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New Number Sequences Obtained with Grouping of Fibonacci Numbers

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Abstract

The relations of the sum and difference of two Fibonacci numbers, in which the difference of subscripts is even, was given by Koshy. In this study, the relations of the sum and difference of two Fibonacci numbers, in which is the difference of subscripts is odd, is introduced by using the generating functions. Similar results are given for Lucas numbers. Also, two new family of Fibonacci numbers are obtained.

Keywords: *Binet Formula, Fibonacci sequence, the generalized Fibonacci sequence, Generating functions, Lucas sequence.*

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INTRODUCTION

Considered one of the greatest mathematicians of the Middle Ages, Fibonacci was born in Pisa, Italy in 1170. His full name is Leonardo Fibonacci.

In 1201, he wrote a mathematics book called "Liber Abaci" which means "abacus book" or "calculation book". With this book, he introduced the Arabic numerals and the number system to Europe. This book also describes a problem that is the basis of the Fibonacci number sequence, which deals with the breeding of a pair of rabbits and their progeny. “A pair of adult rabbits breed a new pair of rabbits every month. These puppies become adults at the end of a month and produce a new pair each month after. Find the number of cubs and adult rabbits at the end of any month. It will be assumed that none of the rabbits died during this time." was in the form. According to this problem, the increase in the number of rabbit pairs corresponds to the numbers of a number sequence such as 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …. These numbers are called Fibonacci numbers, and the sequence formed by these numbers is called the Fibonacci sequence. In this sequence, every term except the first two terms is equal to the sum of the two terms that precede it.

As time progressed, the importance of the Fibonacci number sequence was understood more and many studies on this sequence were made and works were revealed [1-11]

In this paper, the relations giving the sum and difference of two Fibonacci numbers where the difference between their subscripts are odd numbers was obtained by using Generator Functions. Similar relations to these relations were found for Lucas numbers. While these relations were obtained, two different Fibonacci families were created.

MATERIALS AND METHODS

Definition 1.

The Fibonacci numbers are the sequence of numbers $\{f_n\}$ defined by the linear recurrence equation, for $n \geq 0$,

$F_{n+2} = F_{n+1} + F_n$

with $F_0 = 0$ and $F_1 = 1$. [11]

Binet formula of this sequence;

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$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Definition 2.
The function $G(x)$ is called the generating function of $\{a_n\}$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$

where $a_0, a_1, a_2, a_3 \cdots$ are terms of a series of real numbers,

The generating function of the Fibonacci sequence; [3]

$$G(x) = \frac{x}{1 - x - x^2}$$

Definition 3
The Lucas numbers are the sequence of numbers $\{l_n\}$ defined by the linear recurrence equation, for $n \geq 0$,

$$L_{n+2} = L_{n+1} + L_n$$

with $L_0 = 2$ and $L_1 = 1$.

Binet formula of this sequence;

$$L_n = \alpha^n + \beta^n.$$

Also, The generating function of the Fibonacci sequence; [3]

$$G(x) = \frac{2 - x}{1 - x - x^2}$$

Theorem 4.
For $n \in \mathbb{N}$ and $m \geq n$, if the difference between subscripts in Fibonacci numbers,$\{(m+n) - (m-n) = 2n\}$, is even number then we have [4]

$$F_{m+n} + F_{m-n} = \begin{cases} L_mF_n, n \text{ is odd number} \\ L_nF_m, n \text{ is even number} \end{cases}$$

Theorem 5. For $n \in \mathbb{N}$ and $m \geq n$, we have the following relation [4]

$$F_{m+n} - F_{m-n} = \begin{cases} L_nF_m, n \text{ is odd number} \\ L_mF_n, n \text{ is even number} \end{cases}$$

RESULTS AND DISCUSSION

Now, for $n \in \mathbb{N}$ and $m \geq n$, when the difference between subscripts in Fibonacci numbers, $\{(m+n+1) - (m-n) = 2n + 1\}$, is odd number, the sum and difference of two Fibonacci numbers will be examined.

