(N_{k+3}(x) - N_{k+2}(x)) + i(N_{k+2}(x) \\ &\quad - N_{k+1}(x)) + \cdots + i(N_4(x) - N_3(x)) + N_3(x)] \\ &\quad + [(N_{k+1}(x) - N_k(x)) + i(N_k(x) \end{aligned}$$

$$
\begin{aligned}
&- N_{k-1}(x)) + \cdots + (N_5(x) - N_3(x)) + iN_3(x)] \\
&= [x(N_{k+3}(x) - N_{k+2}(x)) + ix(N_{k+2}(x) \\
&- N_{k+1}(x)) + \cdots + xi(N_4(x) - N_3(x)) + xN_3(x)] \\
&+ [(N_{k+1}(x) - N_k(x)) + i(N_k(x) \\
&- N_{k-1}(x)) + \cdots + (N_5(x) - N_3(x)) + iN_3(x)] \\
&= (N_{k+4}(x) - N_{k+3}(x)) + (N_{k+2}(x) \\
&- N_k(x)) + \cdots + (N_5(x) - N_3(x)) + i(N_{k+3}(x) \\
&- N_{k+2}(x) + N_k(x) - N_{k-1}(x) + \ldots + N_3(x))
\end{aligned}
$$

Here,

$$
\begin{aligned}
GN_{k+3}(x) = (N_{k+4}(x) - N_{k+3}(x)) &+ i(N_{k+3}(x) \\
&- N_{k+2}(x)) + (N_{k+2}(x) - N_k(x)) + i(N_k(x) - N_{k-1}(x)) + (N_{k-1}(x) \\
&- N_{k-3}(x)) + \cdots + (N_5(x) - N_3(x)) + iN_3(x)
\end{aligned}
$$

As desired. □

An example for $n = 6$

$$
\begin{aligned}
GN_6(x) &= (N_7(x) - N_6(x)) + i(N_6(x) - N_5(x)) + (N_5(x) - N_3(x)) + iN_3(x) \\
&= x^4 + x^3 i + x^2 + 2x + i.
\end{aligned}
$$

Proposition 4.7. *For $n \geq 3$ we have*

$$
\begin{vmatrix}
GN_{n+1}(x)GN_{n-1}(x) & GN_n(x) \\
GN_n(x) \;\; GN_{n-2}(x) & GN_{n-1}(x) \\
GN_{n-1}(x)GN_{n-3}(x) & GN_{n-2}(x)
\end{vmatrix} = x^2 - 3x + 2 + i(x - 4).
$$

Proof. We give a proof by induction on n.

For $n = 3$, we have

$$
\begin{vmatrix}
x^2 + xi + 11 & x + i \\
x + i \quad 1 & 1 \\
1 \quad i & 1
\end{vmatrix} = x^2 - 3x + 2 + i(x - 4)
$$

Assume it is true for $n = k$, that is,

$$
\begin{vmatrix}
GN_{k+1}(x)GN_{k-1}(x) & GN_k(x) \\
GN_k(x) \;\; GN_{k-2}(x) & GN_{k-1}(x) \\
GN_{k-1}(x)GN_{k-3}(x) & GN_{k-2}(x)
\end{vmatrix} = x^2 - 3x + 2 + i(x - 4) \tag{4.1}
$$

Now, prove that it is true for $n = k + 1$. That is,

$$\begin{vmatrix} GN_{k+2}(x) & GN_k(x) & GN_{k+1}(x) \\ GN_{k+1}(x) & GN_{k-1}(x) & GN_k(x) \\ GN_k(x) & GN_{k-2}(x) & GN_{k-1}(x) \end{vmatrix} = x^2 - 3x + 2 + i(x-4)$$

If we use elementary row, column operations and properties of the determinates in (4.1)

$$\begin{vmatrix} GN_{k-1}(x) & GN_{k+1}(x) & GN_k(x) \\ GN_{k-2}(x) & GN_k(x) & GN_{k-1}(x) \\ GN_{k-3}(x) & GN_{k-1}(x) & GN_{k-2}(x) \end{vmatrix} = -(x^2 - 3x + 2 + i(x-4)),$$

$$\begin{vmatrix} xGN_{k+1}(x) + GN_{k-1}(x) & GN_{k+1}(x) & GN_k(x) \\ xGN_k(x) + GN_{k-2}(x) & GN_k(x) & GN_{k-1}(x) \\ xGN_{k-1}(x) + GN_{k-3}(x) & GN_{k-1}(x) & GN_{k-2}(x) \end{vmatrix}$$

$$= \begin{vmatrix} GN_{k+2}(x) & GN_{k+1}(x) & GN_k(x) \\ GN_{k+1}(x) & GN_k(x) & GN_{k-1}(x) \\ GN_k(x) & GN_{k-1}(x) & GN_{k-2}(x) \end{vmatrix} = -(x^2 - 3x + 2 + i(x-4))$$

$$= \begin{vmatrix} GN_{k+2}(x) & GN_k(x) & GN_{k+1}(x) \\ GN_{k+1}(x) & GN_{k-1}(x) & GN_k(x) \\ GN_k(x) & GN_{k-2}(x) & GN_{k-1}(x) \end{vmatrix} = x^2 - 3x + 2 + i(x-4).$$

As desired. □

5. Conclusion

We presented a new definition of Narayana polynomials and showed that there is a relationship between the coefficient of the Narayana polynomials and Pascal's triangle. Similarly, we defined the Gauss Narayana numbers and their polynomials. Then we showed a relationship between the Gauss Narayana polynomials and the Narayana polynomials. We studied on determinants of the Narayana and Gauss Narayana polynomials. Further we showed that there is a relationship between the coefficients of first and second derivatives of the Narayana polynomials and Pascal' triangle. We also obtained the relation between the new Narayana polynomials and the known Pell numbers with a diagram and a definition. Finally, we gave the Hankel transform of the new Narayana polynomials. In addition, we presented some theorems and propositions. These identities can be used to develop new identities of polynomials.

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