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Contributed Paper

Fibonacci Sequences in Nilpotent Groups with Class *n*

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ABSTRACT

We have given formulas to find the αth term of the sequence constituted 3-step Fibonacci sequences in a nilpotent group with exponent p and nilpotency class n (p is a prime number).

Keywords : Fibonacci sequences, nilpotency class, nilpotent group.

1. INTRODUCTION

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Takahashi gave a fast algorithm which is based on the product of Lucas number to compute large Fibonacci numbers [1]. Fibonacci sequences have been an interesting subject in applied mathematics. West has shown using transfer matrices that the number $|S_n(123,3214)|$ of permutations avoiding the patterns 123 and 3214 is the Fibonacci number F_{2n} [2].

The study of Fibonacci sequences in groups began with the earlier work of Wall [3] where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid eighties Wilcox extended the problem to abelian groups [4]. Prolific co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups [5]. Aydin and Smith proved in [6] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. The theory has been generalized in [7] to the ordinary 3-step Fibonacci sequences in finite nilpotent groups of nilpotency class 2 and exponent p. Furthermore, it is shown that for the *3*-step general Fibonacci recurrence and any finite p-group of exponent p and nilpotency class *2*, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step general Fibonacci sequence in the field *GF(p)* [8]. Then it is given formulas to find α. term of the sequence constituted 3-step general Fibonacci sequences in a nilpotent group with exponent p and nilpotency class 4 [9].

One of the latest works in this area is [10] where it is constituted 3-step Fibonacci sequences in nilpotent group with exponent p and nilpotency class n. The aim of this work is to generalize [10] to cover works in [9].

2. THREE STEP RECURRENCES

Let G be a finite group. We define a family of recurrences, parameterized by natural numbers n, via equations

$$x_i = x_{i-1}x_{i-2}...x_{i-n} \qquad (R_n)$$

We fix $n>1$. We may 'start' the recurrence with any initial data $g_0, g_1, ..., g_{n-1}$ and then the recurrence will define a periodic bi-infinite sequence (g_i) indexed by the integers. We use the loop to describe a bi-infinite sequence satisfying the recurrence. Notice that,

if $\alpha \in \mathbb{Z}^+$, the set of posotive integers, then putting $h_i = g_{i+\alpha}$ for $0 \le i < n$ gives another set of initial data. The recurrence R_n then gives us another loop (h_i) of G. Notice that (h_i) is just (g_i) shifted through α steps. We say that (h_i) is a rotation of (g_i).

The fundamental period of a loop is clearly a significant number, and sometimes is called the Wall's number.

Let us consider the more general recurrences written multiplicatively as

$$x_i = x_{i-1}^{a_1} x_{i-2}^{a_2} \ldots x_{i-n}^{a_n}$$

or additively as

$$x_i = \sum_{r=1}^{n} a_r x_{i-r} \qquad (A_n).$$

The exponents(coefficients) are deemed to be integers. Such a recurrence may not be so well behaved as R_n, for it may be singular in particular group. This means that initial data may not define a unique bi-infinite periodic sequence; the problem arises when attempting to extend the sequence through negative integer subscripts. Certainly all is well if $a_n \in \{1,-1\}$. Equally well, if we are working in a p-group all is well provided $lcm(p,a_n)=1$. We now press on to discuss the recurrences R_3. We work in the additive group H of the a finite field $GF(p^n)$.

Now, let $s=(s_i)$ denote the ordinary 3-step Fibonacci sequence in *GF(p)* defined by the recurrence $s_{i+3} = s_i + s_{i+1} + s_{i+2}$ and the initial data $s_0 = 0$, $s_1 = 0$ and $s_2 = 1$. This is a bi-infinite periodic sequence or loop indexed by the integers. The shortest period of this sequence is called the fundamental period and it will be denoted by k. As before, the period of any loop must divide k for the usual reasons.

Definition 2.2: F/R is called to be relatively free group if F is a free group and R is a fully invariant subgroup of F.

