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MECHANISMS OF MASS GENERATION FOR NEUTRINOS IN NON-SUPERSYMMETRIC GUTS BASED ON SU(5)

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PREFACE / ACKNOWLEDGEMENT

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ÖZET

SÜPERSİMERTİK OLMAYAN SU(5) BÜYÜK BİRLEŞİM TEORİLERİNDE NÖTRİNO KÜTLELERİNİN OLUŞUM MEKANİZMALARI

Doğada kütleçekimi dışında temel üç kuvvet olan güçlü, zayıf ve elektromanyetik kuvvetler Standard Model (SM) ayar teorisini oluşturmaktatır. SM altında birleşen elektromanyetik ve zayıf etkileşim sabitleriyle yüksek enerjilerde güçlü etkileşim sabitinin de birleşmesi mümkün olmadığı için Büyük Birleşim Teorileri geliştirilmeye başlanmıştır. Georgi-Glashow modeli ilk büyük birleşim teorisi olup SM'de birleşimi mümkün olan elektromenyetik ve zayıf etkileşimlerine güçlü etkileşimi de ekleyerek tek bir etkileşim sabitine dönüşmesini amaçlayarak ortaya atılan ilk büyük birleşim teorisidir. Georgi-Glashow (GG) SU(5) ayar teorisinde bahsedilen üç temel kuvvet yüksek enerjilerde ($10^{15\text{-}16}$ GeV) birleşmemeye devam etmektedir. Günümüzde kütleli oldukları bilinen nötrinoların kütlesi SM ayar teorisinde doğal bir yolla açıklanamamakta ve GG modelinde de nötrinolar kütlesiz kalmaya devam etmektedir. GG modelinin diğer bir eksik yönü ise yüklü lepton ve kuark kütle oranlarının deneylerle uyumlu olmamasıdır.

Tezimizde GG modeline SU(5)'teki 45 ve iki 75 temsillerinde sırasıyla skaler ve fermiyonlar eklenerek süpersimetrik olmayan ve renormalize edilebilir bir model oluşturulacaktır. 45 skaler alanı ile yüklü lepton ve kuark kütle oranları düzeltilirken iki 75 fermiyon alanları ile en az iki nötrino kütlesi hesaplanarak nötrino kütle ilişkilerinin normal ya da tersine hiyerarşiden hangisine sahip olduğu ortaya konacaktır. Nötrino kütleleri Tip-1 tahteravalli mekanizması ile açıklanmaya çalışılacaktır. 75 fermiyon alanı gerçek bir temsil olduğu için modelimiz anomali içermemektedir. 45 skaler temsilinde SM'deki Higgs ile aynı kuantum sayılarına sahip bölünmüş temsil olması modelimizin iki-Higgs-ikilisi modeli yapmaktadır. SM'de ve GG modelinde birleşmeyen etkileşim sabitleri 45 ve 75 temsillerindeki skaler ve fermiyonların kütle ilişkileri kullanılarak farklı ara ölçeklerde akışa dahil olmaları ile birleşmeleri sağlanacaktır. Proton bozunması deneyinden BBT ölçeğine ve kozmolojik hesaplamalardan nötrino kütlelerine gelen kısıtlar göz önünde bulundurulacaktır.

ABSTRACT

MECHANISMS OF MASS GENERATION FOR NEUTRINOS IN NON-SUPERSYMMETRIC GUTS BASED ON SU(5)

The strong, weak and electromagnetic forces, which are the three fundamental forces in nature other than gravity, form the Standard Model (SM) gauge theory. Since it is not possible to combine the electromagnetic and weak interaction constants under SM with the strong interaction constant at high energies, Grand Unification Theories have begun to be developed. The Georgi-Glashow model is the first major unification theory that put forward by adding the strong interaction to the electromagnetic and weak interactions that are possible to combine in the SM, aiming to transform it into a single interaction constant. The three fundamental forces mentioned in Georgi-Glashow (GG) SU(5) gauge theory continue to not coalesce at high energies ($10^{15\text{-}16}$ GeV). The mass of neutrinos, which are known to have mass today, cannot be explained in a natural way in the SM gauge theory, and neutrinos continue to be massless in the GG model. Another shortcoming of the GG model is that the mass ratios of charged leptons and quarks are not compatible with the experiments. In this thesis, a non-supersymmetric and renormalizable model will be constructed by adding scalars and fermions in the 45 and two 75 representations in SU(5), respectively, to the GG model. While correcting the mass ratios of charged lepton and quark with 45 scalar fields, at least two neutrino masses will be calculated with two 75 fermion fields, and it will be revealed whether the neutrino mass relations have normal or inverted hierarchy. Neutrino masses will be explained by the Type-1 seesaw mechanism. Since the 75 fermion fields are a real representation, our model does not contain anomalies. The split multiplet in the 45 scalar representation with the same quantum numbers as the Higgs in SM makes our model a two-Higgs-doublet model. Interaction constants that do not unify in the SM and GG model will be provided to do by the inclusion of scalars and fermions in the 45 and 75 representations in the flow at different intermediate scales using mass relations. Constraints from proton decay experiment to GUT scale and from cosmological calculations to neutrino masses will be considered.

SYMBOLS

α : Fine structure constant

α_i : Fine structure constants for i = 1,2,3 for $U(1)_Y$, $SU(2)_L$, $SU(3)_C$ respectively

g_i : Coupling constants for i = 1,2,3 for $U(1)_Y$, $SU(2)_L$, $SU(3)_C$ respectively

$\theta_{\square}$: Weinberg angle

γ : Gamma matrices

λ : Gell-Mann matrices

ν : Vacuum expectation value

σ : Pauli matrices

β : Beta function

I : Intermediate scale

Σ : SU(5) adjoint scalar

Φ : Scalar field

Ψ : Fermionic field

J : Current density

ℓ : Lepton

$\mathcal{L}$: Lagrangian density

Q : Quark

X : Gauge boson of SU(5)

y : Yukawa coupling

Γ : Decay rate

H : Higgs field

$\hbar$: Planck constant

t : Time (s)

ABBREVIATIONS

CKM : Cabibbo – Kobayashi -Maskawa

GG : Georgi-Glashow Model

GUT : Grand Unified Theory

PMNS : Pontecorvo – Maki – Nakagawa – Sakata

SM : Standard Model

SSB : Spontaneous Symmetry Breaking

SUSY : Supersymmetric

VEV : Vacuum Expectation Value

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1. INTRODUCTION

1.1.Aim

Physics is comprised of four forces which are electromagnetic force that is very well known, weak force that takes place according to particles' helicity, strong force which seems particles with colors and finally gravity. Those forces are effective in different energy scales. High energy physics is interested with subatomic particles like quarks, electron, proton, etc. These particles interactions become fact with energy scales lower than the energy scale of the gravity force which is about Planck scale (10^{19} GeV). For many years high energy physicists have been doing researches and experiments to find a resonable and natural relationships between electromagnetic, weak and strong forces since finding a unique force which explains subatomic particle interactions will be fabulous.

For many years various experiments have been conducting to find whether there is a energy scale in which the three forces merge and become a unique force. Experiments conducted in LHC, ATLAS, and Hyper-Kamiokande have given some clues for the unifying energy scale is around 10^{15-16} GeV. Many theorethical studies have been trying to explain particle interactions in this energy scale naturally.

Until now, particles and their interactions in low energy scale (~M_W) is explained in Standard Model (SM) successfully. Even though the gauge couplings of weak and electromagnetic forces unify in SM gauge theory, strong force does not give a unique coupling constant with them in higher energies. If one look at the lepton content of the SM, it is seen that neutrinos are left-handed and anti-neutrinos are right-handed particles and antiparticles respectively. SM does not have enough parameters to explain the small mass of neutrinos in a natural way. Experiments on solar neutrinos have shown that neutrinos are oscillating and they have very small mass ~O(1) eV.

The discovery of neutrino oscillations opens a door for the existence of the right-handed neutrinos. After these clues and evidences the question comes to the mind that a theory beyond the SM can be built? The answer is yes, Grand Unified Theories (GUTs). The aim of GUTs is to find a single coupling constant which is at GUT scale (10^{15-16}

GeV). In this thesis the main aim will be using GUTs to unify three coupling constant in GUT scale and also explain the small mass of neutrinos, naturally. Evidences of neutrino oscillations results in neutrino mixing. There are three types of neutrinos; electron neutrino, muon neutrino, and tau neutrino. Experiments on solar and atmospheric neutrinos have demostrated that these three flavors of neutrinos transforms amongst themselves. Due to the mixing of three flavors of neutrinos, the mass relations between the difference of neutrino masses can be predicted as normal ordering (normal hierarchy) or reverse ordering (inverted hierarchy). To explain the mass relations of neutrinos, different mechanisms are created. However, the important part in this thesis is that the mechanism should explain the small mass of neutrinos naturally. Ensuring that criteria limits options. Seesaw mechanism, named by Tsutomu Yanagida in 1981, generated to explain the relative sizes of neutrino masses compared to quarks that are millions of times heavier.

Before the discovery of the top and bottom quarks, Cabibbo realized that the strange and down quarks decay into the charm or up quarks. After doing some calculations he came up with a 2×2 rotation matrix called Cabibbo matrix. Since the CP (Charge-Parity) relations had not explained by using only four quarks, the same matrix models applied into six quarks model by Nicola Cabbibbo, Makoto Kobayashi, and Toshihide Maskawa. The new 3×3 mixing matrix called CKM (Cabibbo-Kobayashi- Maskawa) matrix is introduced in 1973. In this manner, the existence of the mixing of neutrinos becomes an evidence for the neutrino or lepton mixing matrix introduced by Ziro Maki, Masami Nakagawa and Shoichi Sakata in 1962. Since neutrino oscillations predicted by the Bruno Pontecorco, the neutrino mixing matrix is called as PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix. By using PMNS matrix in this thesis it is aimed to find one of mass orderings of neutrinos.

Above the CP violation is mentioned as universal. In SM there are some symmetries such as lepton flavor number and baryon number. These symmetries are global symmetries in the SM, therefore, the proton decay cannot be expected. However, in GUTs, proton can decay into neutral pion and positron or anti-muon and neutral pion. There are other decay possibilities. The essential point is that in GUTs baryon (B) and lepton (L) numbers are not separetely global symmetries anymore but their difference

(B-L) is. Up till now, the lower limit of the proton lifetime has been predicted as 10^{34} years. In the next chapter the calculation of the proton lifetime will be explained clearly. Proton decay was predicted by various GUTs in 1970. Owing to the fact that the unification scale in GUTs appears in the calculation of the proton lifetime, the lower bound of the proton lifetime measured in the Super-Kamiokande experiment sets a lower limit to the GUT scale. Thereby, finding and appropriate unification scale is another criterion of this thesis.

At the beginning of this part, it is mentioned that various GUT models exist. They can be grouped as supersymmetric (SUSY) and non-supersymmeric GUTs. Supersymmetric GUTs are developed as a solution for the hierarchy problem. There are many examples of hierarchy problem but in particle physics the problem is about naturalness. The problem is that the mass of the Higgs ($\sim 10^2$ GeV) is so much lighter than the Planck mass (10^{19} GeV). There has been no natural explanation for that problem yet; however, supersymmetry can find a solution for this problem. In the supersymmetry there are many additional particles to SM particles therefore these models are not economical. Finding a solution for the hierarchy problem is not an aim of this thesis. In this study due to the fact that a natural explanation for both unification and neutrino mass are expected, non-supersymmetric GUT model is selected to create a model. It means that not many additional particles other than the SM particles are used to create a GUT model. For many years many non-supersymmetric GUTs have been studied. To protect the naturality condition for non-supersymmetric GUTs brings a model which still lives and limits additional particle numbers. SU(5) GUTs is a gauge theory which many high-energy physicits have been studying for years. The most important reason to choose SU(5) GUTs is that it is the smallest model containing SM particles. Extending the size of the gauge group requires a valid reason to retain its naturality.

To sum up, aims of the thesis are the following; first using the SU(5) GUT to unify coupling constants of electromagnetic, weak and strong interactions at GUT scale that ensure the lower limit comes from the proton decay, and second, explaining the small mass of neutrino naturally via seesaw mechanism.

1.2.Literature Review

In 1974 Howard Georgi and Sheldon Glashow introduced the first GUT model called Georgi-Glashow (GG) Model [1]. This model is based on the simple SU(5) Lie Groups.

The first assumption of the GG model is it is a gauge-invariant theory with spontaneous symmetry breaking (SSB) with vector bosons mediate weak and electromagnetic forces. The model which is describing the interactions of weak and electromagnetic forces was first introduced by Sheldon Glashow and then perfected by Steven Weinberg and Mohammad Abdus Salam. In 1979 they were awarded Nobel Prize in physics due to presenting a unification of weak and electromagnetic interactions of elementary particles. Their model is known as Glashow-Weinberg-Salam (GWS) model. GG model assumes that there are no unobserved elementary particles. With this assumption, the leptonic content of the model consists of left-handed particles and charge conjugate of them. Hadrons can be included in this model via Glashow-Iliopoulos-Maiani (GIM) mechanism generated by Howard Georgi, John Iliopoulos, and Luciano Maiani [2]. To avoid anomalies of lepton and hadrons three quartets of fractionally charges quarks model is suggested by Claude Bouchiat, John Iliopoulos, and Philippe Meyer [3]. Since the anomalies created by lepton and hadron cancel out, the theory of the weak and electromagnetic interactions becomes renormalizable. At the beginning of this chapter it is said that there is another interaction occurs between elementary particle, the strong interaction. After the theory of weak and electromagnetic interactions are generated, strong interactions must be inserted into the theory. Strong interactions are mediated by the neutral vector gauge bosons called gluons. Gluons mediate SU(3) interactions between the particles carrying color charges. It is seen that unification works if $SU(3)_C$ is also included in $SU(2)_L$ and U(1) coupling. These three fundamental forces generate SM.

Since the right-handed neutrino is colorless and neutral, it does not interact strogly, weakly, and electrically. It is singlet under SM. Singlet under these forces is represented as (1, 1, 0).

The SSB of SM occurs through the Higgs in the 5 representation of SU(5). The Higgs in 5 acquire vacuum expectation value (VEV). At the end of the spontaneous symmetry

breaking of strong, weak and electromagnetic interactions the only unbroken symmetries left must be the direct product of SU(3) strong and U(1) electromagnetic local symmetries. The spontaneous symmetry breaking gives all gauge fields masses except photon and gluons through Higgs mechanism. In GG model SU(5) is an anomaly free complex group with rank four same as the SM. According to the theory particles and antiparticles live in $\bar{5} + 10$ and $5 + \overline{10}$ representations respectively.

The Higgs in the 24 representation, 24_H of SU(5) acquires VEV and breaks the SU(5) symmetry. After this symmetry breaking surprisingly 12 of the gauge bosons becomes goldstone bosons eaten by gauge bosons which must be consistent with SM. After goldstone bosons are eaten the remaining ones become superheavy gauge bosons.

The scalar of Higgs sector of GG model includes 24_H and 5_H. The former is responsible for the SU(5) symmetry breaking in GUT scale and the latter is to give masses to particles except neutrino, gluons and photon in weak energy scale (M_W). One of the beauties of the SU(5) model is that with the right choice of the VEV of 24_H it directly breaks into the SM.

Today there are three families (or generations) in SM. The same equation can be written for other families either. GG model is also known as minimal SU(5) model since there is no additional particles to SM. According to the theory mixing angle (or weak angle) is predicted as $\sin^2 \theta_W = \frac{3}{8}$.

Today it is known that GG model needs to be improved due to some experimental results. One of them is that in GG model neutrinos are not massive, however as it was explained before they have small masses. Second, quarks and charged fermions mass relations are not coherent with the experimentally measured values. Lastly, there is no exact unification of gauge couplings in it.

In 1991, Hihoshi Murayama and Tsutomu Yanagida introduced a new model with additional scalars in 10 and $\overline{10}$ representations of SU(5) and Higgs doublets [4]. According to the theory there is no explanation for neutrino mass. The unification scale in that model is not viable today since the experimental lower limit of the proton lifetime is $\sim 10^{34}$ years.

For many years GG models have tried to be progressed by adding new particle(s). One of them was done by Ilja Dorsner and Pavel Fileviex Pèrez in 2005 [5]. Their model adds additional Higgs of 15 representation of SU(5) to the GG model.

One of the problems with the both SM and GG model is that the neutrinos are massless. According to the study this problem can be easily fixed through adding three right-handed neutrinos. Since in the SM right-handed neutrinos do not exist, they are added as singlets in it by using Type-Ⅰ seesaw which will be explained in the next chapter. On the other hand in the article adding a Higgs of the 15 representation makes Type-Ⅱ seesaw as a useful option. It can be used both Type-Ⅰ and Type-Ⅱ seesaw mechanisms to get small masses of neutrinos if one adds (1, 1, 0) as a right-handed neutrino.

In the GG model the mass ratios of quarks and charged fermions are not consistent with the experimental values. Dorsner and Pérez mentions two solutions to solve that problem. One of them is adding higher-dimensional operators in the Yukawa gauge couplings. The other is to extend the Higgs sector by the use of extra scalars. Since their aim is to keep the model as minimal as it can be, the first option is chosen as a solution. Even though it is decided to use higher-dimensional operators, only use non-renormalizable terms are used in the model.

Unification of the gauge couplings does not occur in GG model which is another and the most important problem to generate a viable GUT scenario. Using the different split-multiplets of the 15 repreentation solves the unification problem. It can also be solved by adding extra scalars; however, particle content must be kept minimal as it is noted before.

Although result of the proton lifetime is still viable, the GUT scale is very low. That is why the study has not been promising yet.

In GG model, Higges are in the representations of 24 (adjoint) and 5 of SU(5). According to the last model (GG + 15_H) the proton lifetime is well predicted, and unification is provided. This theory was improved by Ilja Dorsner, Pavel Fileviez Pérez, and Ricardo González Felipe with two-loop adjustments to the unification of gauge couplings [6]. The model aims to explain observed baryon asymmetry in the universe with the scalar triplets multiplet of 15. Leptogenesis constraints are considered to

explain the baryon asymmetry. Coupling of the matter with the Higgs in the 15 representations give small mass to neutrinos and the scalar leptoquarks in the adjoint is relevant with the proton lifetime which has calculated upper limit as 10^{39} years.

Today we know that at least two neutrinos are massive to generate PMNS matrix. In a study proceeded by Borut Bacj and Goran Senjanović introduces new model that includes GG model and additional fermions lives in the 24, represented as 24_F [7]. The theory can be still seen as minimal. Adding new fermionic particles fixes the unification problem and gives small masses to neutrinos. In 24_F there are both singlet and triplet split multiplets. Thereby, it is possible to have massive neutrinos with the help of two different seesaw mechanisms. The singlet (1, 1, 0) can be treated as a right-handed neutrino and used in the Type-I seesaw mechanism. On the other hand, the triplet (1, 3, 0) can be used in the Type-Ⅲ seesaw. This is remarkable to generate a successful spectrum of neutrino masses. Two important predictions of this model is that the mass of fermionic triplet in 24_F and scalar triplet in 24_H below the TeV scale by the restrictions come from both proton decay and unification. In this model getting a consistent mass ratio of charged leptons, higher-dimensional operators are used in the Yukawa sector. As a result, two neutrinos get small masses and the other one remains massless. Another conclusion is the mass of the fermionic SU(2) triplet which has a role in the Type-Ⅲ seesaw must be close to M_Z (~91.2 GeV). The model cannot make comments for mass ordering or neutrinos since one neutrino is massless. Both mass orderings, normal and inverted hierarchies, are possible between two light neutrinos. Using the fermionic mass splitting with higher-dimensional factors the unification occurs at $\sim 10^{15.5}$ GeV. This scale is really close to the lower limit for the GUT scale but still lower than it. This study is very interesting due to the combination of the Type- I and Type-Ⅲ seesaw. However, it cannot predict mass relations of neutrinos. The scale for the fermionic triplet responsible for Type-Ⅲ seesaw must be higher for thermal leptogenesis.