Let's first examine the sum of two Fibonacci numbers whose subscript difference is 3.

$$\begin{aligned} F_{m+2} + F_{m-1} &= F_{m+1} + F_m + F_{m-1} \\ &= 2F_m + 2F_{m-1} \\ &= 4F_{m-1} + 2F_{m-2} \\ &= 6F_{m-2} + 4F_{m-3} \\ &= 10F_{m-3} + 6F_{m-4} \cdots \end{aligned}$$

When it is continued similarly, it is seen that the coefficients in the form of 2,4,6,10,16,26,…, as in the Fibonacci numbers, each term progresses as the sum of the two terms preceding it.

If the sum of two Fibonacci numbers is examined when the difference between their subscripts is 5, we get

$$\begin{aligned} F_{m+3} + F_{m-2} &= F_{m+2} + F_{m+1} + F_{m-2} \\ &= 2F_{m+1} + F_m + F_{m-2} \\ &= 3F_m + F_{m-1} + F_{m-1} + F_{m-2} \\ &= 4F_m + F_{m-1} \\ &= 5F_{m-1} + 4F_{m-2} \\ &= 9F_{m-2} + 5F_{m-3} \ldots \end{aligned}$$

In a similar way, it is seen that the coefficients obtained in the form of 4,5,9,14,23,… progress as the sum of the two terms that precede each term, as in the Fibonacci numbers.

When the sum of two Fibonacci numbers whose difference between subscripts is 7 is examined, the coefficients proceed as 4,9,13,22,35,….

Let's show the row number with "*n*" and column number with "*p*",

When the difference between the subscripts of two Fibonacci terms is 1 (n = 0), let's write the coefficients from the sum of the two Fibonacci numbers into the first row, and when the difference is 3, (n = 1), let's write the coefficients from the sum of the two Fibonacci numbers on the second row. If we continue in this way, we obtain Table 1.

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Table 1: *Coefficients obtained from the sum of two Fibonacci numbers where the difference between their subscripts is odd.*

p / n	1	2	3	4	5	6
0	2	3	5	8	13	21
1	2	4	6	10	16	26
2	4	5	9	14	23	37
3	4	9	13	22	35	57
4	10	12	22	34	56	90
5	10	23	33	56	89	145
6	26	31	57	88	145	233
7	26	60	86	146	232	378
8	68	81	149	230	379	609
9	68	157	225	382	607	989

When Table 1 is examined, the numbers in each row are equal to the sum of the two terms that precede it, as in the Fibonacci sequence. Thus, a new family of Fibonacci number sequences is obtained. Notice that when $n = 0$, the known Fibonacci sequenceis obtained.

Let's find the generating function of the numbers 2,2,4,4,10,10,26,26,… in the first column in Table1. Let's denote this function with $G_1(z)$. So we have

$$G_1(z) = 2 + 2z + 4z^2 + 4z^3 + 10z^4 + 10z^5 + 26z^6 + 26z^7 + \quad (1)$$

Multiply both sides of the equation in (3.1) by z^4 and then $-3z^2$.

$$z^4G_1(z) = 2z^4 + 2z^5 + 4z^6 + 4z^7 + 10z^8 + 10z^9 + 26z^{10} + 26z^{11} + \quad (2)$$

$$-3z^2G_1(z) = -6z^2 - 6z^3 - 12z^4 - 12z^5 - 30z^6 - 30z^7 - 786z^8 - \quad (3)$$

From (1), (2) and (3), we get

$(z^4 - 3z^2 + 1)G_1(z) = 2 + 2z - 2z^2 - 2z^3.$

So we obtain

$$G_1(z) = \frac{2 + 2z - 2z^2 - 2z^3}{z^4 - 3z^2 + 1}.$$

Now let's break the function $G_1(z)$ into simple fractions.

$$G_1(z) = \frac{-z - 1}{z^2 + z - 1} + \frac{-z - 1}{z^2 - z - 1}$$

If each simple fractionis examined separately then we have

$$\frac{-z - 1}{z^2 + z - 1} = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \cdots = \sum_{n=0}^{\infty} z^n F_{n+2}$$

and

$$\frac{-z - 1}{z^2 - z - 1} = 1 + z^2 - z^3 + 2z^4 - 3z^5 + 5z^6 - 8z^7 + \cdots = \sum_{n=0}^{\infty} z^n (-1)^n F_{n-1}$$