Definition 2.3: Let $H \lhd G, K \lhd G$ and $K \le H$. If H/K is contained in the center of G/K, then H/K is called a central factor of G. A group G is called nilpotent if it has finite series of normal subgroups

$$G = G_0 \ge G_1 \ge \cdots G_r = 1$$

such that G_{i-1}/G_i is a central factor of G for each $i = 1,2,\cdots,r$. The smallest possible r is called the nilpotency class of G.

Let G be the group which has the following presentation

$$G = < x_1, x_2, \ldots, x_n, x_{n+1} : [x_2,x_1] = x_3, [x_3,x_1] = x_4, \ldots, [x_n,x_1] = x_{n+1} >,$$

where those pairs of generators with unspecified commutators are implicitly deemed to commute. This is a group of nilpotecy class n and exponent p. Each element of the group will have a unique representation as $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n+1}^{a_{n+1}}$ where $a_1, a_2, a_3, \ldots a_{n+1} \in Z/pZ$. The element $x_{n+1}^{a_{n+1}}$ is central and the subgroup $H = < x_2, x_3, \cdots, x_n, x_{n+1} >$ is abelian. Let $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n+1}^{a_{n+1}}$, $x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots x_{n+1}^{b_{n+1}}$ and $x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots x_{n+1}^{c_{n+1}}$ be elements of G. Having distinguished this way of writing elements we can even think of groups elements as vectors of dimension $n+1$ over *GF(p)*, i.e. as $(a_1, a_2, a_3, \ldots a_{n+1}) \in \mathbb{Z}/p\mathbb{Z}$. So triple multiplication is determined by the following rules;

That is, $(x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n+1}^{a_{n+1}})(x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots x_{n+1}^{b_{n+1}})(x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots x_{n+1}^{c_{n+1}}) = (x_1^{d_1} x_2^{d_2} x_3^{d_3} \cdots x_{n+1}^{d_{n+1}})$

As to knowledge in above we can give the following lemma.

Lemma 2.4: If G is a group of exponent p, nilpotency class n and with the presentation

$$G = < x_1, x_2, \ldots, x_n, x_{n+1} : [x_2,x_1] = x_3, [x_3,x_1] = x_4, \ldots, [x_n,x_1] = x_{n+1} >$$

then entries of the 3-step Fibonacci sequences formed by three elements of G have the following form.

$$d_1 = a_1 + b_1 + c_1$$
$$d_2 = a_2 + b_2 + c_2$$
$$d_3 = a_3 + b_3 + c_3 + b_1a_2 + (a_2 + b_2)c_1$$
$$d_4 = a_4 + b_4 + c_4 + b_1a_3 + \binom{b_1}{2}a_2 + \binom{c_1}{1}(a_3 + b_3 + b_1a_2) + (a_2 + b_2)\binom{c_1}{2}$$

$$d_5 = a_5 + b_5 + c_5 + b_1a_4 + \binom{b_1}{2}a_3 + \binom{b_1}{3}a_2 + \binom{c_1}{1}(a_4 + b_4 + b_1a_3 + a_2\binom{b_1}{2})$$
$$+\binom{c_1}{2}(a_3 + b_3 + b_1a_2) + \binom{c_1}{3}(a_2 + b_2)$$

$$d_6 = a_6 + b_6 + c_6 + b_1a_5 + \binom{b_1}{2}a_4 + \binom{b_1}{3}a_3 + \binom{b_1}{4}a_2$$
$$+\binom{c_1}{1}(a_5 + b_5 + b_1a_4 + \binom{b_1}{2}a_3 + \binom{b_1}{3}a_2) + \binom{c_1}{2}(a_4 + b_4 + b_1a_3 + a_2\binom{b_1}{2})$$
$$+\binom{c_1}{3}(a_3 + b_3 + b_1a_2) + \binom{c_1}{4}(a_2 + b_2)$$