A new renormalizable SU(5) GUT model with additional matter in the adjoint representation was introduced by Pavel Fileviez Pérez in 2007 [8]. This new model can be considered as renormalized version of [7]. In this model getting a consistents mass ratio of charged leptons and down quarks, which was first proposed by Howard Georgi

and Cecilia Jarlskog [9], is successed by including extra Higgs in 45 representation that is symbolized as 45_H. The Higgs sector of this model comprises 5_H, 24_H, and 45_H where the first two come from GG model. Up to now the particle content is the same as the minimal SU(5) model [1]. To have massive neutrinos there are three options: first one is adding extra two SU(5) fermionic singlets to have at least two massive neutrinos through Type-Ⅰ seesaw mechanism, second introducing Higgs in 15 representation as in the [5] Type-Ⅱ seesaw mechanism can be used, and third is through the combination of Type-Ⅰ and Type-Ⅲ seesaw mechanisms at least two neutrinos can acquire mass with extra two 24_F as explained in [7]. To avoid complicated Higgs sector since 45_H already exists, the study focuses on the third possibility. Split multiplets (1, 3, 0) and (1, 1, 0) of 24_F are $SU(2)_L$ triplets and singlets that proceed Type-Ⅲ and Type-Ⅰ seesaw mechanisms respectively. The matter content of this model is constructed by $\bar{5}$, 10, and 24, that of antimatter is done by 5, $\overline{10}$. The main difference between this model and [7] is that the higher-dimensional terms are not needed to give masses to fermions including neutrinos. That can be achived by using only two Higgses, 5_H and 45_H.

In this theory one of the neutrinos is predicted as massless. In addition, both normal and inverted hierarcies are possible. Different scenarios are studied in this theory. Among them the most promising unification scale found in this study is $M_{GUT} = 1.2\times10^{17}$ GeV and proton lifetime $\tau_p \gtrsim 10^{41}$ years.

It is possible to give masses to neutrinos through the Zee mechanism [10]. In the Zee Model massive neutrinos can be generated by adding two extra Higgs to SM Higgs sector. This model works in one loop level to generate masses for neutrinos. The additional two Higgses are in the representations of 10_H and 45_H with quantum numbers (1, 2, 1/2), in which they are of strong, weak, hypercharge forces respectively. 10_H is an antisymmetric representation in SU(5) which contains the singlet (1, 1, 1). Since 45_H includes Higgs with quantum numbers same as the SM, there is no need to add extra Higgs. This model adds only scalars of 10_H and 45_H to GG model. In the study there is no prediction for neutrino masses. The additional 45_H also helps to get succesful mass ratios of quarks and charged leptons.

A promising unification scale is achieved in the study [11] around 10^{16}GeV. In this model there are extra scalars in 5_H, 10, and 10 representaions of SU(5) additional to the GG model.

In a study neutrino gain small masses at two-loop level [12] naturally. In this study there are some criterias that must be achived: (1) it is a renormalizable model, (2) the unique symmetry is SU(5) symmetry, (3) only fermions in GG model are counted, (4) neutrinos do not have mass at both tree and one loop levels, (5) model does not contain SU(5) singlet, (6) dimensions of the contributed representations must be smaller than 100. Other than the Higgs in the GG model, this study introduces 45_H and 50_H representations.

In the study five cases with different particles with various intermediate scales are investigated. The lowest and the highest M_{GUT} scales of these cases are 1.88×10^{15} GeV and 2.2×10^{17} GeV respectively. Proton decay lifetime is predicted as 7.2×10^{32} years and 1.74×10^{34} years with these unification scales. The former one is ruled out due to the experimental lower bound; however, the latter is extremely promising value for the proton lifetime.

Ilja Dorsner and Shaikh Saad introduced a new model in 2020 [13]. The particle content of this model extended compared to GG model with exta scalar in 35 representation and fermions in $15\oplus\overline{15}$ representations of SU(5). Generating masses for neutrinos is done through radiative neutrino mass mechanism. Unification of gauge couplings of strong, weak and electromagnetic interactions is obtained at one-loop level. The results shows that it occurs at 5×10^{15} GeV that ensures the lower bound.

Neutrinos in that theory can acquire mass via Majorana term in Yukawa Lagrangian at tree level and one-loop level. In tree level dimension 7 (d=7) operators are used to generate masses to neutrinos. On the other hand, at one-loop level d=5 operators are used to get massive neutrinos. If one look at the contributions of tree and one-loop level considering unification, it is seen that the contribution of the former one is very low compared to the latter. Quark mass matrices under unification at one-loop level bring results consistent with experimental values with observation of non-equality between mass ratios of quarks and charged leptons. Moreover, two of neutrinos become massive

and one remeains massless. The mass relations of neutrinos are calculated via PMNS matrix.

A recent study investigates the phenomenology of minimal realistic SU(5) gauge theory [14] with the same particle content as [13]. Three predictions are made in this model as: (1) neutrinos Majorana fermions and one of them remains massless, (2) the mass ordering of neutrinos exhibits normal hierarchy, (3) proton lifetime calculated from its decay of $p \rightarrow \pi^0 e^+$. This model consists of eighteen real parameters and fourteen phases such as masses, mixing angles, CP violation and Majorana phases. Particles of this model live in the first five minimal SU(5) representations which are 5, 10, 15, 24, and 35.

In GUT model, the beauty lies in the simplest one. This model has a beauty since there is an obvious relation between neutrino mass matrix and down-type quarks and charged leptons. While two of Majorana neutrinos get masses at one-loop level, one is stays massless. The unification analysis of this model shows that gauge couplings unify at scales $M_{GUT} > 6\times10^{15\text{-}16}$ GeV and the proton lifetime is predicted as 1.0×10^{35}-1.3×10^{35} years.

According to the study [15] successful SU(5) unification with radiative neutrino model is observed in model F8 which has scalar content ϕ_5, ϕ_{24}, ϕ_{45}, ϕ_{40} and fermionic content $\psi_{\bar{5}}$, ψ_5, ψ_{10}, $\psi_{\overline{10}}$. The largest unification scale observed about 10^{18} GeV and lower bound of the proton lifetime is predicted as $\tau_p \gtrsim 1.6\times10^{34}$ years, lower than the current limit.

1.3. The Proposed Model

In the previous part most of the minimal non-supersymmetric SU(5) GUT models are explained. One of the aims of this thesis is to construct a model with unification occurs at $M_{GUT}>5\times10^{15}$ GeV that also predicts proton decay lifetime consistent with the experimental bound $\tau_p>2.4\times10^{34}$ years. Since it is also aimed to explain small masses of neutrinos naturally and generating PMNS unitary mixing matrix by a new model many representations are considered.

In the study [15] leaves a model consists of extra fermions live in the represantation of 75_F, which is not investigated, and extra scalars in 45_S for the GG model. Neutrinos in this model can be massive through the Type- I seesaw mechanism. It is known that at least two neutrinos must be massive, therefore two 75_F representations should be added as fermionic content of the GG model. Other models mentioned in [15] are not promising for the aim ot this thesis since neutrinos get mass at the loop level, however it is aimed to do that at the tree level to explain its smallness naturally. Moreover, having a 45_S can result in successful mass ratios of down-type quarks and charged leptons explained in [10]. Our model that has not been studied before, is non-supersymmetric renormalizable SU(5) GUT that will contain two 75 fermionic and a 45 scalar fields additional to GG model.

75_F is a real representation of SU(5) generated with the direct product of 5 and 45 representations of SU(5). 75_F contains a split multiplet which is a singlet under $SU(3)_C$ and $SU(2)_L$ subgroups. Writing a mass term with two 75_F representations may help to predict neutrino masses if there will be proper limitations on the masses of the other split multiplets of it. In this thesis that mass relations of both 75_F and 45_S are used to find mases of split multiplets of these representations.

2. MATERIALS AND METHOD

2.1. The Standard Model

Standard Model is a renormalizable gauge theory of strong, weak, and electromagnetic interactions. This means that at some energy or momentum scale we can gauge the interaction constants of these interactions. The name ‘Standard Model’ was given in 1970 as it is due to the fact that the model includes fundamental particles and their interactions among each other. In 2012 owing to the discovery of the Higgs boson the acquired mass to fundamental particles in SM is understood.

SM particles are grouped as fermions and bosons. Fermions are particles which obey Fermi-Dirac statistics. On the other hand, bosons are particles that obey the Bose-Einstein statistics which says that the wave functions of those particles are symmetric

under interchange of identical particles. Bosons have integer spin or no spin such as Higgs particle. We grouped the SM particles as fermions and bosons according to their spin.

Symmetries that are conserved in SM are listed below as:

1. Energy-momentum,
2. Angular momentum,
3. Charges (Electric, weak, color),
4. Lepton family number,
5. Baryon number.

Baryon and lepton number symmetries are global symmetries in SM. The baryon number are +1 for a baryon and -1 for a antibaryon. On the other hand, because baryons are genereted by three quarks the baryon number of a quarks is +1/3 and it is -1/3 for antiquark. Moreover, since mesons are particles consists of one quark an one antiquark, there is no conservation of meson number or meson symmetry in SM. On the other hand, there is no explanation for the charge quantization in SM. That means there is no need the down quark (d) has to carry electric charge which is 1/3 of the electron's.

We can represent SM gauge group as G_{SM} that consists of strong, weak and elecromagnetic interactions. In group theory [16], we represent these interactions as shown in the Table 2.1.

Table 2.1 The Standard Model Interactions and Corresponding Gauge Symmetries

Interaction	**Group of Gauge Symmetry**
Electromagnetic	$U(1)_{EM}$
Weak	$SU(2)_L$
Strong	$SU(3)_c$

So that the SM group should satisfy

$$G_{SM} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \tag{2.1}$$

The indices C represents the color charge, L represents left-handed particles, and Y represents the hypercharce. The symmetry group of the strong interaction $SU(3)_C$ has color indice because strong interactions are seen between quarks which carry color charge. Its mediators are gluons and it is colorless to be invariant under SU(3) symmetry. In the above pages it is said that only left-handed particles can interact weakly with mediators $W^{\pm}$ and Z. There is a relation between charge and hypercharge of a particle and in SM we are interested with the hypercharge of particles.

$$Q = T_3 + Y \tag{2.2}$$

Here Q is the electric charge, T_3 is the third component of the weak isospin, and Y is the hypercharge. There is a situation differs the U(1) group from SU(2) and SU(3) groups. $U(1)_Y$ is an abelian group, so its generators commute with each other. On the other hand, $SU(2)_L$ and $SU(3)_C$ are nonabelian groups therefore their genarators do not commute with each other.

$U(1)_Y$

Gauge theories for gauge groups vary according the the characteristics of being an abelian or nonabelian gauge theory. The hypercharge $U(1)_Y$ is an abelian gauge theory however, weak and strong interaction tranformations, $SU(2)_L$ and $SU(3)_C$ are nonabelian gauge theories. Since it is easier to understand abelian gauge theories, it is better to explain gauge transformation by starting with elecromagnetism.

Gauge symmetry of quantum electrodynamics (QED) can be defined as independent symmetry transformation at every point in spacetime. Beginning with defining $\phi(x)$ as a complex scalar field, it must be invariant under the transformation [17]

$$\phi(x) \rightarrow e^{i\alpha(x)}\phi(x). \tag{2.3}$$

where $\alpha(x)$ is the space-dependent angle. This is the phase rotation for the field $\phi(x)$. To be invariant under this transformation, after the phase rotation it must stay as $\psi(x)$. Same is valid for the Lagrangian. To write down a gauge invariant Lagrangian, this phase rotation invariance should be satisfied.

Let us define A_μ as the new vector field called as connection which connect the transformation of the field with the infinitesimal local symmetry transformation [18]. The covariant derivative is defined as

$$D_\mu = \partial_\mu + ieA_\mu \tag{2.4}$$

The covariant derivative of the Dirac field $\phi(x)$ is then

$$D_\mu\phi(x) = \partial_\mu\phi(x) + ieA_\mu\phi(x) \tag{2.5}$$

The transformation of the connection A_μ is

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \tag{2.6}$$

Now we can check the transformation of the covariant derivative of the Dirac field.

$$D_\mu\phi(x) \rightarrow e^{i\alpha(x)}(\partial_\mu + ieA_\mu(x))\phi(x) = e^{i\alpha(x)}D_\mu\phi(x) \tag{2.7}$$

It is seen from equation (2.7) that covariant derivative transforms in the same way as the field does under gauge transformation. The electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.8}$$

which is invariant under the transformation of A_μ. The commutator of covariant derivatives is

$$[D_\mu, D_\nu] = \frac{F_{\mu\nu}}{ie} \tag{2.9}$$

$SU(3)_C$

Fermions are spin-1/2 particles with left-handed and right-handed parity according to Weyl fields. A Dirac fermion has left- and right-handed components. According to quantum choromodynamics quarks carry color charge such as red, green, and blue other than helicity. The left-handed fermionic content of the first generation of the SM particles is [18-19]

$$u^r , u^g , u^b , \bar{u}^r , \bar{u}^g , \bar{u}^b , d^r , d^g , d^b , \bar{d}^r , \bar{d}^g , \bar{d}^b , e^- , e^+ , \nu_e \quad (2.10)$$

Table 2.2 Masses of Quarks

Quarks	Mass (MeV)
u	2.16
d	4.67
s	93
c	1270
b	4180
t	172900

The masses of quarks are shown in the table above [20] where they are measured at weak energy scale. It is seen that the mass of the particles is getting heavier when the generation number increases because the mass of the particles and detection of it is inversely related. What it means is that it is more difficult to detect havier particles.

From the section SU(N) it is learnt that there are 3^2-1 = 8 generators T^a of SU(3) group which means there are 8 gauge bosons, called gluons, represented as A^a_μ. According to the quantum field theory the Lagrangian term for the u quark is [19]

$$L=\bar{u}i\gamma^u\left(\partial_\mu - igA^a_\mu\lambda^a\right)u \quad (2.11)$$

where $D_\mu=\partial_\mu - igA_\mu$ and λ_a are Gell-Mann matrices given by

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\ \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\qquad \lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\ \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix} \quad (2.12)$$

Gluons are responsible for interactions between quarks such as a transformation of an up quark carrying red color charge to an up quark carrying blue color charge and a down quark carrying a blue color charge to down quark carrying red color charge as shown below.

$$u^r+d^b\rightarrow u^b+d^r \quad (2.13)$$

Strong interactions occur due to the emission and absorbtion of corresponding gluons of u^r and d^b respectively. If we consider the first family of the SM particles u and d quarks lie in the fundamental representation 3 of SU(3) group and antiquarks $\bar{u}$ and $\bar{d}$ lie in the complex representation $\bar{3}$ of SU(3) group shown as below.

$$u=\begin{pmatrix} u^r \\ u^g \\ u^b \end{pmatrix} \qquad \bar{u}=\begin{pmatrix} \bar{u}^r \\ \bar{u}^g \\ \bar{u}^b \end{pmatrix} \tag{2.14a}$$

$$d=\begin{pmatrix} d^r \\ d^g \\ d^b \end{pmatrix} \qquad \bar{d}=\begin{pmatrix} \bar{d}^r \\ \bar{d}^g \\ \bar{d}^b \end{pmatrix} \tag{2.14b}$$

Particles which do not carry color charges do not transform under SU(3) transformation, or it can be said that they do not tranform under strong force. An example of these particles such as e^-, e^+, and ν_e lie in the 1 (singlet) representation of SU(3). The first generation of the SM 15 particles shown in equation (2.25a, 2.25b) are in the representations of SU(3) respectively as $3 \oplus \bar{3} \oplus 3 \oplus \bar{3} \oplus 1 \oplus 1 \oplus 1$.

$SU(2)_L$

Now we repeat the analysis for two fermion fields, in other words, fermionic doublet fields. First, consider $\psi(x)$ is generated by a doublet of $\psi_1(x)$ and $\psi_2(x)$ fields [18].

$$\psi=\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.15}$$

To ensure the local symmetry invariance $\psi(x)$ must transform under a phase depends on x as

$$\psi(x) \rightarrow V(x)\psi(x) \tag{2.16}$$

in which V(x) can be defined as

$$V(x) = e^{i\alpha^i(x)\frac{\sigma^i}{2}} \tag{2.17}$$

In the above equation σ^i are the Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad , \quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad , \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.18}$$

which are commutative on the other hand, which is a specific situation of non-Abelian gauge theory, is called the Abelian gauge theory [18-19].

Since Pauli matrices are SU(2) local symmetry generator matrices, the new covariant derivative will be associated with SU(2) as

$$D_\mu=\partial_\mu - igA_\mu^i \frac{\sigma^i}{2} \tag{2.19}$$

$$D_\mu \rightarrow V(x)\left(A_\mu^i(x)\frac{\sigma^i}{2}+\frac{i}{g}\partial_\mu\right)V^\dagger(x) \tag{2.20}$$

The new transformation for commutator of covariant derivatives is shown below.

$$[D_\mu,D_\nu]\rightarrow V(x)[D_\mu,D_\nu]V^\dagger \tag{2.21}$$

Inserting the definition of the covariant derivative into the (2.21) the commutator of covariant derivative depending on the electromagntic field tensor is obtained as given below [18].

$$[D_\mu,D_\nu]=\frac{i}{g}F_{\mu\nu}^i\frac{\sigma^i}{2} \tag{2.22}$$

In the Lie Algebra of the SU(2) gauge theory the commutator of SU(2) generators is given in the equation below.

$$\left[\frac{\sigma^i}{2},\frac{\sigma^j}{2}\right]=i\epsilon^{ijk}\frac{\sigma^k}{2} \tag{2.23}$$

Therefore, we can write $F_{\mu\nu}^i$ as in the following equation.

$$F_{\mu\nu}^i=\partial_\mu A_\mu^i-\partial_\nu A_\mu^i+g\epsilon^{ijk}A_\mu^j A_\nu^k. \tag{2.24}$$

The Lagrangian is defined as

$$L=\bar{\psi}\left(i\gamma^\mu D_\mu\right)\psi-\frac{1}{4}\left(F_{\mu\nu}^i\right)^2-m\bar{\psi}\psi. \tag{2.25}$$

In equation (2.34), the commutator of the covariant derivative is written for the SU(2) gauge theory. Lagrangian in equation (2.25) is an example of the non-Abealian gauge theory and describres also an interacting field theory called Yang-Mills theory [18].

$SU(2)_L$ group has 2^2-1 = 3 gauge bosons that mediate weak interactions. To give an example of a weak interaction, starting to consider β-decay (beta decay) which is

$$n \rightarrow p + e^- + \bar{\nu}_e \tag{2.26}$$

Since a neutron is composed of two down and one up quarks, and a proton is that of two up and one down quarks, the equation (2.27) can be rewritten as following.

$$ddu \rightarrow uud + e^- + \bar{\nu}_e \tag{2.27}$$

The transformation that takes place between quarks (due to the fact that there are d,u and u,d quarks at the both left- and right-hand) is the tranformation of down and up quarks shown below.

$$d \rightarrow u + e^- + \bar{\nu}_e \tag{2.28}$$

This interaction is possible in the relevance of SU(2) gauge theory by putting up and down quarks in a doublet representation of SU(2). Three gauge bosons of SU(2) can be represented as $W_\mu{}^a$ where a = 1,2,3. The doublet representation of SU(2) is 2 that is the fundamental representation. SU(2) generators are 2-by-2 matrices.