Thus, the generator function of the first column in Table 1 is

$$G_1(z) = \sum_{n=0}^{\infty} z^n (F_{n+2} + (-1)^{n+2} F_{n-1}).$$

If similar steps are applied for the second column of Table.1 then we have

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$$G_2(z) = \sum_{n=0}^{\infty} z^n (F_{n+3} + (-1)^{n+3} F_{n-2})$$

If we continue like this, the generating function for *p*th column is as follows

$$G_p(z) = \sum_{n=0}^{\infty} z^n \left(F_{n+p+1} + (-1)^{n+p+1} F_{n-p}\right)$$

Theorem 6.
There is a relationship between the columns in Table 1 as follows;

$G_k(z) = G_{k-1}(z) + G_{k-2}(z)$

Proof: Let's do the proof by induction on *k*.
For *k*=1, we have

$$\begin{aligned} G_1(z) + G_2(z) &= \sum_{n=0}^{\infty} z^n (F_{n+2} + (-1)^{n+2} F_{n-1}) + \sum_{n=0}^{\infty} z^n (F_{n+3} + (-1)^{n+3} F_{n-2}) \\ &= \sum_{n=0}^{\infty} z^n (F_{n+2} + (-1)^{n+2} F_{n-1} + F_{n+3} + (-1)^{n+3} F_{n-2}) \\ &= \sum_{n=0}^{\infty} z^n \left(F_{n+3} + F_{n+2} + (-1)^{n+2} (F_{n-1} - F_{n-2})\right) \\ &= \sum_{n=0}^{\infty} z^n \left(F_{n+4} + (-1)^{n+2} (F_{n-3})\right) \\ &= \sum_{n=0}^{\infty} z^n \left(F_{n+4} + (-1)^{n+4} (F_{n-3})\right) \\ &= G_3(z) \end{aligned}$$

Now, assume that the theorem holds for $p = k$, that is

$$G_k(z) = G_{k-1}(z) + G_{k-2}(z).$$

Then, for $p = k + 1$, we have

$$\begin{aligned} G_{k+1}(z) &= G_k(z) + G_{k-1}(z) = 2G_{k-1}(z) + G_{k-2}(z) \\ &= 2\left(\sum_{n=0}^{\infty} z^n \left(F_{n+k} + (-1)^{n+k} F_{n-k+1}\right)\right) + \sum_{n=0}^{\infty} z^n (F_{n+k-1} + (-1)^{n+p-1} F_{n-k+2}) \\ &= \sum_{n=0}^{\infty} z^n [(F_{n+k} + F_{n+k} + F_{n+k-1}) + (-1)^{n+k+2} (F_{n-k+1} + F_{n-k+1} - F_{n-k+2})] \\ &= \sum_{n=0}^{\infty} z^n [(F_{n+k} + F_{n+k+1}) + (-1)^{n+k+2} (F_{n-k+1} - F_{n-k})] \\ &= \sum_{n=0}^{\infty} z^n \left(F_{n+k+2} + (-1)^{n+k+2} F_{n-k-1}\right) \blacksquare \end{aligned}$$

Theorem 6. shows us that for each column the generating function of that column can be written as the sum of the generating functions of the first two columns preceding it.