$$\vdots$$

$$d_{n+1} = a_{n+1} + b_{n+1} + c_{n+1} + \binom{b_1}{1}a_n + \binom{b_1}{2}a_{n-1} + \binom{b_1}{3}a_{n-2} + \ldots + \binom{b_1}{n-1}a_2$$
$$+\binom{c_1}{1}(a_n + b_n + b_1a_{n-1} + \binom{b_1}{2}a_{n-2} + \ldots + \binom{b_1}{n-2}a_2)$$
$$+\binom{c_1}{2}(a_{n-1} + b_{n-1} + b_1a_{n-2} + \binom{b_1}{2}a_{n-3} + \ldots + \binom{b_1}{n-3}a_2) + \ldots.$$
$$+\binom{c_1}{n-2}(a_3 + b_3 + \binom{b_1}{1}a_2) + \binom{c_1}{n-1}(a_2 + b_2) \qquad \ldots..[10]$$

3. Main Results

Now we want to generalize the formulas in the above lemma for a group of exponent p and nilpotency class n.

Theorem 3.1: If G is a group of exponent p, nilpotency class n and with the presentation

$$G = < x_1, x_2, \ldots, x_n, x_{n+1} : [x_2, x_1] = x_3, [x_3, x_1] = x_4, \ldots, [x_n, x_1] = x_{n+1} >$$

then entries of the 3-step Fibonacci sequences formed by three elements of G have the following form.

$$a_\alpha = \sum_{i=0}^{\alpha-1} t_i = \sum_{i=0}^{\alpha-1} (s_{i+1} - s_i)$$

$$b_\alpha = \sum_{i=0}^{\alpha-1} s_i$$

$$c_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}))$$

$$d_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+1} c_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} s_i \binom{t_{i+1}}{2} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+2}(c_i + c_{i+1} + s_i t_{i+1})$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2}(s_i + s_{i+1})$$

$$\vdots$$

$$y_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} v_i \binom{t_{i+1}}{1} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} u_i \binom{t_{i+1}}{2} + \ldots + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} c_i \binom{t_{i+1}}{n-3} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} s_i \binom{t_{i+1}}{n-2}$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{1} \left[v_i + v_{i+1} + u_i \binom{t_{i+1}}{1} + r_i \binom{t_{i+1}}{2} + \ldots + c_i \binom{t_{i+1}}{n-4} + s_i \binom{t_{i+1}}{n-3} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2} \left[u_i + u_{i+1} + r_i \binom{t_{i+1}}{1} + \ldots + c_i \binom{t_{i+1}}{n-5} + s_i \binom{t_{i+1}}{n-4} \right] + \ldots$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-4} \left[d_i + d_{i+1} + c_i \binom{t_{i+1}}{1} + s_i \binom{t_{i+1}}{2} \right] + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-3} \left[c_i + c_{i+1} + s_i \binom{t_{i+1}}{1} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-2}(s_i + s_{i+1})$$

and

$$z_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} y_i \binom{t_{i+1}}{1} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} v_i \binom{t_{i+1}}{2} + \ldots + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} c_i \binom{t_{i+1}}{n-2} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} s_i \binom{t_{i+1}}{n-1}$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{1} \left[y_i + y_{i+1} + v_i \binom{t_{i+1}}{1} + u_i \binom{t_{i+1}}{2} + \ldots + c_i \binom{t_{i+1}}{n-3} + s_i \binom{t_{i+1}}{n-2} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2} \left[v_i + v_{i+1} + u_i \binom{t_{i+1}}{1} + r_i \binom{t_{i+1}}{2} \ldots + c_i \binom{t_{i+1}}{n-4} + s_i \binom{t_{i+1}}{n-3} \right] + \ldots$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-3} \left[d_i + d_{i+1} + c_i \binom{t_{i+1}}{1} + s_i \binom{t_{i+1}}{2} \right] + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-2} \left[c_i + c_{i+1} + s_i \binom{t_{i+1}}{1} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-1}(s_i + s_{i+1})$$