Electron and electron neutrino are put in a doublet in the fundamental representation of SU(2). The number of 2 is analogous to spin-up and spin-down quantum representation meaning 2 degrees of freedom. The linear combination of the first three-gauge bosons mediates the transformation of up quark to down quark or electron neutrino to electron or vice versa.

The SU(2) doublets of the first generation of the SM particles can be written as

$$\begin{pmatrix} u \\ d \end{pmatrix}_L , \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \tag{2.29}$$

Particles which do not interact weakly lie in the singlet, 1, representation of SU(2) such as e^-_R, u_R, d_R. All right-handed particles are $SU(2)_L$ in the 1 representation of $SU(2)_L$. They do not transform under weak interaction. Only left-handed particles transform under weak interactions via SU(2) gauge bosons.

If we write down the summation of the $SU(2)_L$ representation of all first generation particles, we will get $2 \oplus 2 \oplus 2 \oplus 2 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1=15$ fields. In SM right-handed neutrino does not exist, however, it can be added as singlet in both SU(3) and SU(2) groups as quantum numbers (1,1,0). Therefore, right-handed electron does not couple with neutrino. It stays in the singlet representation.

The important point in weak interactions is while gauge bosons mediate the transformation of d quark to up quarks the color charges of quarks do not change. Owing to the fact that $SU(2)_L$ gauge bosons do not have color charge, they cannot interact with the color charges of quarks. Therefore, it is concluded that $SU(2)_L$ and $SU(3)_C$ groups have totally distinct languages. We cannot add or subtract these symmetries to each other. However, the direct product of strong and weak interactions generates a gauge group.

15 Weyl fields under the direct product of strong and weak interactions, $SU(3)_C \otimes SU(2)_L$ are represented as [19]

$$\begin{pmatrix} u^r \\ d^r \end{pmatrix}, \begin{pmatrix} u^g \\ d^g \end{pmatrix}, \begin{pmatrix} u^b \\ d^b \end{pmatrix}, \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \left(\bar{u}^r, \bar{u}^g, \bar{u}^b\right), \left(\bar{d}^r, \bar{d}^g, \bar{d}^b\right), \bar{e} \qquad (2.30)$$

There are 3 generations (families) in SM. In total, there are 15 fields with their antifields in each three generations in SM as shown in the Table 2.3.

Table 2.3 Generations of The Standard Model Particles

Generation	Particles
1st generation	$\begin{pmatrix} u^i \\ d^i \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, u^i_R, d^i_R, e^-_R$ + antiparticles
2nd generation	$\begin{pmatrix} c^i \\ s^i \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, c^i_R, s^i_R, \mu^-_R$ + antiparticles
3rd generation	$\begin{pmatrix} t^i \\ b^i \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, t^i_R, b^i_R, \tau^-_R$ + antiparticles

where the index i stands for color (i = red, green, blue); L and R indices denote left-handed and right-handed particles respectively.

There are left-handed doublets in each generation. Particles do not carry color charge cannot interact strongly and right-handed particles cannot interact weakly. On the other hand, for a right-handed electron, since it is not a doublet but a single particle its quantum number in SU(2) group is 1, and its hypercharge is also -1. The quark doublet in the table interacts strongly and because they carry color charge, the SU(3) quantum number is 3. Moreover, since it is a doublet, it transform as 2 under SU(2) group. Again, L denotes the left-handed helicity which can be represented by

$$\Psi_{L,R}=\frac{1\mp\gamma_5}{2}\,\Psi \tag{2.31}$$

We said before that the gauge bosons mediate the transformation of particles in doublets one another is the linear combination of the first two-gauge bosons of $W_\mu{}^a$ as $W^\pm$ [19]. What about the third component, W^3? Since W^3 couples to the neutrino, which is electrically neutral, with strength (-1/2) it cannot be a photon, the gauge boson of U(1) electromagnetism. W^3 is related to the third component of the weak isospin T^3.

Standard Model Gauge Couplings

It is known that the summation over the T^3 quantum numbers of particles in the $SU(3)_C \otimes SU(2)_L$ gauge group must give 0 since it is traceless. The hypercharge quantum numbers of particles are Q- T^3. Doing that calculation for particles in the first generation (u, d, e^-, ν) says that the electric charges are +2/3, -1/3, -1, 0, and their T^3 values are 1/2, -1/2, -1/2, +1/2 respectively. Hypercharges are calculated as 1/6, 1/6, -1/2, -1/2 for SU(2) doublets. For particles in the singlet representations of $SU(2)_L$ have hypercharges equal to their electric charges. The SM Particles can be represented with additional hypercharge quantum numbers as

$$\left(3,\,2,\,\tfrac{1}{6}\right)\oplus\left(\bar{3},\,1,\,-\tfrac{2}{3}\right)\oplus\left(\bar{3},\,1,\,\tfrac{1}{3}\right)\oplus\left(1,\,2,\,-\tfrac{1}{2}\right)\oplus(1,\,1,\,1) \tag{2.32}$$

The sum over hypercharge quantum numbers is 1-2+1-1+1= 0. It is not a surprise because hypercharge, $U(1)_Y$, is a generator. Photon emerges from the linear

combination of W^3 and gauge boson of $U(1)_Y$. On the other hand Z boson is orthogonal to this linear combination. Gluons, W, Z, the orthogonal linear combination of the photon and Z boson couples respectively to the first, second, and the third numbers

in equation (2.32) [19]. To give a short review, first $SU(3)_C \otimes SU(2)_L$ gauge group is set which is an irreducible gauge group, then by considering the third component of the SU(2) gauge bosons the missing point is found out as hypercharge which is associated with electric charge and weak isospin component. The end product is the known gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, the Standard Model.

The fields listed in the Table 2.4 have quantum numbers related to $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ groups shown below.

Table 2.4 Standard Model Particles under $SU(2)_L$ Gauge Group

Standard Model	$\begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix}_L$	$(\mathbf{u^c})_L$	$(\mathbf{d^c})_L$	$\begin{pmatrix} \boldsymbol{\nu_e} \\ \mathbf{e} \end{pmatrix}_L$	$(\mathbf{e^c})_L$
$\mathbf{SU(3)_C}$	3	$\bar{3}$	$\bar{3}$	1	1
$\mathbf{SU(2)_L}$	2	1	1	2	1
$\mathbf{U(1)_Y}$	$\frac{1}{6}$	$-\frac{2}{3}$	$+\frac{1}{3}$	$-\frac{1}{2}$	1

For g_i are the coupling constants of a gauge group

$$\frac{d\alpha_i}{d \ln q^2} = b_i \alpha_i^2 + O(\alpha_i^3) \tag{2.33}$$

where q^2 is the renormalization scale squared and $\alpha_i = \frac{g_i^2}{4\pi}$ for i=1,2,3 refers to SU(3), $SU(2)_L$, $U(1)_Y$ [18]. The b_i coefficients take values depending on vector boson, scalar boson, and fermionic loops contributions.

In the above equation (2.33) the second term on the right-hand side of the equation represents the 3rd power of the α and we will ignore these terms in beta functions. The variable g_i is the coupling constant; however, we will use α_i in beta functions for

simplicity. Moreover, there is a derivative in equation (2.44) with respect to the ln (q^2), where q^2 denotes the energy or the momentum scale at which α_i are mesasured. The coefficient b_i in equation (2.33) gets contributions from interaction loops such as scalar, vector, and fermion loops.

The number of contributions to b_i come from vector and fermion loops is [18]

$$b_i = -\frac{1}{4\pi}\left[\frac{11}{3}C_2(G_i) - \sum_f \frac{4}{3}T(R)_f\right], \tag{2.34}$$

when i=1,2,3 (from now on I will call α_i as coupling constants) α_1, α_2, α_3 are fine tunings of gauge groups of $U(1)_Y$, $SU(2)_L$, and $SU(3)_C$ symmetries respectively.

For groups SU(2) and SU(3) C_2 (G_i) is 2 for each generation n_g. So, if we write down all these variables in the beta function then we will have the following equations.

$$b_1 = -\frac{1}{4\pi}\left[\frac{11}{3}C_2(U(1)) - \frac{4}{3}\frac{1}{2}\frac{10}{3}n_g\right] = -\frac{1}{4\pi}\left[0 - \frac{20}{9}n_g\right] \tag{2.35a}$$

$$b_2 = -\frac{1}{4\pi}\left[\frac{11}{3}C_2(SU(2)) - \frac{4}{3}\frac{1}{2}2n_g\right] = -\frac{1}{4\pi}\left[\frac{22}{3} - \frac{4}{3}n_g\right] \tag{2.35b}$$

$$b_3 = -\frac{1}{4\pi}\left[\frac{11}{3}C_2(SU(3)) - \frac{4}{3}\frac{1}{2}2n_g\right] = -\frac{1}{4\pi}\left[11 - \frac{4}{3}n_g\right] \tag{2.35c}$$

As it's seen from equation (2.35a) the Casimir value for U(1) is zero. For SU(2) and SU(3) that value equal to the generator number of its group, which are 2 and 2 respectively.

In the beta functions above, equations (2.35a), (2.35b), and (2.35c), we have ignored the contributions from Higgs. From 2012, we know that in SM there is one Higgs doublet. If we add the Higgs contributions to the beta functions, there will be a $-\frac{1}{6}n_H$ factors.

$$\frac{1}{\alpha_1(q)} = \frac{1}{\alpha_{1_{GUT}}} + \frac{1}{2\pi}\left(\frac{41}{10}\ln\frac{M_X}{q}\right) \tag{2.36a}$$

$$\frac{1}{\alpha_2(q)} = \frac{1}{\alpha_{2_{GUT}}} + \frac{1}{2\pi}\left(-\frac{19}{6}\ln\frac{M_X}{q}\right) \tag{2.36b}$$

$$\frac{1}{\alpha_3(q)} = \frac{1}{\alpha_{3_{GUT}}} + \frac{1}{2\pi}\left(-7\ln\frac{M_X}{q}\right) \tag{2.36c}$$

In the above equations (2.36a, 2.36b, 2.36c) the parameters on the left-hand-side are coupling constants of $U(1)_Y$, $SU(2)_L$, and $SU(3)_C$ in measured energy scale q. These constants take values in energy scale of the Z boson (M_Z) as 58.996, 29.579, and 8.382 respectively [20]. The flow of the beta functions that shows contributions of SM gauge couplings are shown in the Figure 2.1.

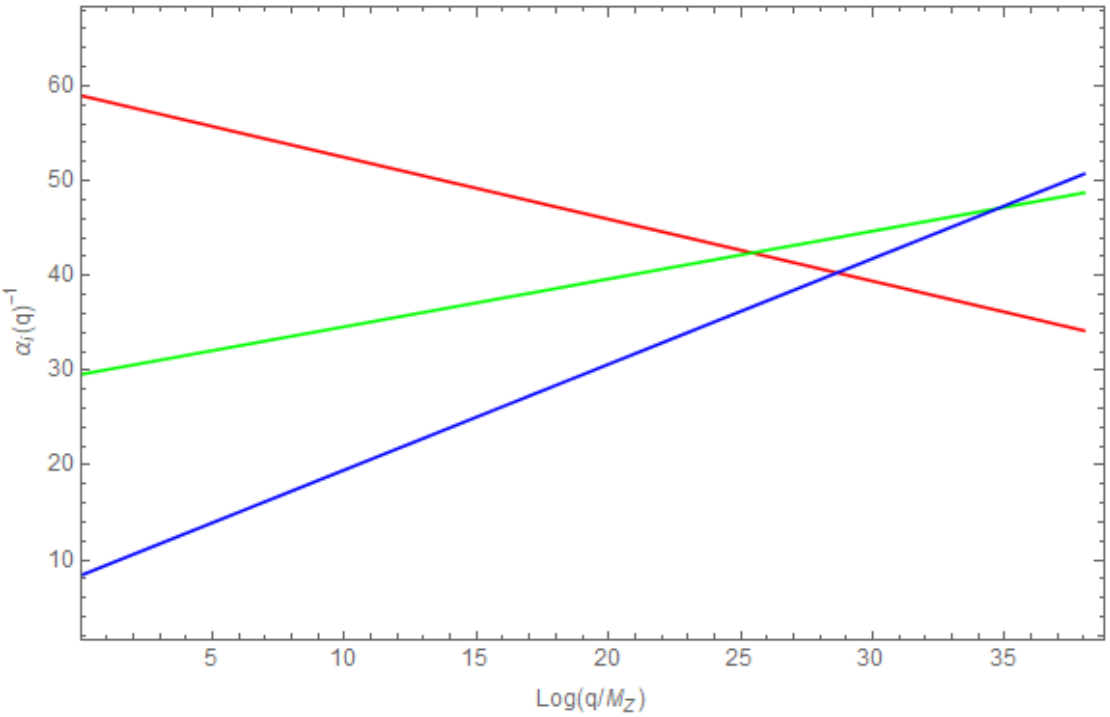


Figure 2.1 The Standard Model gauge couplings. Red, green and blue lines represents $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ gauge couplings respectively.

The states are unphysical eigenstates up to now since the Higgs doublet have not gained a VEV. Therefore, a mechanism, which must carry correct quantum numbers to procure the SM, is needed to get masses and supply a VEV for the Higgs doublet in SM. This process is called spontaneous symmetry breaking as demonstrated below.

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \xrightarrow[\text{SSB at weak energy scale}]{} SU(3)_C \otimes U(1)_{EM} \tag{2.37a}$$

We can look at the breaking symmetry in equation (2.37a) as

$$SU(2)_L \otimes U(1)_Y \xrightarrow[\text{SSB at wak energy scale}]{} U(1)_{EM} \tag{2.37b}$$

From eqution (2.37b) it is clear that the broken symmetry is the electroweak symmetry. Sometimes the $SU(3)_C \otimes U(1)_{EM}$ symmetry is called the exact gauge symmetry [17]. We have not encountered any symmetry breaking of $U(1)_{EM}$. This symmetry behaves like identity matrices in every gauge and Lorentz transformations. After SSB, 3 out of the 12-gauge bosons get mass. The bosons which stay massless are photon (A^{μ}) and gluons. From Gell-Mann matrices, which are generators of $SU(3)_c$, we know that there are 8 gluons. Since the Higgs doublet which has a VEV does not carry color charge, gluons and the Higgs do not interact with each other at tree level. As a result, gluons remain massless. The same result is also valid for A^{μ} due to the fact that Higgs does not carry electric charge in SM.

After the SSB of the SM we get $U(1)_{EM}$ and $SU(3)_C$ symmetries, which have different coupling constants than SM. Since this symmetry is embedded in SM, its quantum numbers are contained in SM. When we consider the coupling constants behavior under increasing energy (or decreasing), at first there must be three different coupling constants α_1, α_2, and α_3 before SSB. While the energy is decreasing, we see that there will be two different coupling constants of $U(1)_{EM}$ and $SU(3)_c$.

As it is mentioned earlier there are gauge bosons, which are mediators of strong, weak and electromagnetic interactions. When these bosons mediate, there will be interactions between fermions and these mediators. The Lagrangian (it will be called the Lagrangian density as Lagrangian) of these interactions can be written as follows [21].

$$L_{int}=g_1 J^{\mu} Y_{\mu}+g_2 J_i^{\mu} W_{\mu_i}+g_3 J_a^{\mu} A_{\mu_a} \tag{2.38}$$

with Y_{μ}, W_{μ_i}, and A_{μ_a} are gauge bosons where i=1, 2, 3 and a= 1, 2, …,8

g_1, g_2, and g_3 are the gauge coupling constants for electromagnetic, weak, and strong interactions respectively. The vector currets are given by

$$J^{\mu}=\frac{1}{6}\left(\bar{u}\bar{d}\right)_L \gamma^{\mu}\begin{pmatrix} u \\ d \end{pmatrix}_L+\frac{2}{3}\bar{u}_R\gamma^{\mu}u_R-\frac{1}{3}\bar{d}_L\gamma^{\mu}d_R+\frac{1}{2}(\bar{\nu_e}\quad \bar{e})_L\gamma^{\mu}\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L-\bar{e}_R\gamma^{\mu}e_R\,, \tag{2.39}$$

$$J_i^{\mu}=\left(\bar{u}\bar{d}\right)_L \gamma^{\mu}\frac{\sigma_i}{2}\begin{pmatrix} u \\ d \end{pmatrix}_L+(\bar{\nu}_e\bar{e})_L\gamma^{\mu}\frac{\sigma_i}{2}\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L\,, \tag{2.40}$$

$$J_a^{\mu}=\left(\bar{u}\bar{d}\right)_L \gamma^{\mu}\frac{\lambda_a}{2}\begin{pmatrix} u \\ d \end{pmatrix}_L\,, \tag{2.41}$$

where σ_i and λ_a denote the Pauli and Gell-Mann matrices respectively.