Theorem7.
For $m, n \in \mathbb{N}$ ve $p \in \mathbb{N}^+$, if we show the sum of two Fibonacci numbers as $F_{m+n+1} + F_{m-n}$ where the difference between the subscales of the terms in the Fibonacci sequence is odd, there is the following relation

$$F_{m+n+1} + F_{m-n} = \left(F_{n+p+1} + (-1)^{n+p+1}F_{n-p}\right)F_{m-p-1} + \left(F_{n+p+2} + (-1)^{n+p+2}F_{n-p-1}\right)F_{m-p}$$

Proof: For $n \in \mathbb{N}^+$, we get

$$F_{m+n+1} + F_{m-n} = \left(F_{n+p+1} + (-1)^{n+p+1}F_{n-p}\right)F_{m-p-1} + \left(F_{n+p+2} + (-1)^{n+p+2}F_{n-p-1}\right)F_{m-p}.$$

If we use Binet formula, we have

$$\left(F_{n+p+1} + (-1)^{n+p+1}F_{n-p}\right)F_{m-p-1} + \left(F_{n+p+2} + (-1)^{n+p+2}F_{n-p-1}\right)F_{m-p}$$

$$= \left(\frac{\alpha^{n+p+1} - \beta^{n+p+1}}{\alpha - \beta} + (-1)^{n+p+1} \cdot \frac{\alpha^{n-p} - \beta^{n-p}}{\alpha - \beta}\right)\frac{\alpha^{m-p-1} - \beta^{m-p-1}}{\alpha - \beta}$$
$$+ \left(\frac{\alpha^{n+p+2} - \beta^{n+p+2}}{\alpha - \beta} + (-1)^{n+p+2} \cdot \frac{\alpha^{n-p-1} - \beta^{n-p-1}}{\alpha - \beta}\right)\frac{\alpha^{m-p} - \beta^{m-p}}{\alpha - \beta}$$

$$= \frac{\alpha^{n+m} - \beta^{n+p+1}\alpha^{m-p-1} + (-1)^{n+p+1}\alpha^{n+m-2p-1} - (-1)^{n+p+1}\alpha^{m-p-1}\beta^{n-p}}{(\alpha-\beta)^2}$$
$$+ \frac{-\alpha^{n+p+1}\beta^{m-p-1} + \beta^{n+p} - (-1)^{n+p+1}\alpha^{n-p}\beta^{m-p-1} + (-1)^{n+p+1}\beta^{n+m-2p-1}}{(\alpha-\beta)^2}$$
$$+ \frac{\alpha^{n+m+2} - \beta^{n+p+2}\alpha^{m-p} + (-1)^{n+p+2}\alpha^{n+m-2p-1} - (-1)^{n+p+2}\alpha^{m-p}\beta^{n-p-1}}{(\alpha-\beta)^2}$$
$$+ \frac{-\alpha^{n+p+2}\beta^{m-p} + \beta^{n+m+2} - (-1)^{n+p+2}\alpha^{n-p-1}\beta^{m-p} + (-1)^{n+p+2}\beta^{n+m-2p-1}}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{n+m}(\alpha^2+1) - \beta^{n+p+1}\alpha^{m-p-1}(\alpha\beta+1) - (-1)^{n+p+1}\alpha^{m-p-1}\beta^{n-p-1}(\beta-\alpha)}{(\alpha-\beta)^2}$$
$$+ \frac{-\alpha^{n+p+1}\beta^{m-p-1}(\alpha\beta+1) + \beta^{n+m}(\beta^2+1) - (-1)^{n+p+1}\alpha^{n-p-1}\beta^{m-p-1}(\alpha-\beta)}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{n+m}(\alpha^2+1) + \beta^{n+m}(\beta^2+1)}{(\alpha-\beta)^2}$$
$$+ \frac{(-1)^{n+p}\alpha^{m-p-1}\beta^{n-p-1}(\beta-\alpha) + (-1)^{n+p}\alpha^{n-p-1}\beta^{m-p-1}(\alpha-\beta)}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{n+m+1}\left(\alpha + \frac{1}{\alpha}\right) + \beta^{m+n+1}\left(\beta + \frac{1}{\beta}\right) + (-1)^{n+p}(\alpha-\beta)(\alpha\beta)^{n-p-1}(\beta^{m-n} - \alpha^{m-n})}{(\alpha-\beta)^2}.$$