Proof. Let G be the group which has the following presentation

$$G = < x_1, x_2, \ldots, x_n, x_{n+1} : [x_2, x_1] = x_3, [x_3, x_1] = x_4, \ldots, [x_n, x_1] = x_{n+1} >,$$

where those pairs of generators with unspecified commutators are implicitly deemed to

commute. This is a group of nilpotecy class n and exponent p. Each element of the group will have a unique representation as $x_1^a x_2^b x_3^c x_4^d \cdots x_{n-3}^r x_{n-2}^u x_{n-1}^v x_n^y x_{n+1}^z$ where $a,b,c,d,\ldots, r,u,v,y,z \in GF(p)$. The element x_{n+1}^z is central and the group $H =< x_2, x_3, \cdots, x_n, x_{n+1} >$ is abelian. Let $x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n+1}^{z_0}$, $x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_{n+1}^{z_1}$ and $x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}$ be elements of G. Then, suppose that $(x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n+1}^{z_0})(x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_{n+1}^{z_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) = (x_1^{a_3} x_2^{b_3} x_3^{c_3} \cdots x_{n+1}^{z_3})$.

Under the group law, we can calculate the entries of 3-step Fibonacci sequences as follows:

$$
\begin{aligned}
&(x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n+1}^{z_0})(x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_{n+1}^{z_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&= [(x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n+1}^{z_0})(x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_{n+1}^{z_1})](x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&= (x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_n^{y_0} x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_{n+1}^{z_0+z_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&= (x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n-1}^{v_0} x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_n^{y_0+y_1} x_{n+1}^{z_0+z_1+y_0 a_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&= (x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n-1}^{v_0-1} x_1 x_{n-1} x_1^{a_1-1} x_n x_{n+1}^{a_1-1} x_2^{b_1} x_3^{c_1} \cdots x_n^{y_0+y_1} x_{n+1}^{z_0+z_1+y_0 a_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&= (x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n-1}^{v_0-1} x_1^2 x_{n-1} x_1^{a_1-2} x_n^2 x_{n+1}^{a_1-1+a_1-2} x_2^{b_1} x_3^{c_1} \cdots x_n^{y_0+y_1} x_{n+1}^{z_0+z_1+y_0 a_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&\vdots \\
&= (x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n-1}^{v_0-1} x_1^{a_1} x_{n-1} x_n^{a_1} x_{n+1}^{a_1-1+\cdots+1} x_2^{b_1} x_3^{c_1} \cdots x_n^{y_0+y_1} x_{n+1}^{z_0+z_1+y_0 a_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&\vdots \\
&= (x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n-2}^{u_0} x_1^{a_1} x_{n-1}^{v_0} x_n^{a_1 v_0} x_{n+1}^{\binom{a_1}{2} v_0} x_2^{b_1} x_3^{c_1} \cdots x_n^{y_0+y_1} x_{n+1}^{z_0+z_1+y_0 a_1})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}) \\
&= (x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n-2}^{u_0} x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_{n-1}^{v_0+v_1} x_n^{y_0+y_1+a_1 v_0} x_{n+1}^{z_0+z_1+y_0 a_1+\binom{a_1}{2} v_0})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2})
\end{aligned}
$$

We can do the same operations between $x_{n-2}^{u_0}$ and $x_1^{a_1}$. Then the last equation is

$$
\begin{aligned}
(x_1^{a_3} x_2^{b_3} x_3^{c_3} \cdots x_{n+1}^{z_3}) = (&x_1^{a_0} x_2^{b_0} x_3^{c_0} \cdots x_{n-3}^{r_0} x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots x_{n-2}^{u_0+u_1} \\
&x_{n-1}^{v_0+v_1+a_1 u_0} x_n^{y_0+y_1+a_1 v_0+\binom{a_1}{2} v_0} x_{n+1}^{z_0+z_1+y_0 a_1+\binom{a_1}{2} v_0+\binom{a_1}{3} u_0})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}).
\end{aligned}
$$