The gauge bosons which gain mass are $(W^{\pm})^{\mu}$ and $(Z^0)^{\mu}$.We can write down equations for $(W^{\pm})^{\mu}$, $(Z^0)^{\mu}$ and A^{μ} as [21]

$$(W^{\pm})^{\mu}=\frac{1}{\sqrt{2}}(W_1^{\mu}\mp iW_2^{\mu}) , \tag{2.42a}$$

$$(Z^0)^{\mu}=\cos\theta_W W_3^{\mu}-\sin\theta_W Y^{\mu} , \tag{2.42b}$$

$$A^{\mu}=\sin\theta_W W_3^{\mu}+\cos\theta_W Y^{\mu} , \tag{2.42c}$$

where θ_W is called as weak angle. When we look at the equations above, we cannot see any mixing between $(W^{\pm})^{\mu}$ and $(Z^0)^{\mu}$ bosons. The reason is the conservation of $U(1)_{em}$ symmetry as we have said earlier. Now, we can find out the Lagrangian again for SSB in equation [21].

$$Y^{\mu}=\cos\theta_W A^{\mu}-\sin\theta_W (Z^0)^{\mu} , \tag{2.43}$$

$$W_3^{\mu}=\cos\theta_W (Z^0)^{\mu}+\sin\theta_W A^{\mu} , \tag{2.44a}$$

$$W_1^{\mu}=\frac{\sqrt{2}}{2}\left((W^+)^{\mu}+(W^-)^{\mu}\right) \text{ and } W_2^{\mu}=\frac{\sqrt{2}}{2i}\left((W^-)^{\mu}-(W^+)^{\mu}\right) , \tag{2.44b}$$

$$L_{int}=\frac{1}{2\sqrt{2}}g_2\left(W^+)^{\mu}J_{-\mu}+\left(W^-\right)^{\mu}J_{+\mu}\right)+\left(\left(g_2\cos\theta_W+g_1\sin\theta_W\right)J_{3\mu}-g_1\sin\theta_W J_{em\,\mu}\right)(Z^0)^{\mu}+(g_1\cos\theta_W J_{em\,\mu}+(g_1 \tag{2.45}$$

The electromagnetic current J_{em}^{μ} must couple only with the photon A^{μ}. Therefore

$$g_1\cos\theta_W-g_2\sin\theta_W=0 . \tag{2.46}$$

We know that g_1 and g_2 are different than zero, so $g_1\cos\theta_W$ and $g_2\sin\theta_W$ must be equal to the same constant like

$$g_1\cos\theta_W=g_2\sin\theta_W=e . \tag{2.47}$$

We can rearrange the Lagrangian by writing e instead of g_1 and g_2.

$$L_{int}=\frac{e}{2\sqrt{2}\sin\theta_W}\left[(W^+)^\mu(J_-)_\mu+(W^-)^\mu(J_+)_\mu\right]+\frac{e}{2\cos\theta_W\sin\theta_W}(J_{NC})_\mu Z^\mu+e(J_{em})_\mu A^\mu\,, \quad (2.48)$$

where

$$(J_{NC})_\mu=2\left[(J_3)_\mu-\sin^2\theta_W(J_{em})_\mu\right] \quad (2.49)$$

is called the neutral current. And the constant ρ is found as the following

$$\rho=\frac{M_W^2}{M_Z^2\cos^2\theta_W} \quad (2.50)$$

Higgs mechanism

For a complex scalar field in an electromagnetic field the Lagrangian is

$$L=-\frac{1}{4}\left(F_{\mu\nu}\right)^2+\left|D_\mu\phi\right|^2-V(\phi) \quad (2.51)$$

where the covariant derivative is $D_\mu=\partial_\mu+ieA_\mu$. Lagrangian in equation (2.51) is invariant under phase transformations which can be seen as the transformation under U(1) symmetry. If we change the potential term in the equation (2.51) as

$$V(\phi)=-\mu^2\phi^*\phi+\frac{\lambda}{2}\left(\phi^*\phi\right)^2 \quad (2.52)$$

where $\mu^2>0$, the negative sign of the mass term in equation (2.10) becomes positive. When the complex scalar field ϕ acquires a non-zero VEV, the Lagrangian is not invariant under the U(1) transformation anymore and that the global U(1) symmetry is spontaneously broken. The Lagrangian with the negative sign potential is

$$L=-\frac{1}{4}\left(F_{\mu\nu}\right)^2+\left|D_\mu\phi\right|^2+\mu^2\phi^*\phi-\frac{\lambda}{2}\left(\phi^*\phi\right)^2\,. \quad (2.53)$$

To find out the minimum value of the potential in the above Lagrangian we must take its derivative which must be zero at the minimum value of the potential.

$$\frac{\partial L}{\partial\phi}=2\mu^2\phi^*-2\lambda\phi^*|\phi|^2=0\Rightarrow\phi_0=\left(\frac{\mu^2}{\lambda}\right)^{1/2}\,, \quad (2.54)$$

$$\langle\phi\rangle = \phi_0 = \left(\frac{\mu^2}{\lambda}\right)^{1/2} . \tag{2.55}$$

In equation (2.55) the term $\langle\phi\rangle$ represent the VEV of the field ϕ. Now, if we rewrite the complex field $\phi(x)$ in terms of real and complex fields separately we find

$$\phi(x) \equiv e^{-i\theta}\left((\phi_0+h)/\sqrt{2}\right) \tag{2.56}$$

$$\phi(x) \rightarrow e^{i\theta}\phi(x) \tag{2.57}$$

Then the Lagrangian can be written as the following

$$L_V = -\mu^2 h^2 - \lambda v h^3 - \frac{1}{4}\lambda h^4 . \tag{2.58}$$

While the real part of the field, acquires a mass $m = \sqrt{2}\mu$, the complex part is massless. After the SSB, massless scalar field related with the symmetry is called Goldstone bosons. We can write the term of the square of the VEV as a mass term as following

$$e^2\phi_0^2 A_\mu A^\mu = \frac{1}{2}(2e^2\phi_0^2)A_\mu A^\mu , \tag{2.59}$$

where the term $2e^2\phi_0^2$ is a mass term for the gauge boson of U(1) symmetry that is photon. The Lagrangian difference due to the mass term has a positive sign, which is correct for a potential term, written as

$$\Delta L = \frac{1}{2} m_A^2 (A_\mu A^\mu). \tag{2.60}$$

If we take $\phi(x)$ as a real field, the complex component will disappear. Then the Lagrangian in equation (2.53) becomes

$$L = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 + \left(\partial_\mu h\right)^2 + e^2\phi_0^2 A_\mu A^\mu - V(h) \tag{2.61}$$

At the minimum of the Lagrangian in the eqaution (2.61) the potential gets a nonzero VEV, the field acquire mass. The remaining massless part becomes a component of the

real field ϕ_1. This analysis shows how gauge bosons becomes massive after SSB. This mechanism created for the gauge theory by Higgs, Kibble Guralnik, Hagen, Brout, and Englert is called shortly as the Higgs mechanism.

Higgs mechanism is a mechanism to provide mass for fermions and gauge bosons except neutrinos, photon, and gluons respectively by gaining VEV to break the symmetry as we said earlier. The doublet field ϕ of Higgs looks like as follows.

$$\phi = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix} \tag{2.62}$$

This potential transforms as (1, 2, -1/2) under $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. Since the upper component of the Higgs potential ϕ^0 carries both $SU(2)_L$ and $U(1)_Y$ quantum numbers, the symmetry is broken to $SU(3)_c \otimes U(1)_{em}$ symmetry. However, it does not transform under the $U(1)_{em}$ symmetry, since it is electrically neutral when the SSB occurs. The VEV of Higgs potential is

$$\langle \phi \rangle = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \tag{2.63}$$

The mass term of gauge bosons comes from the kinetic term of the Lagrangian

$$L_{mass} = \left(\frac{g_2 v}{2}\right)^2 W_+^\mu W_\mu^- + \frac{1}{8} v^2 \left(W_3^\mu Y^\mu\right) \begin{pmatrix} g_2^2 & -g g_1 \\ -g g_1 & g_1^2 \end{pmatrix} \begin{pmatrix} W_3^\mu \\ Y^\mu \end{pmatrix}. \tag{2.64}$$

The mass eigenvalues of the gauge bosons are

$$M_{W^\pm}^2 = \frac{g_2^2 v^2}{4}, \tag{2.65a}$$

$$M_Z^2 = \frac{1}{4}\left(g_2^2 + g_1^2\right) v^2, \tag{2.65b}$$

$$M_A^2 = 0, \tag{2.65c}$$

For $\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W}$ we find

$$\rho=\frac{\frac{g_2^2v^2}{4}}{\frac{1}{4}(g_2^2+g_1^2)v^2\cos\theta_W}=1\,. \tag{2.66}$$

Experimentally, ρ is not exactly, but very close to 1 as 0.99. When we study other gauge groups, we must consider the ρ value, which must be almost 1.

The equation (2.64) is a mass term for gauge bosons. We said earlier that the SSB cause massive fermions either. The mass term for fermions can be written as [18]

$$L_Y=\Gamma^l_{ij}\bar{L}_{ij}\tilde{\phi}l_{jR}-\Gamma^d_{ij}\bar{Q}_{iL}\tilde{\phi}d_{jR}+\Gamma^u_{ij}\bar{Q}_{iL}\phi u_{jR}+\text{h.c.} \tag{2.67}$$

where i, j are indices of generations and L_L and Q_L are lepton and quarks weak isodoublets. The term Γ^l_{ij} is arbitrary Yukawa coupling constant. $\tilde{\phi}$ that is the charge conjugate of ϕ, can be defined by the following equation.

$$\tilde{\phi}=i\tau_2\phi=\begin{pmatrix}\phi^+\\ -(\phi^0)^*\end{pmatrix} \tag{2.68}$$

The equation (2.67) is Yukawa Lagrangian where fermion masses, except neutrinos, come from. Equation (2.67) is Lagrangian of quarks and leptons where no SSB of the SM has been occured yet. After SSB, the equation (2.67) becomes [18]

$$L^F_{mass}=\bar{l}_{iL}M^l_{ij}l_{iR}+\bar{d}_{iL}M^d_{ij}d_{jR}+\bar{u}_{iL}M^u_{ij}u_{jR}+\text{h.c.} \tag{2.69}$$

where l_{iL} and l_{iR} are left and right-handed fields of i^{th} generation lepton and M^f_{ij} (f = l,f,u, represents fermions) is

$$M^f_{ij}=\frac{1}{\sqrt{2}}v\Gamma^f_{ij}\,. \tag{2.70}$$

The current eigenstates of the L_{int} are related with the mass eigenstates of the L^f_{mass} as

$$\psi^f_{L,R}=U^f_{L,R}\psi^{0f}_{L,R}. \tag{2.71}$$

In equation (2.71) $U^f_{L,R}$ are unitary matrices that diagonalize M^f_{ij} with no summation over the indice i.

$$U_L^f M_{ij}^f U_R^f = \left(M_D^f\right)_i \delta_{ij} \,, \tag{2.72}$$

where the subscript "D" means diagonalized matrix.

The symmetry breaking of the SM via Higgs mechanism is an economical way to give masses to fermions (except neutrinos) and gauge bosons (saving photons and gluons). The determination of the number of Higgs, its potential, and the way we choose to use the mechanism, Yukawa couplings can be arranged. We determine these vaiables to get a successful VEV and get massive fermions and gauge bosons.

2.2.SU(5) Grand Unification

The question in mind after setting up SM is, is it possible to unify these three forces? The answer is yes, and the name of the theory which aims the question is called Grand Unification Theories (GUTs). GUTs can be categorized into two main braches which are SUSY GUTs and non-SUSY GUTs. In the first part of this thesis, it is mentioned that non-SUSY GUT model will be studied. Therefore, it will be resonable to explain only non-SUSY GUTs in this part. The beauty in the theory is its simplicity. For many years physicists search the simplest explanation for what is observed in the nature.

When we try to embed the group into the SM group, G_{SM}, into a larger group G, G_{SM} become a subgroup of the G, thereby the generators of its subgroup should be normalized in the same way. For example, the normalization of the U(1) hypercharge must be normalized as $SU(2)_L$.

The particle content of $SU(3)\otimes SU(2)\otimes U(1)$ irreducible representations can be written as:

$$\left(3,2,\tfrac{1}{6}\right),\ \left(\bar{3},1,-\tfrac{2}{3}\right)\ ,\ \left(\bar{3},1,\tfrac{1}{3}\right)\ ,\ \left(1,2,-\tfrac{1}{2}\right)\ ,(1,1,1) \tag{2.73}$$

The most satisfying GUT model must be the simplest or it can be said the smallest one. In the previous page it is said that Grand Unification must obey SM constructions. That represents containing the fundamental forces (of course gravity is not included). Therefore the research must be focused on the smallest group that contains $SU(3)\otimes SU(2)\otimes U(1)$. The result of this research shows us that SU(5) is the smallest

appropriate group contains $SU(3)\otimes SU(2)\otimes U(1)$. To find out a gauge group G that includes the $SU(3)_C\otimes SU(2)_L\otimes U(1)_Y$ must have a gauge boson other than gauge bosons of SM gauge theory. G must be a larger group than the SM to include it. This gauge boson must be couple to particles in the equation (2.73). The minimal group must at least have the rank equals to the SM which equals to 4.

SU(5) is a gauge group has rank 4 and, 5^2-1 = 24 gauge bosons composed of 5-by-5 traceless matrices. If we break the SU(5) symmetry the remaining is the SM symmetry.

$$SU(5) \xrightarrow[\text{Higgs in 24}]{} SU(3)_C\otimes SU(2)_L\otimes U(1)_Y \tag{2.74}$$

The fundamental representation 5 of SU(5) can be demonstrated as ψ^μ where μ=1, 2, 3, 4, 5. Since $SU(3)_C$ and $SU(2)_L$ are subgroups of SU(5) and its elements can be seen as two parts, one comes from SU(3) and the other from SU(2). Mathematically this separation can be made with ψ^μ where μ=1,2,3 represents SU(3) subgroups and ψ^μ where μ= 4,5 represents SU(2) subgroups [19]. It can also be done by considering adjoint representation A. The matrix form of A can be written as [21] when i=1 to 24,

$$A= \sqrt{2}\sum_i^{24} A^i \frac{\lambda_i}{2} \tag{2.75}$$

Let us define G_i are SU(5) generator matrices. When i = 1-8 the adjoint SU(5) matrices are [21]

$$G_{1\text{-}8}=\begin{pmatrix} \left[\frac{\lambda}{2}\right]_{3\times3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{5\times5}, \tag{2.76a}$$

when i = 9-11

$$G_{9\text{-}11}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left[\frac{\sigma}{2}\right]_{2\times2} \end{pmatrix}_{5\times5}, \tag{2.76b}$$

which is 2+1+1 = 4.

When i = 24

$$G_{24}=\frac{1}{\sqrt{15}}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 3/2 \end{pmatrix}_{5\times 5} . \qquad (2.76c)$$

Other values of i are in the coset of SU(5). It is seen from the equation (2.76c) that $U(1)_Y$ lie along the diagonal. It transforms like identity matrix in both SU(3) and SU(2) transformations. These matrices are orthogonal to each other. Remembering the Lie Algebra of SU(N) groups [16], we can write down

$$Tr(\lambda_a\lambda_b)=2\delta_{ab} \qquad (2.77)$$

The adjoint representation of SU(5) is constructed by the direct product of fundamental and complex fundamental representations of SU(5). The fundemental representation 5 of SU(5) is $5 \supset \left(3, 1, -\frac{1}{3}\right) \oplus \left(1, 2, +\frac{1}{2}\right)$ and its complex representation $\bar{5} \supset \left(\bar{3}, 1, +\frac{1}{3}\right) \oplus \left(1, 2, -\frac{1}{2}\right)$. Direct product of them

$$5 \otimes \bar{5} = 24 \oplus 1 , \qquad (2.78)$$

where 1 is the singlet representation of SU(5). Analogous to the SU(2) and SU(3) singlet representation SU(5) singlet means that the field does not transform under SU(5). 24 is traceless and real representation. After doing the direct product calculation in the equation (2.78), 24 is written explicitly as

$$24 \supset (8 , 1 , 0) \oplus (1 , 3 , 0) \oplus (1 , 1 , 0) \oplus (3 , 2 , -5/6) \oplus (\bar{3} , 2 , 5/6) , \qquad (2.79)$$

where (8, 1, 0) carry color charge (3, 2, -5/6), and $(\bar{3}, 2, 5/6)$ split multiplets carry both color and weak isospin charges. (1, 3, 0) transform only under weak isospin. (1, 1, 0) does not transform under any fundamental interactions. It is neutral and singlet in SU(3) and SU(2) groups. We can call these split multiplets as $G^{\alpha}_{\mu\beta}$, $\left(W^{\pm}_{\mu}, W^{3}_{\mu}\right)$, B_{μ}, $A^{\alpha}_{\mu r}$, $A^{r}_{\mu\alpha}$ respectively where $\alpha, \beta = 1, 2, 3$ like color indices of SU(3) and $r = 4, 5$ like weak isospin indices of SU(2) as subgroups of SU(5) [21].

Using the equation (2.75) gauge bosons of SU(5) can be placed in the matrix [21]

$$A=\begin{pmatrix} G_1^1-\sqrt{\frac{2}{15}}B & G_2^1 & G_3^1 & \bar{X}^1 & \bar{Y}^1 \\ G_1^2 & G_2^2-\sqrt{\frac{2}{15}}B & G_3^2 & \bar{X}^2 & \bar{Y}^2 \\ G_1^3 & G_2^3 & G_3^3-\sqrt{\frac{2}{15}}B & \bar{X}^3 & \bar{Y}^3 \\ X_1 & X_2 & X_3 & \frac{-W^3}{\sqrt{2}}+\frac{3}{2}\sqrt{\frac{2}{15}}B & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & \frac{-W^3}{\sqrt{2}}+\frac{3}{2}\sqrt{\frac{2}{15}}B \end{pmatrix} \quad (2.80)$$

Matrices above are chosen such that we want to break SU(5) into the SM leaving the other symmetries intact. The naturality lying behind the SU(5) symmetry is that it breaks into SM just in one symmetry breaking. Other symmetries such as SO(10) is a group of rank 5. SO(10) symmetry breaks into SM under two SSB [22].

According to GG model all SM particles are placed in $\bar{5}\oplus 10$ representations of SU(5) [1]. At the beginning of this page split multiplets of $\bar{5}$ is demonstrated. 10 antisymmetric representation of SU(5)

$$10 \supset (3,\ 2,\ 1/6) \oplus (\bar{3},\ 1,\ -2/3) \oplus (1,\ 1,\ 1) \quad (2.81)$$

comes from the direct product of two fundamental representations as

$$5\otimes 5=15\oplus 10 \quad (2.82)$$

We know that the direct sum of these representations must give us a singlet, 1 of SU(5). It is analogous to getting a particle with its antiparticle together.

Looking at split multiplets of the $\bar{5}$ which are $(\bar{3},\ 1,\ 1/3)\oplus(1,\ 2,\ -1/2)$ and 10 having split multiplets of $(3,\ 2,\ 1/6)\oplus(\bar{3},\ 1,\ -2/3)\oplus(1,\ 1,\ 1)$, it is seen that they marvelously fix the fermionic fields with quantum numbers of particles in the SM. What about antiparticles of fermions? Clearly, they are placed in the complex representations of $\overline{(\bar{5}\oplus 10)} = 5\oplus\overline{10}$. Taking $\bar{5}$ as a column vector ψ_μ and 10 as a 5-by-5 antisymmetric matrix $\psi^{\mu\nu}$ we can write them as in the folowing equations [19].

$$\psi_\mu = \begin{pmatrix} d^c \\ d^c \\ d^c \\ e^- \\ \nu_e \end{pmatrix}, \tag{2.83a}$$

$$\psi^{\mu\nu} = \begin{pmatrix} 0 & u^c & -u^c & -u & -d \\ -u^c & 0 & u^c & -u & -d \\ u^c & -u^c & 0 & -u & -d \\ u & u & u & 0 & -e^+ \\ d & d & d & e^+ & 0 \end{pmatrix}, \tag{2.83b}$$

In equations (2.83a) and (2.83b) color charges and handedness are not indicated. These matrices can be taken as both right-handed or left-handed, independently. For instance, if they are taken as left-handed, their complex fields must be right-handed for parity conservation. At the end, these matrices must include all fermions in one generation. Today, it is known that there are three generations of quarks and leptons. Even though, up to now there have not been found any scientific explanation that falsify it, the whole analysis for the first generation can be repeated for the following generations (2^{nd} and 3^{rd}).

Returning back to the particles representations $\bar{5}$ and 10, with vector representations ψ_μ and $\psi^{\mu\nu}$ respectively, it is possible to write down these vectors under corresponding SU(3) and SU(2) subgroups. If we write down two subgroups of $\bar{5}$, ψ_μ can be written as [19,21]

$$\psi_\mu = \begin{pmatrix} \psi_\alpha \\ \psi_i \end{pmatrix} \tag{2.84}$$

where $\alpha = 1, 2, 3$ and $i=4, 5$. For the antisymmetric tensor 10, $\psi^{\mu\nu}$ can be rewritten as [19,21]

$$\psi^{\mu\nu} = \{\psi^{\alpha\beta}, \psi^{\alpha i}, \psi^{ij}\} \tag{2.85}$$

where $\beta=1, 2, 3$ and $j=4, 5$.

Charge quantization

In SU(5) gauge theory fermions and leptons are placed in the $\bar{5} \oplus 10$ representation where the sum of the electric charges of particles must be 0 since the hypercharge symmetry transformation is SU(5) invariant. The relationship between hypercharge and electric charge was given as $Q=T_3+ Y$. Looking at the 5 of SU(5) the particles are $3(d^c)$, e , and ν_e with charges 1/3, -1, and 0 respectively. The sum of the electric charges of these particles is 0. It can be concluded that by replacing the particles as in the GG model it is possible to explain the electric charge quantization. In SM it cannot be related electric charges of the d-type quarks with the electron, since the electric charge is a free parameter under SM gauge transformation. GUTs are successful to give an explanation to quantization of the electric charge.