If we use the flowing equations,

$$\frac{1}{\alpha} = -\beta, \alpha\beta = -1$$

We obtain

$$= \frac{\alpha^{m+n+1}(\alpha-\beta) - \beta^{m+n+1}(\alpha-\beta) + (-1)^{2n-1}(\alpha-\beta)(\beta^{m-n} - \alpha^{m-n})}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{m+n+1} - \beta^{m+n+1}}{(\alpha-\beta)} + \frac{\alpha^{m-n} - \beta^{m-n}}{(\alpha-\beta)}$$

$$= F_{m+n+1} + F_{m-n} \blacksquare$$

Theorem 8.
For $m, n \in \mathbb{N}$ and $p \in \mathbb{N}^+$, we have

$$L_{m+n+1} + L_{m-n} = \left(F_{n+p+1} + (-1)^{n+p+1}F_{n-p}\right)L_{m-p-1} + \left(F_{n+p+2} + (-1)^{n+p+2}F_{n-p-1}\right)L_{m-p}$$

Proof: We have

$L_{m+n+1} + L_{m-n} = \left(F_{n+p+1} + (-1)^{n+p+1}F_{n-p}\right)L_{m-p-1} + \left(F_{n+p+2} + (-1)^{n+p+2}F_{n-p-1}\right)L_{m-p}.$

If we use Binet Formula, we get

$$\left(F_{n+p+1} + (-1)^{n+p+1}F_{n-p}\right)L_{m-p-1} + \left(F_{n+p+2} + (-1)^{n+p+2}F_{n-p-1}\right)L_{m-p}$$

$$= \left(\frac{\alpha^{n+p+1} - \beta^{n+p+1}}{\alpha-\beta} + (-1)^{n+p+1} \cdot \frac{\alpha^{n-p} - \beta^{n-p}}{\alpha-\beta}\right)(\alpha^{m-p-1} + \beta^{m-p-1})$$

$$+ \left(\frac{\alpha^{n+p+2} - \beta^{n+p+2}}{\alpha-\beta} + (-1)^{n+p+2} \cdot \frac{\alpha^{n-p-1} - \beta^{n-p-1}}{\alpha-\beta}\right)(\alpha^{m-p} + \beta^{m-p})$$

$$= \frac{\alpha^{n+m} - \beta^{n+p+1}\alpha^{m-p-1} + (-1)^{n+p+1}\alpha^{n+m-2p-1} - (-1)^{n+p+1}\alpha^{m-p-1}\beta^{n-p}}{\alpha-\beta}$$

$$+ \frac{\alpha^{n+p+1}\beta^{m-p-1} - \beta^{n+m} + (-1)^{n+p+1}\alpha^{n-p}\beta^{m-p-1} - (-1)^{n+p+1}\beta^{n+m-2p-1}}{\alpha-\beta}$$

$$+ \frac{\alpha^{n+m+2} - \beta^{n+p+2}\alpha^{m-p} + (-1)^{n+p+2}\alpha^{n+m-2p-1} - (-1)^{n+p+2}\alpha^{m-p}\beta^{n-p-1}}{\alpha-\beta}$$

$$+ \frac{\alpha^{n+p+2}\beta^{m-p} - \beta^{n+p+2} + (-1)^{n+p+2}\alpha^{n-p-1}\beta^{m-p} - (-1)^{n+p+2}\beta^{n+m-2p-1}}{\alpha-\beta}$$

$$= \frac{\alpha^{n+m}(\alpha^2+1) - \alpha^{m-p-1}\beta^{n+p+1}(\alpha\beta+1) - (-1)^{n+p+1}\alpha^{m-p-1}\beta^{n-p-1}(\beta-\alpha)}{\alpha-\beta}$$

$$+ \frac{\alpha^{n+p+1}\beta^{m-p-1}(\alpha\beta+1) - \beta^{n+m}(\beta^2+1) + (-1)^{n+p+1}\alpha^{n-p-1}\beta^{m-p-1}(\alpha-\beta)}{\alpha-\beta}$$

$$= \frac{\alpha^{n+m}(\alpha^2+1) - \beta^{m+n}(\beta^2+1)}{\alpha-\beta}$$

$$+ \frac{(-1)^{n+p}\alpha^{m-p-1}\beta^{n-p-1}(\beta-\alpha) - (-1)^{n+p}\alpha^{n-p-1}\beta^{m-p-1}(\alpha-\beta)}{\alpha-\beta}$$