So, we can do the same operations between $x_{n-3}^{r_0}, \ldots, x_2^{b_0}$ and $x_1^{a_1}$. Then last equation is

$$
\begin{aligned}
(x_1^{a_3} x_2^{b_3} x_3^{c_3} \cdots x_{n+1}^{z_3}) = (&x_1^{a_0+a_1} x_2^{b_0+b_1} x_3^{c_0+c_1+a_1 b_0} \cdots x_n^{y_0+y_1+a_1 v_0+\binom{a_1}{2} v_0+\cdots+\binom{a_1}{n-2} b_0} \\
&x_{n+1}^{z_0+z_1+y_0 a_1+\binom{a_1}{2} v_0+\binom{a_1}{3} u_0+\cdots+\binom{a_1}{n-1} b_0})(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2}).
\end{aligned}
$$

So we can also apply the same operations between $(x_1^{a_0+a_1} x_2^{b_0+b_1} x_3^{c_0+c_1+a_1 b_0} \cdots x_n^{y_0+y_1+a_1 v_0+\binom{a_1}{2} v_0+\cdots+\binom{a_1}{n-2} b_0}$ $x_{n+1}^{z_0+z_1+y_0 a_1+\binom{a_1}{2} v_0+\binom{a_1}{3} u_0+\cdots+\binom{a_1}{n-1} b_0})$ and $(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots x_{n+1}^{z_2})$. Thus the last equation is

$$
\begin{aligned}
(x_1^{a_3} x_2^{b_3} x_3^{c_3} \cdots x_{n+1}^{z_3}) = (&x_1^{a_0+a_1+a_2} x_2^{b_0+b_1+b_2} x_3^{c_0+c_1+c_2+\binom{a_1}{1} b_0+(b_0+b_1)\binom{a_2}{1}} \cdots \\
&x_n^{y_0+y_1+y_2+\binom{a_1}{1} v_0+\binom{a_1}{2} u_0+\binom{a_1}{3} r_0+\cdots+\binom{a_1}{n-3} c_0+\binom{a_1}{n-2} b_0+\binom{a_2}{1}(v_0+v_1+\binom{a_1}{1} u_0+\binom{a_1}{2} r_0+\cdots+c_0\binom{a_1}{n-4}+b_0\binom{a_1}{n-3})} \\
&x_n^{+\binom{a_2}{2}(u_0+u_1+\binom{a_1}{1} r_0+\cdots+\binom{a_1}{n-4} b_0)+\cdots+\binom{a_2}{n-4}(d_0+d_1+\binom{a_1}{1} c_0+\binom{a_1}{2} b_0)+\binom{a_2}{n-3}(c_0+c_1+\binom{a_1}{1} b_0)+\binom{a_2}{n-2}(b_0+b_1)} \\
&x_{n+1}^{z_0+z_1+z_2+\binom{a_1}{1} y_0+\binom{a_1}{2} v_0+\cdots+\binom{a_1}{n-2} c_0+\binom{a_1}{n-1} b_0+\binom{a_2}{1}(y_0+y_1+\binom{a_1}{1} v_0+\binom{a_1}{2} u_0+\cdots+c_0\binom{a_1}{n-3}+b_0\binom{a_1}{n-2})} \\
&x_{n+1}^{+\binom{a_2}{2}(v_0+v_1+\binom{a_1}{1} u_0+\cdots+\binom{a_1}{n-3} b_0)+\cdots+\binom{a_2}{n-3}(d_0+d_1+\binom{a_1}{1} c_0+\binom{a_1}{2} b_0)+\binom{a_2}{n-2}(c_0+c_1+\binom{a_1}{1} b_0)+\binom{a_2}{n-1}(b_0+b_0)}).
\end{aligned}
$$