We know that generator of $U(1)_Y$ is the hypercharge (Y), we will denote the normalized Y as Y' as below:

$$\mathrm{Tr}\left(T_3^2\right)=\mathrm{Tr}\left(Y'^2\right), \tag{2.86}$$

If we consider only one family, then

$$\mathrm{Tr}\left(T_3^2\right)|_{\text{for one family}}=\left(\frac{1}{4}2\right)4=2=\mathrm{Tr}\left(Y'^2\right), \tag{2.87}$$

since we have two doublets and 2 fermions for each doublets, we get the above result. Adding up the hypercharges (Y) of these particles ends up with 0. The reason of that since U(1) obeys the simple Lie Algebra the matrix of the generator must be traceless. The same is valid for hypercharges.

The renormalization of the hypercharge symmetry group is

$$\mathrm{Tr}\left(Y^2\right)=\frac{1}{36}6+\frac{1}{4}2+\left(\frac{4}{3}+\frac{1}{9}\right)3+1=\frac{10}{3}, \tag{2.88a}$$

$$Y'^2=cY^2=2\Rightarrow c\frac{10}{3}=2, \tag{2.88}$$

where c is a constant, then

$$c=\frac{3}{5}. \tag{2.89}$$

So we get the relationship as in the equation.

$$Y'=\sqrt{\frac{3}{5}}Y\,. \tag{2.90}$$

Thereby we can rewrite the eqation (2.2) in terms of Y'.

$$Q=T_3+Y=T_3+\sqrt{\frac{5}{3}}Y \tag{2.91}$$

This renormalization relation between primed and unprimed hypercharges is seen also between their coupling constants. If we call the coupling constant of Y' as g_1', then

$$g_1'=\sqrt{\frac{5}{3}}g_1 \tag{2.92}$$

The example mentioned above is clearly seen as the particles in 5 of SU(5) , given in (2.83a), have electric charges 1/3+1/3+1/3-1+0=0.

Weinberg angle

Besides the prediction of the lifetime of the proton SU(5) gauge theory is a successful theory for unification and estimation of the $\sin^2\theta_W$ where θ_W is called Weinberg (or weak) angle. SU(5) includes SU(3) and SU(2) groups as expressions Q and T_3 respectively. Nomalized SU(3) and SU(2) operators $\widehat{Q}$ and $\widehat{T_3}$ under the 5 of SU(5) symmetry can be written as [21]

$$\widehat{Q}=\sqrt{\frac{3}{8}}\,\mathrm{diag}\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},-1,0\right)=\sqrt{\frac{3}{8}}Q\,, \tag{2.93a}$$

$$\widehat{T_3}=\mathrm{diag}(0,0,0,+1/2\,,-1/2\,)=T_3\,. \tag{2.93b}$$

The relation between SU(3) - SU(5), and SU(2)-SU(5) operators are written as [21]

$$e^2Q^2=g_5^2\widehat{Q}^2\,, \tag{2.94}$$

$$g_2^2 T_3^2 = g_5^2 \widehat{T_3}^2 \,. \tag{2.95}$$

where e, g_2, g_5 are coupling constants of $U(1)_{em}$, $SU(2)_L$, and SU(5) gauge theories respectively. Using the equality of $\widehat{Q}^2 = \widehat{T_3}^2$ the following eqautions are found in GUT scale.

$$\frac{e^2}{g_2^2} = \frac{\mathrm{Tr}\left(T_3^2\right)}{\mathrm{Tr}\left(Q^2\right)}, \tag{2.96}$$

$$\sin^2(\theta_W) = \frac{e^2}{g_2^2} = \frac{\mathrm{Tr}\left(T_3^2\right)}{\mathrm{Tr}\left(Q^2\right)}. \tag{2.97}$$

For the fundamental representation 5 of SU(5) these traces are calculated by using the equation (2.93a) and (2.93b) as

$$\mathrm{Tr}\left(T_3^2\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \tag{2.98a}$$

$$\mathrm{Tr}\left(Q^2\right) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + 1 = \frac{4}{3}, \tag{2.98b}$$

$$\Rightarrow \sin^2(\theta_W) = \frac{1}{2}\frac{3}{4} = \frac{3}{8}. \tag{2.99}$$

The result in equation (2.99) is consistent with the experimental values. For the grand unification, the value of $\sin^2(\theta_W)$ at GUT scale is calculated as following

$$\sin^2\theta_W(q^2) = \frac{\alpha(q^2)}{\alpha_2(q^2)}, \tag{2.100}$$

$$\frac{1}{\alpha_3^2} - \frac{1}{\alpha_2^2} = \left(b_3 - b_2\right) \ln\left(\frac{M_X^2}{q^2}\right), \tag{2.101}$$

where q is the energy scale. The value of b_2-b_1 for number of Higgs (N_H) is 1 as $-\frac{11}{12\pi}$. Therefore

$$\frac{1}{\alpha_2^2} = \frac{1}{\alpha_3^2} + \frac{11}{12\pi} \ln\left(\frac{M_X^2}{q^2}\right), \tag{2.102}$$

$$\sin^2\theta_W(q^2)=\alpha(q^2)\left[\frac{1}{\alpha_3^2}+\frac{11}{12\pi}\ln\left(\frac{M_X^2}{q^2}\right)\right],$$

(2.103)

$$\frac{1}{\alpha(q^2)}=\frac{8}{3}\frac{1}{\alpha_3(q^2)}+\frac{11}{2\pi}\ln\left(\frac{M_X^2}{q^2}\right), \tag{2.104}$$

$$\sin^2\theta_W(q^2)=\alpha(q^2)\left[\frac{3}{8}\left[\frac{1}{\alpha(q^2)}-\frac{11}{2\pi}\ln\left(\frac{M_X^2}{q^2}\right)\right]+\frac{11}{12\pi}\ln\left(\frac{M_X^2}{q^2}\right)\right], \tag{2.105a}$$

$$\sin^2\theta_W(q^2)=\frac{3}{8}\left[1-\frac{\alpha}{4\pi}\frac{110}{9}\ln\left(\frac{M_X^2}{q^2}\right)\right]. \tag{2.105b}$$

According to experimental results the very recent result [20] for $\alpha(M_Z)\cong 0.1193$. At $q=M_Z$

$$\sin^2\theta_W\left(M_Z^2\right)\cong 0.23126 \tag{2.106}$$

To check the ρ value by taking $M_W\cong 80.398$ GeV/c^2 and $M_Z\cong 91.188$ GeV/c^2 we find that

$$\rho=\frac{M_W^2}{M_Z^2\left(1-\frac{9}{16}\right)}\cong 0.91. \tag{2.107}$$

The result given in the equation (2.107) is very close to 1.

Pattern of Spontaneous Symmetry Breaking

Gauge bosons that mediate interactions of both leptons and quarks are called leptoquarks. X and Y are gauge bosons of SU(5) [21]. Leptoquarks carry both color and weak isospin charge, in other words they are not singlet under SU(3) and SU(2) transformations. If we look at the interactions mediated by X and Y gauge bosons, we will see there is a violation of lepton and baryon numbers.

We have the Lagrangian(s) for particle interactions and the role of Higgs mechanism. Now we can break the SU(5) symmetry into the SM.

$$SU(5)\xrightarrow[M_X]{}SU(3)_C\otimes SU(2)_L\otimes U(1)_Y, \tag{2.108a}$$

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \xrightarrow[M_W]{} SU(3)_C \otimes U(1)_{EM} . \quad (2.108b)$$

Equations (2.108a) and (2.108b) are the symmetry breakings of SU(5) and the SM respectively. From these equations it is explicitly seen that the energy scales of these symmetry breakings are about M_X and M_W where the former is the energy scale of the grand unification, and the latter is the weak energy scale.

If we look at the equations (2.108a) and (2.108b) we can see that the symmetry breaking pattern between the weak and hypercharge interactions are below,

$$SU(2)_L \otimes U(1)_Y \xrightarrow[M_W]{} U(1)_{EM} . \quad (2.109)$$

The Higgs scalar responsible for the SU(5) symmetry breaking by acquiring a VEV include particles that are not in the SM. It is known that the rank of SU(5) is the same as the rank of the SM, that is why after the SSB of SU(5) the rank should not change. The Higgs which breaks that symmetry therefore belongs to the adjoint represantation of SU(5). Remembering the gauge boson matrix, we can also write the matrix form of the 24 representations as [21]

$$\phi_a^b = \sum_{i=1}^{24} \varphi_i \frac{\left(\lambda^i\right)_a^b}{\sqrt{2}} , \quad (2.110)$$

which is a traceless matrix. There are larger real representations of SU(5) which can be written as a power of the adjoint representation that transforms as [21]

$$\phi \rightarrow U^{\dagger} \phi U , \quad (2.111)$$

where U is a SU(5) element. The most general potential for the Higgs in a real representation must be a renormalizable function as written in the equation below.

$$V = -\frac{\mu^2}{2} Tr\left(\phi^2\right) + \frac{1}{4} a \left(Tr\left(\phi^2\right)\right)^2 + \frac{1}{2} b Tr\left(\phi^4\right) + \frac{1}{3} Tr\left(\phi^3\right) . \quad (2.112)$$

For conserving a symmetry for $\phi \rightarrow -\phi$ cubic terms in the equation (2.112) is omitted. Keeping in mind that the aim is breaking the SU(5) symmetry into the

$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, hence the VEV of the adjoint Higgs in the SU(5) can always be chosen to be a traceless diagonal matrix [21], as given below

$$\langle\phi\rangle = \begin{pmatrix} v & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & -3v/2 & 0 \\ 0 & 0 & 0 & 0 & -3v/2 \end{pmatrix}. \tag{2.113}$$

The essential point is what happens if we ensure the matrix is traceless and looks like [21]

$$\langle\phi\rangle = \begin{pmatrix} v & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & -4v \end{pmatrix} \tag{2.114}$$

With the choice of Higgs VEV as in the equation (2.114) the SSB results a breaking pattern as following

$$SU(5) \xrightarrow[M_X]{} SU(4) \otimes U(1) \tag{2.115}$$

which is not desired.

The minimum value of the equation (2.112) is the minimum value of the VEV of Higgs given in the equation (2.116) below [21]

$$\langle V\rangle = -\frac{\mu^2}{2}\frac{15}{2}v^2 + \frac{1}{4}a\frac{225}{4}v^4 + \frac{1}{2}b\frac{105}{8}v^4 . \tag{2.116}$$

To find out the minimum of the VEV function given above its derivative must be taken and equated to zero, giving

$$\frac{\partial V}{\partial v} = -\mu^2\frac{15}{2}v + \frac{225}{4}a\,v^3 + \frac{1}{2}b\frac{105}{2}v^3 = 0 , \tag{2.117}$$

$$v\left(-\mu^2\frac{15}{2} + \frac{225}{4}a\,v^2 + b\frac{105}{4}v^2\right) = 0 . \tag{2.118}$$

For nonzezo VEV value we must set the squared term equals to zero

$$-\mu^2\frac{15}{2}+\left(\frac{225}{4}a+b\frac{105}{4}\right)v^2=0\,, \tag{2.119a}$$

$$\left(\frac{15}{2}a+b\frac{7}{2}\right)v^2=\mu^2\,, \tag{2.119b}$$

$$v^2=\frac{2\mu^2}{15a+7b}\,. \tag{2.119c}$$

The SSB of the SM occurs due to the fact that the Higgs in the fundamental representation 5 of SU(5) gains a VEV. Remembering the split multiplets of 5, $5\supset\left(3,1,-\frac{1}{3}\right)\oplus\left(1,2,+\frac{1}{2}\right)$, it is concluded that the Higgs scalar in 5 must contain a color triplet due to SU(3) the quantum number in the first term and an isospin doublet, again becauce of the SU(2) quantum number in the second term. The triplet and the doublet must be singlet under SU(2) and SU(3) symmetries, respectively. The colorless isospin doublet part of the Higgs in the 5 of SU(5) has the same quantum numbers as the Higgs doublet in the SM. The most general potential for the Higgs in 5 can be written as [21]

$$V(H)=-\frac{\mu_5^2}{2}H^\dagger H+\frac{\lambda}{4}\left(H^\dagger H\right)^2. \tag{2.120}$$

The symmetry breaking of $SU(3)_C\otimes SU(2)_L\otimes U(1)_Y\xrightarrow[M_W]{}S(3)_C\otimes U(1)_{EM}$ says that the color symmetry must stay unbroken. The value of the VEV at the minimum of potential V(H) is found as [21]

$$-\mu_5^2v+\lambda v^3=0\,, \tag{2.121a}$$

$$\lambda v^2-\mu_5^2=0\,, \tag{2.121b}$$

$$v^2=\frac{\mu_5^2}{\lambda}\,, \tag{2.121c}$$

and if we define $v_0=2v$ then the VEV of Higgs is

$$\langle H_5\rangle=\begin{pmatrix}0\\0\\0\\0\\v_0/\sqrt{2}\end{pmatrix}, \tag{2.122}$$

where the subscript 5 is used to the denote the Higgs in the representation. If we define a potential V that is V=V(ϕ)+V(H), we can see the VEV of the Higgs given in the equation (2.112), the color triplet of Higgs has the same mass as φ given in the equation (2.110). The reason is the combination of the color triplet of H_5 with the ϕ [21]. On the other hand, the doublet part of the Higgs in 5 has mass at the weak energy scale. It is called doublet-triplet splitting problem.

The potential V = V(ϕ, H) can be written in terms of the combination of these scalars [21]

$$V(\phi,H) = \alpha H^{\dagger} H \mathrm{Tr}\left(\phi^2\right) + \beta H^{\dagger}\phi^2 H \,, \tag{2.123}$$

which is SU(5) gauge invariant (again considering $\phi \rightarrow -\phi$). Now defining a new potential V_T as

$$V_T = V(\phi) + V(H) + V(\phi,H) \,, \tag{2.124}$$

where at the minimum of the potential occurs at the VEV of ϕ [21] as

$$\langle\phi\rangle = \mathrm{diag}\left(v, \quad v, \quad v, \quad \left(-\frac{3}{2}-\frac{\varepsilon}{2}\right)v, \quad \left(-\frac{3}{2}+\frac{\varepsilon}{2}\right)v\right) . \tag{2.125}$$

The VEV is of the form above in the equation (2.125) since SU(3) symmetry must stay unbroken and SU(2) symmetry must spontaneously broken. The term ε is put into the VEV due to considering the symmetry breaking of $SU(2)_L$. If α and β are 0, we have a boundary for ε, since it must vanish at that point. The value comes from the boundary gives a solution for ε as [21]

$$\varepsilon = \frac{3}{20}\frac{\beta v_0^2}{b v^2} + O\left(\frac{v_0^4}{v^4}\right) . \tag{2.126}$$

Since both v_0 and v are about the energy scale of SU(5) unification, it is found that the εv is much more smaller than the value of v_0. In other words, without the SSB of SU(3) the SU(2) symmetry breaking are more likely to occur due to the VEV of H_5 [21]. By using the equation (2.121c) we can write a new expression for μ considering the condition of $v_0 \ll v$ as [21]

$$\mu^2=\frac{15}{2}av^2+\frac{7}{2}bv^2+\alpha v_0^2+\frac{9}{30}\beta v_0^2. \quad (2.127)$$

Since it is expected that v is of the order M_X and v_0 is that of M_W the new expression for μ_5 that is expressed as [21]

$$\mu_5^2=\frac{1}{2}\lambda v_0^2+15\alpha v^2+\frac{9}{2}\beta v^2-3\epsilon\beta v^2 , \quad (2.128)$$

is not a desired result [21]. The above equation requires extreme fine tuning to make the right-hand side of the equation is the order of M_W^2. This is called the hierarchy problem [21].

To find a solution of the hierarchy problem at each order of the perturbation of the VEV of Higgs potential must be a fine tuning of the values α and β which seems unnatural. The fine-tuning coefficient must be in the $O(v_0^2/v^2)$ that is $\sim\frac{10^4}{10^{30}}=10^{-26}$. After SSB of SU(5) with the application of the fine tuning, the superheavy gauge boson Y gets mass by "eating" the combination of the H_5 and the fifth column of the ϕ which is ϕ_5^α. The state of Y [21]

$$\phi_5^\alpha-\frac{\sqrt{2}v_0}{5v}H^\alpha+O\left(\frac{v_0^2}{v^2}\right). \quad (2.129a)$$

and their orthogonal combination ($\phi^a H^\beta=\delta^{\alpha\beta}$) gives us a physical color triplet Higgs field [21]

$$H_3\equiv H^\alpha+\frac{\sqrt{2}v_0}{5v}\phi_5^\alpha+O\left(\frac{v_0^2}{v^2}\right) , \quad (2.129b)$$

which stays unbroken after the SSB of the SM. The mass of the color triplet Higgs m_{H_3} is given in the equation (2.129) [21] as

$$m_{H_3}=-\frac{5}{3}\beta v^2+O(v_0^2) . \quad (2.130)$$

Proton decay

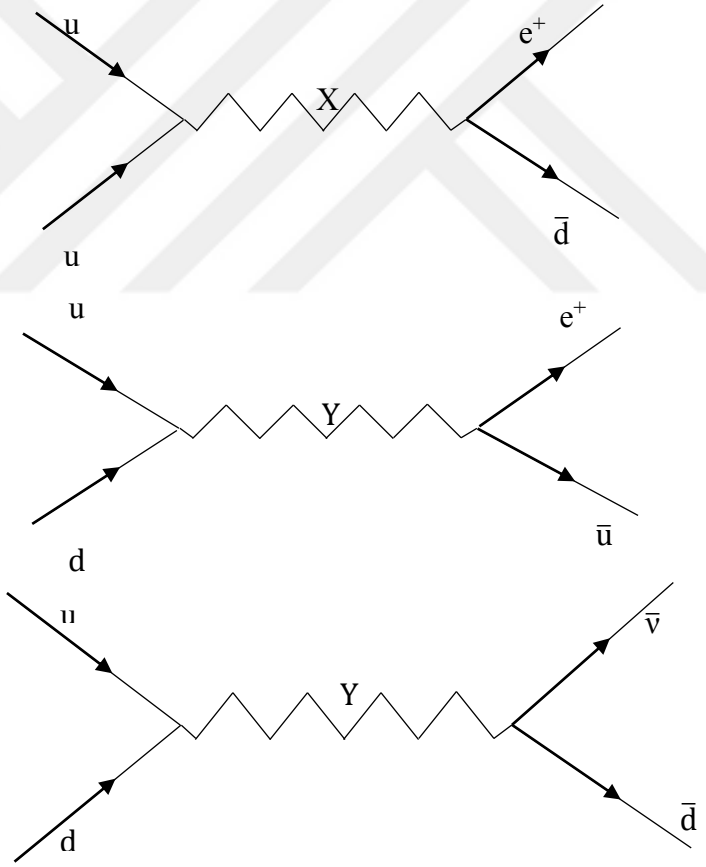
The lightest baryon which is the proton has quark content uud. According to SU(5) gauge theory the proton is allowed to decay as [19]

$$p \rightarrow \pi^0 + e^+. \tag{2.131}$$

If we look at the inner interaction between quarks, the decay is shown in the following equation as (2.132)

$$uud \rightarrow u\bar{u} + e^+. \tag{2.132}$$

The decay of the proton has not been observed yet; however Super-Kamiokande experiments have been conducted to find a lower bound for that [20]. Today the lower bound for the proton lifetime is found as 10^{34} years. To find that lower limit for the proton lifetime it is reasonable to write down the possible decays in the light of the Feynmann Rules [21].