$$= \frac{\alpha^{n+m+1}\left(\alpha+\frac{1}{\alpha}\right) - \beta^{m+n+1}\left(\beta+\frac{1}{\beta}\right) + (-1)^{n+p+1}(\alpha-\beta)(\alpha\beta)^{n-p-1}(\alpha^{m-n} + \beta^{m-n})}{\alpha-\beta}$$

$$= \frac{\alpha^{m+n+1}(\alpha-\beta) + \beta^{m+n+1}(\alpha-\beta) + (-1)^{2n}(\alpha-\beta)(\alpha^{m-n} + \beta^{m-n})}{\alpha-\beta}$$

$$= \alpha^{m+n+1} + \beta^{m+n+1} + \alpha^{m-n} + \beta^{m-n}$$

$= L_{m+n+1} + L_{m-n}$. ■

Now let's examine the difference of two Fibonacci numbers when the difference between subscales of terms in the Fibonacci sequence is odd.

When the difference between the subscripts of two Fibonacci terms is 1 (n = 0), let's write the coefficients from the difference of the two Fibonacci numbers in the first row, when the difference is 3 (n = 1), let's write the coefficients obtained from the difference of the two Fibonacci numbers in the second row. If we continue in this way, we obtain Table 2.

Table 2: *Coefficients derived from the difference of two Fibonacci numbers where the difference between their subscripts is odd numbers.*

p / n	1	2	3	4	5	6
0	2	2	4	6	10	16
1	2	5	7	12	19	31
2	6	7	13	20	33	53
3	6	14	20	34	54	88
4	16	19	35	54	89	143
5	16	37	53	90	143	233
6	42	50	92	142	234	376
7	42	97	139	236	375	611

When Table 2 is examined, the numbers in each row are equal to the sum of the two terms that precede it, as in the Fibonacci sequence. Thus, a new family of Fibonacci number sequences is obtained.

The generating function for *p*th column is as follows

$$G_p(z) = \sum_{n=0}^{\infty} z^n \left(F_{n+p+2} + (-1)^{n+p+1} F_{n-p+1}\right)$$

Theorem 9.
There is a relationship between the columns in Table2 as follows

$G_k(z) = G_{k-1}(z) + G_{k-2}(z)$.

Proof: The proof is done similarly to the method used in Theorem 6. ■

Theorem 10.
For $m, n \in \mathbb{N}$ ve $p \in \mathbb{N}^+$, if we show the difference of two Fibonacci numbers as $F_{m+n+1} - F_{m-n}$ where the difference between the subscales of the terms in the Fibonacci sequence is odd, there is the following relation

$$F_{m+n+2} - F_{m-n-1} = \left(F_{n+p+2} + (-1)^{n+p+1} F_{n-p+1}\right) F_{m-p-1} + \left(F_{n+p+3} + (-1)^{n+p+2} F_{n-p}\right) F_{m-p}$$

Proof:

$$F_{m+n+2} - F_{m-n-1} = \left(F_{n+p+2} + (-1)^{n+p+1} F_{n-p+1}\right) F_{m-p-1} + \left(F_{n+p+3} + (-1)^{n+p+2} F_{n-p}\right) F_{m-p}$$

If we use Binet formula, we have

$$\left(F_{n+p+2} + (-1)^{n+p+1} F_{n-p+1}\right) F_{m-p-1} + \left(F_{n+p+3} + (-1)^{n+p+2} F_{n-p}\right) F_{m-p}$$

$$= \left(\frac{\alpha^{n+p+2} - \beta^{n+p+2}}{\alpha - \beta} + (-1)^{n+p+1} \cdot \frac{\alpha^{n-p+1} - \beta^{n-p+1}}{\alpha - \beta}\right) \frac{(\alpha^{m-p-1} - \beta^{m-p-1})}{\alpha - \beta}$$