So we can write,

$$a_3 = a_0 + a_1 + a_2$$
$$b_3 = b_0 + b_1 + b_2$$
$$c_3 = c_0 + c_1 + c_2 + b_0 a_1 + (b_0 + b_1)a_2$$
$$d_3 = d_0 + d_1 + d_2 + c_0 a_1 + b_0 \binom{a_1}{2} + a_2(c_0 + c_1 + b_0 a_1) + \binom{a_2}{2}(b_0 + b_1)$$
$$\vdots$$
$$y_3 = y_0 + y_1 + y_2 + v_0 \binom{a_1}{1} + u_0 \binom{a_1}{2} + \ldots + c_0 \binom{a_1}{n-3} + b_0 \binom{a_1}{n-2}$$
$$+ \binom{a_2}{1}\left[v_0 + v_1 + u_0 \binom{a_1}{1} + r_0 \binom{a_1}{2} + \ldots + c_0 \binom{a_1}{n-4} + b_0 \binom{a_1}{n-3} \right]$$
$$+ \binom{a_2}{2}\left[u_0 + u_1 + r_0 \binom{a_1}{1} + \ldots + c_0 \binom{a_1}{n-5} + b_0 \binom{a_1}{n-4} \right] + \ldots + \binom{a_2}{n-4}\left[d_0 + d_1 + c_0 \binom{a_1}{1} + b_0 \binom{a_1}{1} \right]$$
$$+ \binom{a_2}{n-3}\left[c_0 + c_1 + b_0 \binom{a_1}{1} \right] + \binom{a_2}{n-2}(b_0 + b_1)$$

and

$$z_3 = z_0 + z_1 + z_2 + y_0 \binom{a_1}{1} + v_0 \binom{a_1}{2} + \ldots + c_0 \binom{a_1}{n-2} + b_0 \binom{a_1}{n-1}$$
$$+ \binom{a_2}{1}\left[y_0 + y_1 + v_0 \binom{a_1}{1} + u_0 \binom{a_1}{2} + \ldots + c_0 \binom{a_1}{n-3} + b_0 \binom{a_1}{n-4} \right]$$
$$+ \binom{a_2}{2}\left[v_0 + v_1 + u_0 \binom{a_1}{1} + \ldots + c_0 \binom{a_1}{n-4} + b_0 \binom{a_1}{n-3} \right] + \ldots + \binom{a_2}{n-3}\left[d_0 + d_1 + c_0 \binom{a_1}{1} + b_0 \binom{a_1}{1} \right]$$
$$+ \binom{a_2}{n-2}\left[c_0 + c_1 + b_0 \binom{a_1}{1} \right] + \binom{a_2}{n-1}(b_0 + b_1).$$

We must consider two types of initial data for loops in G. We have a loop ς of type I with initial data

$$\varsigma_0 = \underbrace{(0,0,0,\ldots,0,0,0)}_{n+1 \text{ times}}$$
$$\varsigma_1 = \underbrace{(1,0,0,\ldots,0,0,0)}_{n+1 \text{ times}}$$
$$\varsigma_2 = \underbrace{(0,1,0,\ldots,0,0,0)}_{n+1 \text{ times}}$$

and another τ of type II with initial data

$$\tau_0 = \underbrace{(1,0,0,\ldots,0,0,0)}_{n+1 \text{ times}}$$
$$\tau_1 = \underbrace{(0,1,0,\ldots,0,0,0)}_{n+1 \text{ times}}$$
$$\tau_2 = \underbrace{(0,0,0,\ldots,0,0,0)}_{n+1 \text{ times}}$$

The analysis of the type II loop is entirely similar to that of type I. Thus the type I loop begins

$$\varsigma_0 = \underbrace{(t_0, s_0, 0,\ldots,0,0,0)}_{n+1 \text{ times}}$$
$$\varsigma_1 = \underbrace{(t_1, s_1, 0,\ldots,0,0,0)}_{n+1 \text{ times}}$$
$$\varsigma_2 = \underbrace{(t_2, s_2, 0,\ldots,0,0,0)}_{n+1 \text{ times}}$$

We focus on the type I loop $\eta_i = \underbrace{(a_i, b_i, c_i, d_i, \ldots, r_i, u_i, v_i, y_i, z_i)}_{n+1 \text{ times}}$. It can be

easily seen that the sequence t_i can be written in terms of s_i as $t_i = s_{i+1} - s_i$. Let $(a_i) = (t_i)$ and $(b_i) = (s_i)$ be.