(2.133)

According to Feynman Rules the contribution to the proton decay rate comes from SU(5) gauge bosons is 1/(mass of gauge bosons)^4=1/M_X^4. The associated Lagrangian of the decay is [21]

$$L_{\text{proton decay}}=\left[\frac{g^2}{8M_X^2}\left(\epsilon_{\alpha\beta\gamma}\bar{u}_{L\gamma}\gamma^{\mu}u_{L\beta}\right)\left(\bar{e^{+}}\gamma^{\mu}d_{\alpha}\right)+\text{h.c}\right]$$

$$+\left[\frac{g^2}{8M_Y^2}\left(\epsilon_{\alpha\beta\gamma}\overline{u_{L\gamma}^{c}}\gamma^{\mu}d_{L\beta}\right)\left(-\bar{e}_L^{+}\gamma^{\mu}u_{L\alpha}-\overline{\nu_R^{c}}\gamma^{\mu}d_{R\alpha}\right)+\text{h.c}\right], \quad (2.134)$$

$$L_{\text{proton decay}}=\frac{g^2}{2M_x}\left[\left(\epsilon_{\alpha\beta\gamma}\bar{u}^{c\gamma}_{L}\gamma^{\mu}u_L^{\beta}\right)\left(2\bar{e}_L^{+}\gamma^{\mu}d_L^{\alpha}+\bar{e}_R\gamma^{\mu}d_R^{\alpha}\right)-\left(\epsilon_{\alpha\beta\gamma}\overline{u_{\gamma L}^{c}}\gamma^{\mu}d_L^{\beta}\right)\left(\overline{\nu_R^{c}}\gamma^{\mu}d_R^{\alpha}\right)\right]+\text{h.c}\,. \quad (2.135)$$

To sum up why is the SU(5) is still valid and successful guage theory and the answers are: first, it is a minimal and simple representation in which SU(3)⊗SU(2)⊗U(1) can be embedded; second, it has a comlex representation; third, it successfully predicts Weinberg angle.

The proton lifetime is estimated as [5, 21]

$$\tau_p\propto\frac{\hbar c^2}{\alpha_{GUT}^2}\frac{M_X^4}{m_p^5}. \quad (2.136)$$

where m_p is the mass of the proton. The equation (2.136) for the proton lifetime when the hadronic matrix element taken into account the following,

$$\tau_p=4.883\times10^{32}\frac{(M_X/10^{16}\text{GeV})^4}{\alpha_{GUT}^2}\text{ years}. \quad (2.137)$$

Neutrino mass

At the beginning of the study it is indicated that in SM right-handed neutrinos do not exist but right-handed antineutrinos exist. Therefore, mass term for neutrinos cannot be written unless the additional higher dimensional operators are 5 at least. In GG model, neutrinos are also stay massless, however they can be massive by adding a right-handed neutrino as a singlet (1,1,0). Even though this is mathematically correct, still is not satisfying to explain the small mass of neutrinos naturally. Explaining massive

neutrinos, naturally actually means that the mass term emerge by itself from the gauge invariant Lagrangian of the SU(5) model. In some studies that can be achived [7], however using non-renormalizable terms is not desired in our model.

Several models or mechanisms are developed to generate massive neutrinos [23-24] such as Zee model [25], radiative neutrino mass models [26-28], neutrino masses from one-loop level parametrization [29-31,], neutrino masses from extra dimensions [32], Majorana neutrino masses [33-34], and the seesaw mechanism [35-37]. As it is explained at in the section 1.2 of the thesis, we will be using the seesaw mechanism to generate massive neutrinos. Since our aim is to find out small masses for neutrinos at tree level, neutrino masses coming from interactions at loop-levels (one-loop and two-loops) are not desired.

Understanding of the massive neutrinos is a good point to start. After the discovery of neutrino oscillation [38], it is understood that at least two neutrinos must have mass different than zero. There are three flavors of neutrinos which are ν_e, ν_μ and ν_τ and in the Sun only the electron neutrino ν_e is created; however fewer of them are detected in the neutrino capture experiments in the deep mines [39-40]. This discovery tells us existence of the right-handed neutrinos. The theory of the SM which says there is no right-handed neutrino is not true anymore.

About neutrinos it can be listed that

1) Masses of neutrinos are very small (m_ν<1.1 eV).
2) Parameters required to construct a neutrino mixing matrice: the 3 mass eigenstates + 3 angles + 1 phase parameter.
3) The origin of mass and mixing of neutrinos is still unknown.

The objective in the seesaw mechanism is to generate a small neutrino mass at tree-level by the addition of minimal number of fields that are not exist in the SM. Neutrino oscillation experiments agree on the massive neutrinos; however, in these experiments squared mass differences of neutrinos are measured. Still, we do not know a single neutrino mass since we cannot measure it. According to experimental results we can group the mass hierarchies of neutrinos into two as normal hierarchy and inverted hierarchy demonstrated at in the Figure-2.2.

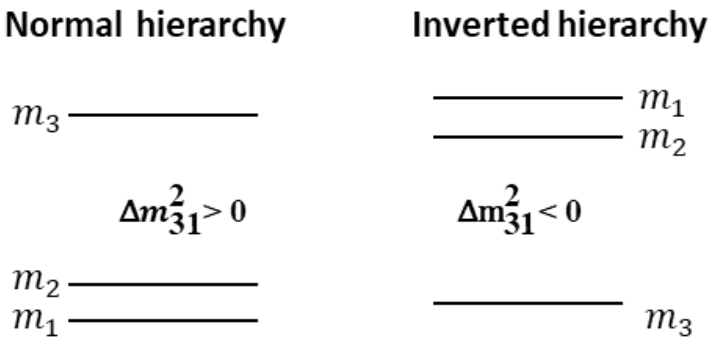


Figure 2.2 Neutrino mass hierarchies

The interesting point about explaning neutrino masses is introducing a right-handed neutrino can explain the massless neutrino since we are assuming at least two of neutrinos have small mass. However, introducing a right-handed neutrino is not enough to determine the smallness of the neutrino mass. To explain the small mass of neutrinos at tree level, or say naturally, seesaw mechanism is needed.

Type-1 seesaw

In order to write a mass term for neutrinos we must introduce a righ-handed neutrino field ν_R to the SM as a singlet. In SM there are three flavors of left-handed neutrinos, therefore the right-handed neutrino field is also containing all three flavors as $(\nu_e)_R$, $(\nu_\mu)_R$, and $(\nu_\tau)_R$. The Yukawa term for neutrino can be written in the SM as following [41-44].

$$L=-\overline{\nu_R} Y_\nu \phi^\dagger l_L + h.c. \qquad (2.138)$$

The diagram for Type-1 seesaw mechanism is shown in the Figure 2.3.

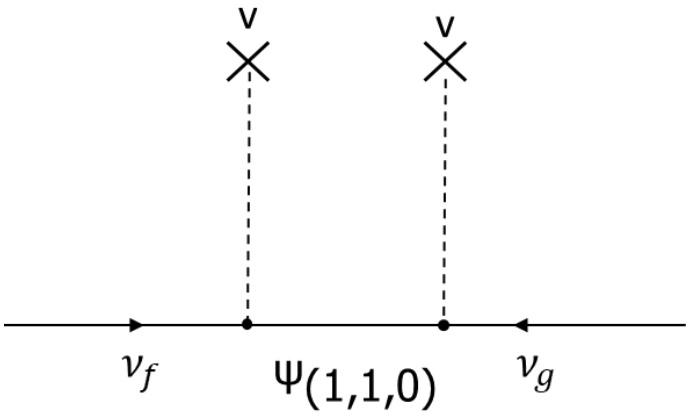


Figure 2.3 The diagram of Type-1 seesaw

In the equation (2.138) the term ϕ is the Higgs scalar which acquires VEV, explained in the previous chapter, $\langle\phi\rangle = \frac{1}{\sqrt{2}}\binom{0}{v}$. Like mass terms of quarks and charged leptons, Dirac mass term can also be written for neutrino as

$$L=-\overline{\nu_R}M_D\nu_L+h.c. \tag{2.139}$$

where $M_D=\frac{Y_\nu v}{2}$. From cosmology [45] there is an upper bound for neutrinos $m_\nu < 1eV$ and the v in the Higgs VEV is $v\simeq 246$ GeV, then we can do the following calculations to find out an upper bound for the Yukawa coupling Y_ν.

$$Y_\nu \frac{246\ \text{GeV}}{\sqrt{2}} < 1\times10^{-9}\ \text{GeV}$$

(2.140a)

$$Y_\nu \lesssim 6\times10^{-12} \qquad \text{GeV}$$

(2.140b)

The result found in the equation (2.140b) does not seem natural compared to a natural constant O(1), it is very small.

In SM the hypercharge quantum value of the left-handed neutrino is -1/2. If we write down a Majorana term for the left-handed field as $\overline{(\nu_L)^c}\nu_L$, the hypercharge value will be violated. Moreover, it is not invariant under SM gauge transformation. However, we can also think about the similar term for the right-handed neutrino which is singlet

under SM gauge transformation, and it does not carry hypercharge quantum number. The accociated Lagrangian is then [41-42]

$$L=-\frac{M_R}{2}\overline{(\nu_R)^c}\nu_R+h.c. \qquad (2.141)$$

where the superscript c is the charge conjugation operator defined as $(\nu_{L,R})^c=(\nu^c)_{R,L}$. Neutrino oscillations on the other hand, cannot be explained by the term in the eqaution (2.141) since right-handed neutrino does not transform under SM gauge transformation. We can combine the Dirac term and Majorana term to form a new Lagrangian for neutrino masses as [41- 42,46]

$$L=-\overline{\nu_R}M_D\nu_L-\frac{M_R}{2}\overline{(\nu_R)^c}\nu_R+h.c.. \qquad (2.142a)$$

Equation (2.142a) can be written in the matrix form as follows [41,46]

$$L=[\bar{\nu}_L \quad \overline{(\nu_R)^c}\,]\begin{bmatrix}0 & M_D\\ M_D^T & M_R\end{bmatrix}\begin{bmatrix}(\nu_L)^c\\ \nu_R\end{bmatrix} \quad . \qquad (2.142b)$$

Defining the matrix M as [41]

$$M=\begin{bmatrix}0 & M_D\\ M_D^T & M_R\end{bmatrix}, \qquad (2.143)$$

we can diagonalize the matrix M as

$$D_M=U^TMU \quad , \qquad (2.144)$$

where U is a unitary matrix defined as

$$U\simeq\begin{bmatrix}1 & M_DM_R^{-1}\\ -M_R^{-1}M_D^T & 1\end{bmatrix}. \qquad (2.145)$$

Now we can define light neutrino as ν and N respectively. Then the primed of the fields can be defined in the matrix form as follows [41,46].

$$\begin{bmatrix} \nu' \\ N' \end{bmatrix} \equiv U^{\dagger} \begin{bmatrix} (\nu_L)^c \\ \nu_R \end{bmatrix} \tag{2.146}$$

If the mass of the N is greater than $10^{14\text{-}15}$ GeV then the light neutrino mass will be smaller than 1 eV [41] with O(1) Yukawa coupling. This mechanism is called the Type-1 seesaw mechanism [7-8,43]. There are also Type-2 [5] and Type-3 [7-8,47] seesaw mechanisms which extend the SM particle content by scalar triplet and fermionic triplet respectively.

Type-2 seesaw

Instead of adding right-handed neutrino to write down a neutrino mass element into the Lagrangian scalar SU(2) triplet Δ with quantum numbers (1, 3, 1) can be used to give mass to neutrino. This mechanism is called Type-2 seesaw mechanism. The potential is [5]

$$V_{seesaw} = -M_{\Delta}^2 Tr\Delta^{\dagger}\Delta - Y_{\nu} l^T C\Delta l + c_3 \psi_D^T \Delta^{\dagger} \psi_D + h.c. \tag{2.147}$$

where

$$\Delta = \begin{pmatrix} \Delta^0 & -\frac{\Delta^+}{\sqrt{2}} \\ -\frac{\Delta^+}{\sqrt{2}} & \Delta^{++} \end{pmatrix}, \qquad l = \begin{pmatrix} \nu \\ e \end{pmatrix}, \qquad \psi_D^T = (H^0 \quad H^+) \tag{2.148}$$

The diagram for neutrino is shown in the Figure 2.4 below.

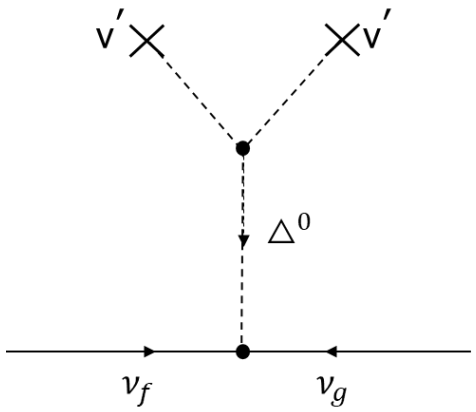


Figure 2.4 Diagram of Type-2 seesaw

The neutrino mass is [5]

$$M_\nu \approx Y_\nu c_3 \frac{\langle H^0 \rangle^2}{M_\Delta^2} \quad (2.149)$$

Type-3 seesaw

Another way to having massive neutrinos at tree level computation is to adding extra colorless fermionic SU(2) triplet Σ carrying zero hypercharge quantum number. This mechanism is called Type-3 seesaw mechanism with Lagrangian [7-8,47]

$$L = L_i y_\nu^i \Sigma H \quad (2.150)$$

The diagram of the Type-3 seesaw mechanism is shown in the Figure 2.5.

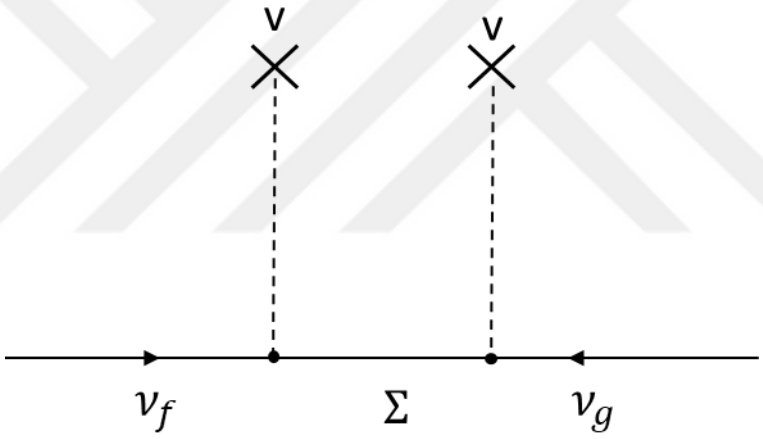


Figure 2.5 Diagram of Type-3 seesaw

In the Lagrangian above the second term is the Majorana mass for neutrino and the third term is the Yukawa coupling. The fermionic triplet Σ can be written in a matrix form as

$$\Sigma = \frac{1}{\sqrt{2}} \sigma_i \Sigma_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \Sigma_3 & \Sigma_1 - i\Sigma_2 \\ \Sigma_1 + \Sigma_2 & -\Sigma_3 \end{pmatrix}. \quad (2.152)$$

Then neutrino mass matrix becomes

$$m_\nu \cong \frac{Y_\Sigma^2 v^2}{M_\Sigma} \tag{2.153}$$

where v is the VEV of Higgs in the SM.

Radiative neutrino masses at one-loop level

The seesaw mechanism is constructed to give mass to neutrino at tree level. It is possible to have massive neutrinos by calculating interactions at loop level by adding extra colorful split multiplet. This mechanism is named as radiative neutrino models [48] at one-loop level shown in the Figure - 2.6.

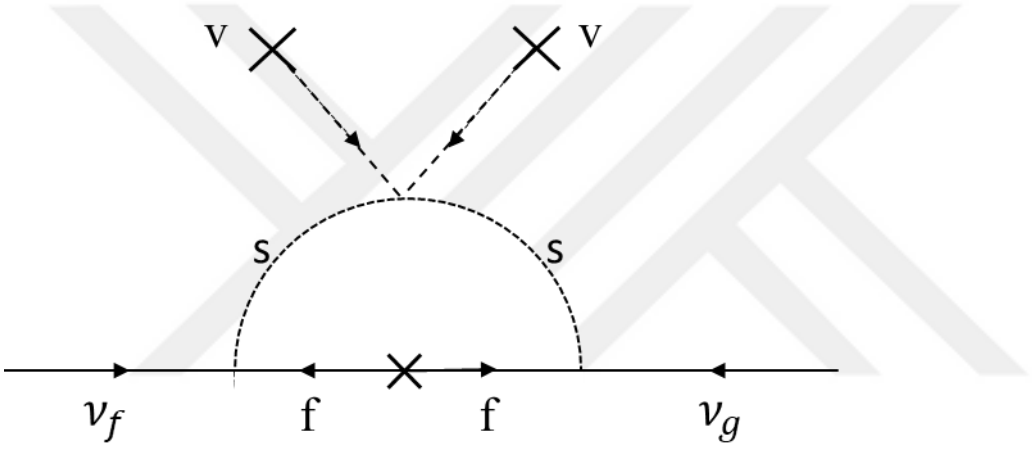


Figure 2.6 One-loop neutrino mass diagram

To avoid anomalies the colorful SU(3) multiplet must lie in its real representation. The minimal real representation (minimal is desired to explain simplicity) is its adjoint 8. We can generate neutrino masses by adding scalar octet [48-50] (8,2,1/2), s_1, and fermionic octet (8,1,0), f_1. Adding two scalar field requires one fermionic field and vice versa. The relevant Lagrangian to generate massive neutrinos is [48]

$$-L_1 = Y_1 l^T C i\sigma_2 s_1 f_1 + M_{f_1} Tr\left(f_1^T C f_1\right) + \lambda_1 Tr\left(s_1^\dagger H\right)^2 + h.c.\,. \tag{2.154}$$

Radiative neutrino mass is also generated at one-loop level by adding fermionic (8,3,0), f_2 instead of fermionic SU(2) singlet in the previous case. Again, adding two (one)

scalar (8,2,1/2) with one (two) fermionic (8,3,0) is necessary. The Lagrangian due to one-loop level interaction is [48]

$$-L_2=Y_2 l^T C i\sigma_2 f_2 s_1+M_{f_2}\mathrm{Tr}\left(f_2^T C f_2\right)+\lambda_2\mathrm{Tr}\left(S_1^\dagger H\right)^2+\text{h.c.}\,. \qquad (2.155)$$

Neutrino Mixing

The CKM matrix are developed to explain the diagonalization inconsistency between the charged leptons and quarks mass eigenstates and flavor eigenstates. This inconsistency is also seen in the mass eigenstates and flavor eigenstates of neutrinos. Like CKM matrix for charged leptons and quaks, PMNS matrix is generated to explain the similar condition for neutrinos as follows [41,46],

$$\begin{bmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{bmatrix}=V_{PMNS}\begin{bmatrix}\nu_1\\ \nu_2\\ \nu_3\end{bmatrix} \qquad (2.156)$$

where the fields $\nu_e, \nu_\mu,$ and ν_τ represents the flavor eigenstates and the fields $\nu_1, \nu_2,$ and ν_3 are considered as mass eigenstates. Since PMNS matrix V is a unitary matrix, it satisfies $V^\dagger V=VV^\dagger=1$. Due to the fact that there are three flavours of neutrinos the mixing matrix PMNS is expected to be 3-by-3 matrix as

$$V_{PMNS}=\begin{bmatrix}V_{e1} & V_{2e} & V_{e3}\\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3}\\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3}\end{bmatrix} \qquad (2.157)$$