$$+\left(\frac{\alpha^{n+p+3}-\beta^{n+p+3}}{\alpha-\beta}+(-1)^{n+p+2}\cdot\frac{\alpha^{n-p}-\beta^{n-p}}{\alpha-\beta}\right)\frac{(\alpha^{m-p}-\beta^{m-p})}{\alpha-\beta}$$

$$=\frac{\alpha^{n+m+1}-\beta^{n+p+2}\alpha^{m-p-1}+(-1)^{n+p+1}\alpha^{m+n-2p}-(-1)^{n+p+1}\alpha^{m-p-1}\beta^{n-p+1}}{(\alpha-\beta)^2}$$

$$+\frac{-\alpha^{n+p+2}\beta^{m-p-1}+\beta^{n+m+1}-(-1)^{n+p+1}\alpha^{n-p+1}\beta^{m-p-1}+(-1)^{n+p+1}\beta^{n+m-2p}}{(\alpha-\beta)^2}$$

$$+\frac{\alpha^{n+m+3}-\alpha^{m-p}\beta^{n+p+3}+(-1)^{n+p+2}\alpha^{n+m-2p}-(-1)^{n+p+2}\alpha^{m-p}\beta^{n-p}}{(\alpha-\beta)^2}$$

$$+\frac{-\alpha^{n+p+3}\beta^{m-p}+\beta^{m+n+3}-(-1)^{n+p+2}\alpha^{n-p}\beta^{m-p}+(-1)^{n+p+2}\beta^{n+m-2p}}{(\alpha-\beta)^2}$$

$$=\frac{\alpha^{n+m+1}(\alpha^2+1)-\alpha^{m-p-1}\beta^{n+p+1}(\alpha\beta+1)-(-1)^{n+p+1}\alpha^{m-p-1}\beta^{n-p}(\beta-\alpha)}{(\alpha-\beta)^2}$$

$$+\frac{-\alpha^{n+p+2}\beta^{m-p-1}(\alpha\beta+1)+\beta^{n+m+1}(\beta^2+1)-(-1)^{n+p+1}\alpha^{n-p}\beta^{m-p-1}(\alpha-\beta)}{(\alpha-\beta)^2}$$

$$=\frac{\alpha^{n+m+2}\left(\alpha+\frac{1}{\alpha}\right)+\beta^{m+n+2}\left(\beta+\frac{1}{\beta}\right)}{(\alpha-\beta)^2}$$

$$-\frac{(-1)^{n+p+1}(\alpha-\beta)(\alpha\beta)^{n-p}(-\alpha^{m-n-1}+\beta^{m-n-1})}{(\alpha-\beta)^2}$$

$$=\frac{\alpha^{m+n+2}(\alpha-\beta)+\beta^{m+n+2}(\beta-\alpha)+(\alpha-\beta)(-\alpha^{m-n-1}+\beta^{m-n-1})}{(\alpha-\beta)^2}$$

$$=\frac{\alpha^{m+n+2}-\beta^{m+n+2}}{(\alpha-\beta)}-\frac{\alpha^{m-n-1}-\beta^{m-n-1}}{(\alpha-\beta)}$$

$$=F_{m+n+2}-F_{m-n-1}.\blacksquare$$

Theorem 11. For m, $n \in \mathbb{N}$ ve $p \in \mathbb{N}^+$, we have

$$L_{m+n+1}-L_{m-n}=\left(F_{n+p+2}+(-1)^{n+p+1}F_{n-p+1}\right)L_{m-p-1}\ +\left(F_{n+p+3}+(-1)^{n+p+2}F_{n-p}\right)L_{m-p}$$

Proof: It is done similarly to the method used in Theorem 10.■

CONCLUSION

In this study, in the case where the difference between the subscales of the terms in the Fibonacci Sequence and the Generalized Fibonacci Sequence is even numbers, the relations related to the sum and difference of two Fibonacci numbers are given. Then, when the difference between the subscripts is odd, the relations giving the sum and difference of two Fibonacci numbers are obtained by using Generator Functions. Similar correlations to these are found for Lucas sequences. With these relations, two new Fibonacci families have been created.

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