So we can write formulas for $c_\alpha, d_\alpha, y_\alpha$ and z_α . Now, let the notation $\sum$ indicate the notation $\sum_{i=0}^{\alpha-1}$. By induction it is

$$a_\alpha = \sum_{i=0}^{\alpha-1} t_i = \sum_{i=0}^{\alpha-1} (s_{i+1} - s_i)$$

$$b_\alpha = \sum_{i=0}^{\alpha-1} s_i$$

$$c_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}))$$

$$d_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+1} c_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} s_i \binom{t_{i+1}}{2} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+2}(c_i + c_{i+1} + s_i t_{i+1})$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2}(s_i + s_{i+1})$$

$$\vdots$$

$$y_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} v_i \binom{t_{i+1}}{1} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} u_i \binom{t_{i+1}}{2} + \ldots + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} c_i \binom{t_{i+1}}{n-3} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} s_i \binom{t_{i+1}}{n-2}$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{1} \left[v_i + v_{i+1} + u_i \binom{t_{i+1}}{1} + r_i \binom{t_{i+1}}{2} + \ldots + c_i \binom{t_{i+1}}{n-4} + s_i \binom{t_{i+1}}{n-3} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2} \left[u_i + u_{i+1} + r_i \binom{t_{i+1}}{1} + \ldots + c_i \binom{t_{i+1}}{n-5} + s_i \binom{t_{i+1}}{n-4} \right] + \ldots$$

$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-4} \left[d_i + d_{i+1} + c_i \binom{t_{i+1}}{1} + s_i \binom{t_{i+1}}{2} \right] + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-3} \left[c_i + c_{i+1} + s_i \binom{t_{i+1}}{1} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-2}(s_i + s_{i+1})$$

and

$$z_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} y_i \binom{t_{i+1}}{1} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} v_i \binom{t_{i+1}}{2} + \ldots + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} c_i \binom{t_{i+1}}{n-2} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} s_i \binom{t_{i+1}}{n-1}$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{1} \left[y_i + y_{i+1} + v_i \binom{t_{i+1}}{1} + u_i \binom{t_{i+1}}{2} + \ldots + c_i \binom{t_{i+1}}{n-3} + s_i \binom{t_{i+1}}{n-2} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2} \left[v_i + v_{i+1} + u_i \binom{t_{i+1}}{1} + r_i \binom{t_{i+1}}{2} \ldots + c_i \binom{t_{i+1}}{n-4} + s_i \binom{t_{i+1}}{n-3} \right] + \ldots$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-3} \left[d_i + d_{i+1} + c_i \binom{t_{i+1}}{1} + s_i \binom{t_{i+1}}{2} \right] + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-2} \left[c_i + c_{i+1} + s_i \binom{t_{i+1}}{1} \right]$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{n-1}(s_i + s_{i+1}).$$

Now we use these formulas to calculate the Fibonacci loop $l(x, y, \ldots)$. For example let $n = 4$. The first few terms are as follows (reduce modulo p of your choice):

(1,0,0,0,0)
(0,1,0,0,0)
(0,0,1,0,0)
(1,1,1,0,0)
(1,2,3,1,0)
(2,3,6,4,1)
(4,6,17,19,15)
⋮

4. CONJECTURE

One can prove theorems similar to those in [8]. That is, there is a relation between the fundamental period (Wall number) of 3-step general Fibonacci sequence and the length of a fundamental period of any loop satisfying the recurrence.

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