Similar to the CKM matrix, the PMNS matrix can be parametrized by 3 angles and 6 phases. However, the crucial point is that if the neutrino masses are written in the Dirac mass, it is posible to reparametrize charged lepton. Therefore, the PMNS matrix must transform under phase transformation as [41,46]

$$V_{\alpha_i}\rightarrow e^{i(\phi_\alpha-\phi_i)}V_{\alpha i} \qquad (2.158)$$

where $\alpha = e, \mu, \tau$ and i=1, 2, 3. Based on the product of rotations $\theta_{12}, \theta_{13}, \theta_{23}$ we can write the V_{PMNS} as [41,46]

$$V_{PMNS}=\begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23}-c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23} \end{bmatrix} P \quad (2.159)$$

where $c_{ij}\equiv\cos\theta_{ij}$, $s_{ij}\equiv\sin\theta_{ij}$ and δ_{CP} is generally called as Dirac phase. In the Majorana case P is diagonal while it is the unit matrix in the Dirac case. For Majorana particle, P can be written as [41,46]

$$P_{M1}=\begin{bmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$P_{M2}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i(\phi_3+\delta_{CP})} \end{bmatrix},$$

$$P_{M3}=\begin{bmatrix} e^{i\rho} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\sigma} \end{bmatrix}. \quad (2.160)$$

Charged lepton and quark mass ratios

At the beginning of the section 2.2 it is mentioned that fermions are placed in the $\bar{5}$ and 10 representations. They can be represented as Dirac spinors ψ_{La} and $\psi_L^{[ab]}$ respectively where a, b = 1, 2, 3, 4, 5. The demonstration of superscript [ab] indicates that the indices in the paranthesis are antisymmetric. Fermions in the SM acquire mass via Higgs in the 5 of SU(5) getting VEV due to the SSB. After Higgs getting VEV, it couples with fermions and make them massive (before the SSB all fermions are massless). The coupling of Higgs with these fermions are represented by the Yukawa terms in the Lagrangian. The fermionic terms in the $\bar{5}$ and 10 representations can be written as following.

$$\psi_{La}^T C\psi_{Lb}\equiv\bar{5}\otimes\bar{5}\,, \quad (2.161a)$$

$$\psi_{La}^T C\psi_L^{bc}=\bar{5}\otimes 10\,, \quad (2.161b)$$

$$\psi_L^{Tab} C \psi_L^{cd} = 10 \otimes 10 \,. \tag{2.161c}$$

where, C denotes the charge conjugation operator which satisfies

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T. \tag{2.162}$$

Since we want to write a mass term which must be SU(2) gauge invariant, the charge conjugation operator is needed. Associated mass terms with two left-handed fermionic fields ψ_L and X_L as

$$\psi_L^T C X_L + h.c = \psi_L^T \left(-X_L^T C\right) + h.c. = -X_L^T \psi_L^T C + h.c. = X_L^T C \psi_L + h.c. = \psi_R^C X_L + h.c. \tag{2.163}$$

We can represent the Higgs in the fundamental representation as H^b. The SU(5) gauge invariant Yukawa Lagrangian [18] is

$$L_Y = g_{1mn} \psi_{Lma}^T C \psi_{Ln}^{ab} \overline{H}_b + \epsilon_{abcde} g_{2mn} \psi_{Lm}^{Tab} C \psi_{Ln}^{cd} H^e + h.c. \tag{2.164}$$

where m and n are family indices m, n=1, 2, 3 and ϵ_{abcde} is SU(5) invariant antisymmetric tensor. $\overline{H}_b$ is the complex field of the Higgs in 5 that acquieres VEV [18]

$$\langle H^e \rangle = \frac{v_0}{\sqrt{2}} \delta^{e5}. \tag{2.165}$$

Also

$$10 \otimes 10 \otimes H \rightarrow \text{mass term of u quark} \tag{2.166a}$$

and

$$\overline{5} \otimes 10 \otimes \overline{H} \rightarrow \text{mass terms of d quark and electron} \tag{2.167b}$$

For a=1, 2, 3 the first term in the equation (2.164) is the mass term for d quarks

$$g_{1mn} \psi_{Lm1}^T C \psi_{Ln}^{15} \langle \overline{H}_5 \rangle = g_{1mn} \psi_{Lm2}^T C \psi_{Ln}^{25} \langle \overline{H}_5 \rangle = g_{1mn} \psi_{Lm3}^T C \psi_{Ln}^{35} \langle \overline{H}_5 \rangle \tag{2.168}$$

and for a=4 it is the mass term for electron,

$$g_{1mn} \psi_{Lm4}^T C \psi_{Ln}^{45} \langle \overline{H}_5 \rangle \,. \tag{2.169}$$

The second term gives masses for u quarks. From equations (2.168) and (2.169) it is seen that the mass of electron is the same as the mass of d quark. And if we repeat that calculation for other generations, we found that

$$m_e = m_d,\ m_\mu = m_s, m_\tau = m_b \tag{2.170}$$

The equation (2.170) is valid for the energy scale greater than the GUT scale in SU(5) gauge theory where the symmetry breaking occurs due to the VEV of Higgs in the 5 of SU(5). However, if the energy scale is around the weak scale, M_W , this equality becomes [21]

$$\frac{m_\tau}{m_b} \cong 3\ . \tag{2.171a}$$

On the other hand

$$\frac{m_e}{m_d} = \frac{m_\mu}{m_s} \tag{2.171b}$$

is very interesting result. The equation (2.171b) is not consistent due to the fact that the mass ratios of the leptons m_e/m_μ is about the 1/9 times of the mass ratio of quarks m_d/m_S. It is possible to fix the unproper mass relation in the equation (2.171b). While doing this, the mass relations of the third generation down-type quark, b-quark, and that of lepton tau at the unification scale will be untouched.

The Higgs in the 45 representation must satisfy the antisymmetry and the trace of 45 as shown below respectively.

$$H_c^{ab} = -H_c^{ba} \tag{2.172a}$$

and

$$H_a^{ab} = 0 \tag{2.172b}$$

The Yukawa Lagrangian including 45-plet Higgs is [21]

$$L_{Y(45)} = e_{1mn}\psi^T_{Lam} C \psi^{bc}_{Ln} \bar{H}^a_{bc} + f_{1mn}\epsilon_{abcef}\psi^{Tab}_{Lm} C \psi^{cd}_{Ln} H^{ef}_d + h.c. \tag{2.173}$$

Taking a=b and do summation over c gives us the 5. The remaining 50-5=45 elements construct the 45 which satisfy equations (2.172a) and (2.172b). While a=b, if we take c=5, then it gives a colorless neutral element with summation over a=1, 2, 3, 4 [21]. Since the color symmetry must stay unbroken and after the SSB the remaining symmetries must be $SU(3)_C \otimes U(1)_{em}$, when a takes color indices a=1, 2, 3 the VEV of 45 Higgs conserving the $SU(3)_C$ symmetry is [18]

$$\langle 45_a^{bc} \rangle = v_{45}\left(\delta_a^b - 4\delta_{a4}\delta^{b4}\right)\delta^{c5}. \tag{2.174}$$

Rewriting the Lagrangian in the eqaution (2.173) including the VEV of Higgs given in the equation (2.174) gives [21]

$$L_Y = e_{1mn}\psi_{Lam}^T C\psi_{Ln}^{bc} v_{45}\left(\delta_b^a - 4\delta^{a4}\delta_{b4}\right)\delta_{c5}. \tag{2.175}$$

The above equation gives for a=1, 2, 3

$$e_{1mn}\psi_{L1m}^T C\psi_{Ln}^{15} v_{45}\left(\delta_1^1 - 4\delta^{14}\delta_{14}\right)\delta_{55} + e_{1mn}\psi_{L2m}^T C\psi_{Ln}^{25} v_{45}\left(\delta_2^2 - 4\delta^{24}\delta_{24}\right)\delta_{55}$$
$$+ e_{1mn}\psi_{L3m}^T C\psi_{Ln}^{35} v_{45}\left(\delta_3^3 - 4\delta^{34}\delta_{34}\right)\delta_{55} \tag{2.176a}$$

and for a =4 [21]

$$e_{1mn}\psi_{L4m}^T C\psi_{Ln}^{45} v_{45}\left(\delta_4^4 - 4\delta^{44}\delta_{44}\right)\delta_{55}. \tag{2.176b}$$

Equation (2.176a) is mass term of d quark and (2.176b) is of the electron with masses $m_d = e_1 v_{45}$ and $m_e = -3e_1 v_{45}$. Since the representation 45 must satisfy [21]

$$\sum_{a=1}^{4} H_a^{a5} = 0 , \tag{2.177}$$

then it is found that

$$m_e = 3m_d , \tag{2.178a}$$

$$m_\mu = 3m_s , \tag{2.178b}$$

$$m_\tau = 3m_b . \tag{2.178c}$$

Adding 45-plet Higgs scalar saves the bad mass ratios in equation (2.171b). From equation (2.178c), it is seen that the mass ratio of the tau-lepton and b quark is getting worse. That means that acquiring a VEV of Higgs in the 45 representation makes better charged lepton and quark mass ratios for the first two generation while spoiling the desired mass ratio of the third-generation lepton and quark. It seems that VEV of Higgs in the 45 is good but not the best solution for charged lepton and quark masses.

It is said that there is another solution to get a desired mass ratios of charged leptons and quarks masses by adding a new interaction, gravity, with combination of SU(5). That is called a "truly unified theory" [21]. Since in this study we are focused on just GUTs, we are not interested with that solution.

In our model both Higgses in 5 and 45 representation acquires vev that makes our model a hybrid model since the VEV of the Higgs in the 45 is required not only to fix the charged lepton and quarks mass ratios but also to have massive neutrinos. The charged lepton and quarks mass ratios are as following.

$$m_b=m_\tau=e_{1mn_{33}}v_5 \tag{2.179a}$$

$$m_s=m_\tau=f_{1mn_{22}}v_{45} \tag{2.179b}$$

$$m_\mu=3f_{1mn_{22}}v_{45} \tag{2.179c}$$

$$m_e=\left(e_{1mn_{12}}v_5\right)^2/m_\mu \tag{2.179d}$$

$$m_d=\left(f_{1mn_{12}}v_5\right)^2/m_s \tag{2.179e}$$

By choosing e_{1mn} and f_{1mn} in a particular way the masses of charged leptons and quarks can be adjusted to fit the experimental values.

75_F and 45_S

There are many representations studied to find out viable GUT model that unify $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ gauge theories embedded as a subgroup of SU(5) as discussed in the section 1.1. The first minimal SU(5) GUT model constructed with particle content containing only SM particles known as GG model [1]. After GG model, other representations of SU(5) are added to this minimal model to construct a viable model

since the GG model does not satisfy experimental bounds and new discoveries such as small mass of neutrino. To review the additional representations from the smallest one to the larger one respectively there are the 10 [4,11], 15 [5,13-14], 24 [7,8], 35[13-14], 40[12], 45[8-9], 50[12] representations have been studied as explained in the literature review. As it is mentioned in the study [15], 75 representation of SU(5) could be count as a small representation which has not been studied. Therefore, in this thesis our aim is to find unification with additional fermionic content of 75_F and a scalar content of 45_S to the GG model.

Beginning with the construction of 75_F representation, 5 and 45 representations are needed. Their direct product gives

$$5 \otimes 45 = 24 \oplus 75 \oplus 126 \tag{2.180}$$

The representation of 75 can be written as $75^{[i,j]}_{[k,l]}$, wthere both i, j and k, l are antisymmetric indices taking values as i, j, k, l = 1, 2, 3, 4, 5. Since 75 is a real and traceless matrix, it satisfies

$$\sum_{i=1}^{5} 75^{[i,j]}_{[i,l]} = 0 \, , \tag{2.181}$$

which also satisfies

$$75^{i,j}_{k,l} = -75^{j,i}_{k,l} = -75^{i,j}_{l,k} = 75^{j,i}_{l,k} . \tag{2.182}$$

The general Lagrangian by including the 75_F and 45_s will be in the form of

$$L \equiv \text{Kinetic terms} + \text{Scalar potential} + \text{Yukawa couplings} + \text{Mass terms} \, . \tag{2.183}$$

The Lagrangian written above must be

- Lorentz invariant,
- SU(5) invariant,
- renormalizable (the dimension of the coupling constants must be non-negative),

as Lagrangian for the minimal SU(5).

The desired mass terms of the 75_F can be written as

$$-\frac{M_{75}}{8}\left(\psi_{75}\right)^{[i,j]}_{[k,l]}\left(\psi_{75}\right)^{[k,l]}_{[i,j]}-\frac{\lambda}{8}\left(\psi_{75}\right)^{[i,j]}_{[k,l]}\left(\psi_{75}\right)^{[k,m]}_{[i,j]}\left(H_{24}\right)^{l}_{m} \qquad (2.184)$$

that satisfies the three conditions written above. λ is a real constant. Since the VEV of Higgs in the adjoint lies in the diagonal as

$$\langle H_{24}\rangle=\frac{\lambda v}{\sqrt{30}}\,\mathrm{diag}(2,2,2,-3,-3) \qquad (2.185)$$

mass terms in the equation (2.184) can be written as

$$-\frac{M_{75}}{8}\left(\psi_{75}\right)^{[i,j]}_{[k,l]}\left(\psi_{75}\right)^{[k,l]}_{[i,j]}-\frac{\lambda}{8}a^{l}\left(\psi_{75}\right)^{[i,j]}_{[k,l]}\left(\psi_{75}\right)^{[k,l]}_{[i,j]} \qquad (2.186)$$

where a comes from the eqaution (2.113) as a value depending on the associated indices. For example, for l=1, 2, 3, a is 2 multiplied by the VEV coefficient of the diagonal, and for i=4, 5, a= -3 times the VEV coefficient. We use the matrix form of SU(5) wthere the indices i, j, k, l, m, n, p, r, s=1,2,3 for $SU(3)_C$ color symmetry and i, j, k, l, m, n, p, r, s=4,5 for $SU(2)_L$ weak isospin symmetry. By considering these transformations we can write down split-multiplets of 75_F and their mass terms as

$$75_F \supset (1,1,0)\oplus\left(3,1,-\frac{5}{3}\right)\oplus\left(\bar{3},1,\frac{5}{3}\right)\oplus\left(3,2,-\frac{5}{6}\right)\oplus\left(\bar{3},2,\frac{5}{6}\right)\oplus\left(6,2,\frac{5}{6}\right)$$
$$\oplus\left(\bar{6},2,-\frac{5}{6}\right)\oplus(8,3,0)\oplus(8,1,0) \qquad .$$
(2.187)

Now it is useful to find out contributions of these split multiplets to beta functions to analyze unification. Since these multiplets are fermions belonging to a real representation the associated formula is [18]

$$\Delta b_i=\frac{1}{2}\frac{8}{3}\frac{1}{2}C(r)_i d_i=\frac{2}{3}C(r)_i d_i\,, \qquad (2.188)$$

where i=1, 2, 3 for $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ gauge theories respectively.In our model we will have 2 of 75_F to make at least two neutrinos massive. Therefore, we need to multiply the formula written in the equation (2.188) with 2. The contributions of the split multiplets of 75_F are given in the Tabşe-2.5.

Table 2.5 Beta Functions Contributions of Split Multiplets of Fermions in 75 of SU(5)

75_F	**(1,1,0)**	$\left(3,1,-\frac{5}{3}\right)+\left(\bar{3},1,\frac{5}{3}\right)$	$\left(3,2,-\frac{5}{6}\right)+\left(\bar{3},2,\frac{5}{6}\right)$	$\left(6,2,\frac{5}{6}\right)+\left(\bar{6},2,-\frac{5}{6}\right)$	**(8,3,0)**	**(8,1,0)**
Δb_1	0	20/3	10/3	20/3	0	0
Δb_2	0	0	2	4	32/3	0
Δb_3	0	2/3	4/3	20/3	6	2
$\Delta b_1-\Delta b_2$	0	20/3	4/3	8/3	-32/3	0
$\Delta b_2-\Delta b_3$	0	-2/3	2/3	-8/3	14/3	-2

The corresponding mass relations of these split multiplets of 75_F are

$$m_{(3,1)}=m_{(8,3)}\ , \tag{2.189a}$$

$$m_{(6,2)}=(m_{(8,3)}+3m_{(8,1)})/4\ , \tag{2.189b}$$

$$m_{(\bar{3},2)}=(5m_{(8,3)}-m_{(8,1)})/4\ , \tag{2.189c}$$

$$m_{(1,1)}=(3m_{(8,3)}-m_{(8,1)})/2\ . \tag{2.189d}$$

75_F generates our additional particle model to the minimal SU(5) model. In this study there will also be extra scalar content 45_S where $45_c^{ab}=-45_c^{ba}$ and $45_b^{ab}=0$. These particle selections are done to first by adding extra 45 scalar we want a better charged lepton and quark mass relation than compared to the GG model [1] and second, we want to give small mass to neutrino which is massless in the GG model [1]. The VEV of Higgs in the 45 satisfies

$$\langle\phi_1^{15}\rangle_0=\langle\phi_2^{25}\rangle_0=\langle\phi_3^{35}\rangle_0=-\tfrac{1}{3}\langle\phi_4^{45}\rangle_0=v_0\ , \tag{2.190}$$

and the VEV of 24 is $\langle H\rangle_0=\text{diag}\left[v,v,v,-\frac{3v}{2},-\frac{3v}{2}\right]$.

Split multiplets of 45 are

$$45_S\supset\left(1,2,\tfrac{1}{2}\right)\oplus\left(3,3,-\tfrac{1}{3}\right)\oplus\left(\bar{3},1,\tfrac{4}{3}\right)\oplus\left(3,1,-\tfrac{1}{3}\right)\oplus\left(\bar{3},2,-\tfrac{7}{6}\right)\oplus\left(\bar{6},1,-\tfrac{1}{3}\right)\oplus\left(8,2,\tfrac{1}{2}\right). \tag{2.191}$$

The potential of 45_S represented as ϕ_c^{ab} and the Σ_b^a ,the adjoint, is [50]

$$V=\alpha\left(\phi_c^{ab}\phi_{ab}^c\right)\left(\Sigma_f^e\Sigma_e^f\right)+\beta_1\phi_c^{ab}\Sigma_a^d\phi_{de}^c\Sigma_b^e+\beta_2\phi_c^{ab}\Sigma_b^c\phi_{ae}^d\Sigma_d^e+\beta_3\phi_c^{ab}\Sigma_d^c\phi_{ae}^d\Sigma_b^e+\beta_4\phi_c^{ab}\Sigma_a^d\phi_{eb}^c\Sigma_d^e+\beta_5\phi_c^{ab}\Sigma_d^c\phi_{ab}^e\Sigma_e^d \quad (2.192)$$

The eqaution (2.192) show us that there are fours leptoquarks in 45_s which are: $\left(3,3,-\frac{1}{3}\right),\left(\bar{3},1,\frac{4}{3}\right),\left(3,1,-\frac{1}{3}\right),\left(\bar{3},2,-\frac{7}{6}\right)$. These leptoquaks' masses play an important role on setting limitations on masses of the other split multiplets due to the proton lifetime bound. There is also second Higgs doublet in 45_S which might also get limitations on its mass due to the experimental results since it is known that there is only one physical Higgs in the SM. Split multiplets of 45_S have contributions calculated as [18]

$$\Delta b_i=\frac{1}{3}C(r)_i d_i\,, \quad (2.193)$$

where the indice i=1, 2, 3 for $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ gauge theories respectively. Corresponding contributions calculated from the equation (2.193) for a single 45_S are given in the Table 2.6.

Table 2.6 Beta Functions Contributions of Split Multipletes of Scalar in 45 of SU(5)

45_S	$\left(1,2,\frac{1}{2}\right)$	$\left(3,3,-\frac{1}{3}\right)$	$\left(\bar{3},1,\frac{4}{3}\right)$	$\left(3,1,-\frac{1}{3}\right)$	$\left(\bar{3},2,-\frac{7}{6}\right)$	$\left(\bar{6},1,-\frac{1}{3}\right)$	$\left(8,2,\frac{1}{2}\right)$
Δb_1	1/10	1/5	16/15	1/15	49/30	2/15	4/5
Δb_2	1/6	2	0	0	1/2	0	4/3
Δb_3	0	1/2	1/6	1/6	1/3	5/6	2
$\Delta b_1-\Delta b_2$	-1/15	-9/5	16/15	1/15	17/15	2/15	-8/15
$\Delta b_2-\Delta b_3$	1/6	3/2	-1/6	-1/6	1/6	-5/6	-2/3

By using the mass terms of 45_S given in the equation (2.192) we can write down mass relations of split multiplets as follows [50].

$$m_{(1,2,1/2)}=-\frac{5v^2}{16}(6\beta_1+\beta_2+4\beta_3+\beta_4+2\beta_5)\,, \quad (2.194a)$$

$$m_{(8,2,1/2)}=-\frac{15v^2}{16}(6\beta_1+5\beta_2+4\beta_3+\beta_4+2\beta_5)\,, \quad (2.194b)$$

$$m_{(\bar{6},1,-1/3)}=-\frac{5v^2}{16}(2\beta_1+15\beta_2+4\beta_3+7\beta_4+6\beta_5) \quad ,$$

(2.194c)

$$m_{(3,3,-1/3)}=-\frac{5v^2}{16}(18\beta_1+15\beta_2+8\beta_3+3\beta_4-2\beta_5) \quad ,$$

(2.194d)

$$m_{(\bar{3},2,-7/6)}=-\frac{5v^2}{16}(2\beta_1+15\beta_2+20\beta_3+7\beta_4-2\beta_5) \quad ,$$

(2.194e)

$$m_{(3,1,-1/3)}=-\frac{5v^2}{16}(-6\beta_1+15\beta_2+20\beta_3-\beta_4+6\beta_5) \quad ,$$

(2.194f)

$$m_{(\bar{3},1,4/3)}=-\frac{5v^2}{16}(10-5\beta_2+6\beta_3+5\beta_4+2\beta_5 \quad .$$

(2.194g)

Then we can write down the two of the split multiplets $m_{(1,2,1/2)}$ and $m_{(8,2,1/2)}$ in terms of others [50].

$$m^2_{(1,2,1/2)}=\frac{1}{5}(m^2_{(3,3,-1/3)}-m^2_{(\bar{3},2,-7/6)}+m^2_{(3,1,-1/3)}+2m^2_{(\bar{3},1,4/3)}) \quad ,$$

(2.195a)

$$m^2_{(8,2,1/2)}=\frac{3}{10}(m^2_{(\bar{6},1,-1/3)}+3m^2_{(3,3,-1/3)}-2m^2_{(\bar{3},2,-7/6)}+2m^2_{(3,1,-1/3)}+2m^2_{(\bar{3},1,4/3)}$$

.(2.195b)

In this model the fermionic and scalar content can be summarized as in the Table 2.3.3.

Table 2.7 Particle and Scalar Content of the Proposed Model

Particles and Antiparticles		**Scalars**		**Total**
Standard Model	75_F	Standard Model	45_S	Standard Model and additional fields
$\bar{5}_F,10_F,5_F,\overline{10}_F$	75_F, 75_F	5_H,24_H	45_S	$\bar{5}_F,10_F,5_F,\overline{10}_F,75_F,75_F,5_H,24_H$ 45_S

3. RESULTS AND DISCUSSION

Since our first aim is to add 45 scalar field to the GG model [1], first, we need to consider contributions to beta functions of its split multiplets. The analysis of unification can be tested according to B-test which is the ratio of subtraction of beta functions which are defined as [5,13-14]

$$B_i=b_i+\sum_I b_{iI}r_I \quad , \tag{3.1}$$

where

$$r_I=\ln(M_{GUT}/M_I)/\ln(M_{GUT}/M_Z) \quad , \tag{3.2}$$

and b_i is the SM beta function contribution, b_{iI} is beta function contributions due to intermediate scales, M_{GUT} is the unification scale, M_Z is the mass of the Z boson, and M_I is the intermediate scales. Then we can write down the difference of beta functions as $B_{ij}=B_i-B_j$ wthere i, j=1, 2, 3. The ratio is

$$\frac{B_{23}}{B_{12}}=\frac{5}{8}\frac{\sin^2\theta_W-\alpha(M_Z)/\alpha_s(M_Z)}{3/8-\sin^2\theta_W} \quad , \tag{3.3}$$

where the experimental values $\sin^2\theta_W \cong 0.231$, M_Z=91.1876 GeV, $\alpha_s(M_Z) \cong 0.118$ [48], and $\alpha^{-1}(M_Z)\cong 127.906$ [20]. The ratio given in the equation (3.3) must be around 0.721 and it is 0.53 for SM beta functions which is low. That improper ratio for the SM can be fixed by adding proper intermediate scales which should decrease the value B_{12}. In 45 representation (1, 2, 1/2) and (3, 3, -1/3) have negative contributions on B_{12} as -1/15 and -9/15 respectively given in the Table-2.4.3. The (1, 2, 1/2) is a scalar field that carries the same quantum numbers as the Higgs in the SM. Therefore, our model is seen as two-Higgs-doublet [51]. According to experimental results there is only one Higgs doublet in SM [20]. Therefore, the second Higgs doublet we are adding must be above the SM Higgs energy scale to explain its existence. On the other hand, the particle (3, 3,

-1/3) is a scalar leptoquark, for that reason it must have higher energy than the SM energy scale (M_W) [52-53]. The contributions of these two 45 split multiplets affect the beta functions flow in desired way, therefore we want to set their mass as minimal as possible such as 1 TeV or 10 TeV. Additionally, making two neutrinos massive requires the addition of 75 fermion fields which include the heavy right-handed neutrino having quantum numbers (1, 1, 0). Experimentally, it is found that the neutrino mass $m_\nu < 1.1$ eV. This upper bound limit set an upper bound for the mass of right-handed neutrino in 75_F as $(m_\nu)_R \leq 1.2\times10^{15}$ GeV. This limit is much lower than the lower bound for unification scale, that is $=5\times10^{15}$GeV. It means that all fermionic multiplets' masses should be below the unification scale and have contributions on beta functions flow. To find out the unification that ensures the boundaries, it is practical to fix masses of (1, 2, 1/2), (8, 2, 1/2), and (1, 3, 0) at 1 TeV and due to experimental bounds (3, 3, -1/3) is at 10^{10} GeV. Then using the computer program[1] written by the author found that unification occurs at 1.018×10^{16} GeV. This calculation is made by using subtraction of the beta functions B_{12} and B_{23}.

$$2\pi\left(\frac{1}{\alpha_1(M_Z)}-\frac{1}{\alpha_2(M_Z)}\right)=b_{12}\left(\ln\frac{M_{GUT}}{M_Z}\right)+\left(-\frac{1}{15}\right)\left(\ln\frac{M_{GUT}}{m_{(1,2)}}\right)$$
$$+\left(\frac{-1}{2}\right)\left(\ln\frac{M_{GUT}}{m_{(8,2)}}\right)+\left(-\frac{1}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(1,3)}}\right)+\left(-\frac{9}{5}\right)\left(\ln\frac{M_{GUT}}{m_{(3,3)}}\right)$$
$$+(0)\left(\ln\frac{M_{GUT}}{m_{(8,1)}}\right)+\left(\frac{16}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(\bar{6},2)}}\right)+\left(-\frac{64}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(8,3)}}\right)$$
$$+\left(\frac{40}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(3,1)}}\right)+\left(\frac{8}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(3,2)}}\right)+(0)\left(\ln\frac{M_{GUT}}{m_{(1,1)}}\right) \quad ,$$

(3.4a)

$$2\pi\left(\frac{1}{\alpha_2(M_Z)}-\frac{1}{\alpha_3(M_Z)}\right)=b_{23}\left(\ln\frac{M_{GUT}}{M_Z}\right)+\left(\frac{1}{6}\right)\left(\ln\frac{M_{GUT}}{m_{(1,2)}}\right)$$
$$+\left(\frac{-2}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(8,2)}}\right)+\left(\frac{1}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(1,3)}}\right)+\left(\frac{3}{2}\right)\left(\ln\frac{M_{GUT}}{m_{(3,3)}}\right)+(-4)\left(\ln\frac{M_{GUT}}{m_{(8,1)}}\right)$$

[1] https://gist.github.com/esraythelite/0bc426b9f58f493d2257ee5f9b5f18d8

$$+\left(-\frac{16}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(\bar{6},2)}}\right)+\left(\frac{28}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(8,3)}}\right)$$

$$+\left(-\frac{4}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(3,1)}}\right)+\left(\frac{4}{3}\right)\left(\ln\frac{M_{GUT}}{m_{(3,2)}}\right)+(0)\left(\ln\frac{M_{GUT}}{m_{(1,1)}}\right) \quad .$$

(3.4b)

In above equation the two of 75_F representations are given as the same energy scale and masses of (3,2), (3,1) ($\bar{6}$,2) is the same as their complex multiplets. Using 75_F mass relations given in the equations (2.190a-2.190d) we can define a ratio r

$$r=\frac{m(8,1)}{m(8,3)} \quad .$$

(3.5)

It is found that for the unification at higher energy scales it requires that r <1. According to mass relations this requirement results the mass of the (1,1) is the highest and mass of the (8,1) is the lowest. Considering the upper bound for (1,1) it is found that r=0.427 gives masses as

$$\psi_1: m(8,1)=1.33\times10^{14}\text{GeV},$$

$$\psi_6: m(\bar{6},2)=1.78\times10^{14}\text{GeV},$$

$$\psi_8, \psi_{(3,1)}: m(8,3)=m(3,1)=3.12\times10^{14}\text{GeV},$$

$$\psi_3: m(3,2)=3.56\times10^{14}\text{GeV},$$

$$\psi_0: m(1,1)=4.01\times10^{14}\text{GeV} \qquad (3.6a)$$

and the scalar content is

$$\phi_1=\phi_8=\Sigma_3 \Rightarrow m(1,2)=m(8,2)=m(1,3)=1\ \text{TeV}. \qquad (3.6b)$$

The unification is $M_{GUT}=1.018\times10^{16}$ GeV. The heavy right-handed neutrino is consistent with the upper bound limit coming from the neutrino mass. This model includes 5 intermediate scales between SM and GUT scale (the contribution of (1,1)

does not count) seen in the Figure-3.1 below. Other split multiplets of 45_S are at or above the GUT scale.

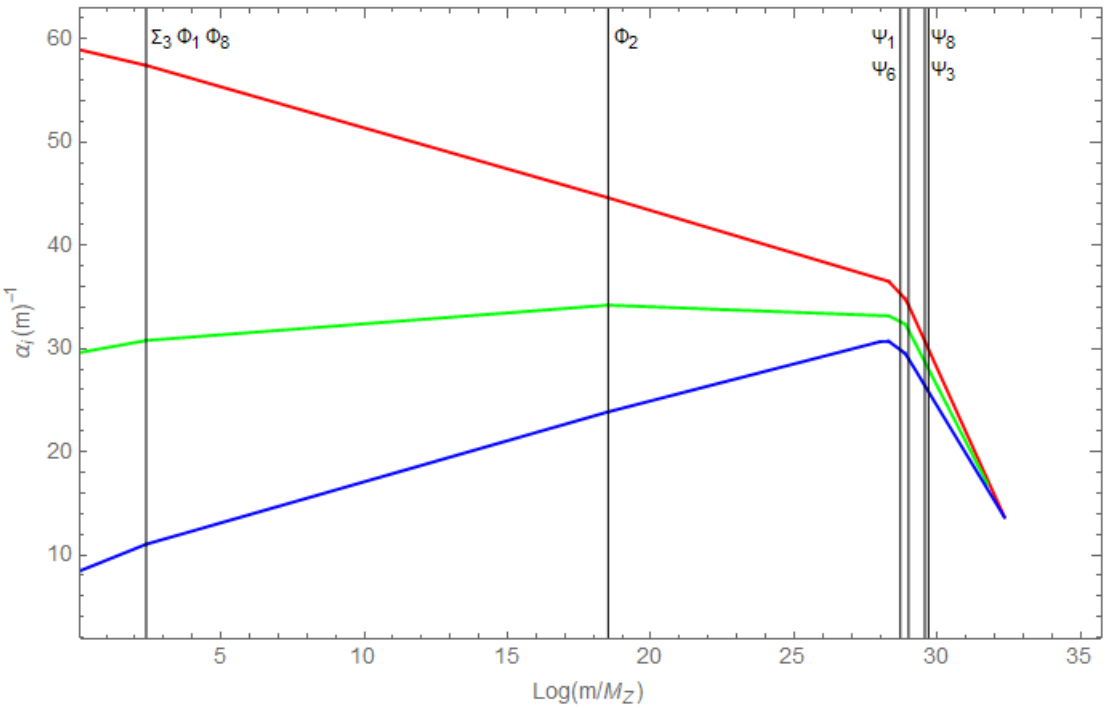


Figure 3.1 Gauge couplings of the proposed model

From the Figure-3.1 it is seen that the coupling constant at GUT scale $\alpha_{GUT}=\frac{1}{17}$ that affect proton lifetime will be calculated later. Now, we can calculate neutrino mass by using the interaction diagram of 5_F, 45_S, and 75_F. The 5_F contains the left-handed neutrino and the Higgs in 45 acquires VEV, $\langle 45\rangle$ results a diagram as shown below.

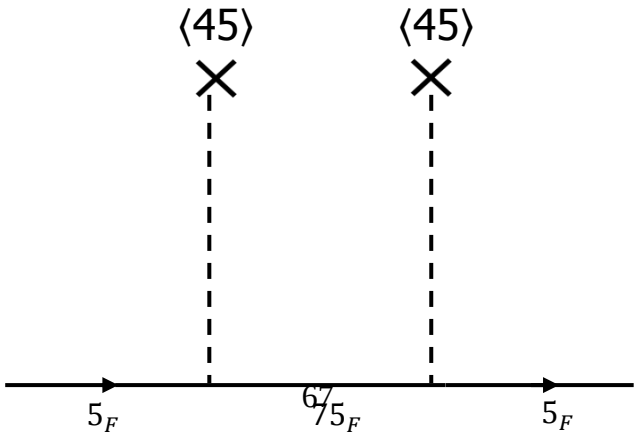


Figure 3.2 Type-1 seesaw diagram for the proposed model

From the diagram above the neutrino mass is calculated from the equation (3.7),

$$m_v = \frac{1}{2}\frac{(gv_{45})^2}{M(1,1)} \quad .$$

(3.7)

In eqaution (3.7) g is the coupling constant comes from the interaction vertex of 5, 45, and 75. v_{45} is the VEV of 45 and M(1,1) is mas of right-handed neutrino in 75. To make constant g natural we are setting them at the O(1). The v_{45} is the VEV of the second Higgs doublet it must satify

$$v^2 = v_5^2 + v_{45}^2 \quad ,$$

(3.8)

where v is total VEV, v=246 GeV, and v_5 is the VEV of the SM Higgs which cause a SSB of SM to $SU(3)_C \otimes SU(1)_{em}$. Since v_5 should be greater than 0, the VEV of 45 can be selected as 150 GeV and 200 GeV to calculate the two masses of neutrinos. Setting these parameters into the eqaution (3.8), mass of the neutrino is found as $m_{v_1}=4.99\times10^{-2}$GeV, $m_{v_2}=8.70\times10^{-3}$eV, and $m_{v_3}=0$.

The mass ordering founded for neutrinos is called normal hierarchy shown below.

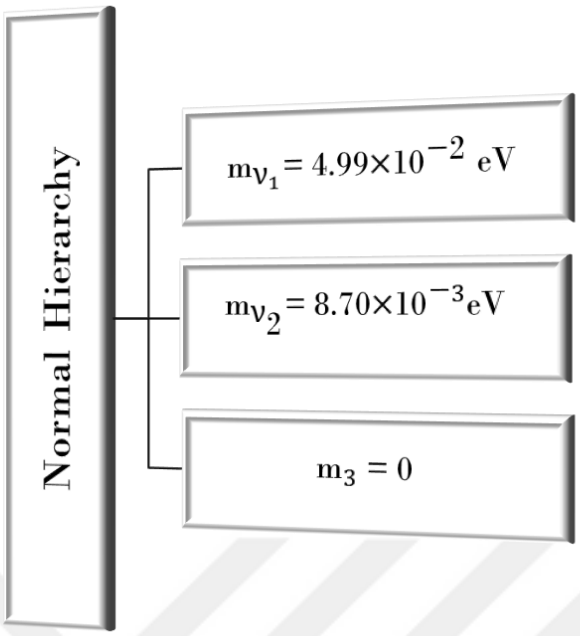


Figure 3.3 Neutrino masses in normal hierarchy

Using the equation (2.137) $\tau_p \cong 4.883\times10^{32}\times\frac{\left(\frac{M_{GUT}}{10^{16}GeV}\right)^4}{\alpha_{GUT}^2}$ years

where α_{GUT} is found as

$$\alpha_{GUT}=0.059\,, \tag{3.9}$$

we can calculate the proton lifetime as

$$\tau_p = 1.5\times10^{35}\text{years}. \tag{3.10}$$

In the Neutrino Mass section, the radiative neutrino mass models are discussed. Interestingly our model which can be shown as $GG+45_S+75_F$ contains particles to generate neutrino mass at one-loop level. Since 45 scalar contains the split multiplet (8,2,1/2) and 75 fermions contain both (8,1,0) and (8,3,0) at some special energy level where the tree-level interaction contributions is suppressed by one-loop level neutrinos become massive. However, in our model one-loop contributions to neutrino mass are supressed and we only consider tree level contributions.

4. CONCLUSIONS

The model we construct is successful at unification scale $M_{GUT}=1.018\times10^{16}$GeV compared to other SU(5) models [4-5, 7, 11, 13-14]. The coupling constant at unification scale $\alpha_{GUT}=\frac{1}{17}$, is lower compared with the studies in general in which it is found as $\frac{1}{\alpha_{Mx}}\cong\frac{1}{38}$. That can be considered as at the unification scale particles interact about two times strongly than in the general GUT models. The lower bound for the proton lifetime $\tau_p \geq 10^{34}$ years is provided. The most amazing point in this study is that all masses of split-multiplets in 75_F and 45_S contribute to beta functions while keeping mass relations [8]. Neutrino mass comes up naturally by using the Type-1 seesaw mechanism [43-44] consistent with experimental bounds [30, 45, 54]. Our model is two-Higgs-doublet model due to the additional Higgs doublet in 45_S field. Since there is one Higgs doublet in SM, the second doublet Higgs has mass greater than the Higgs in the SM. This model can be seen as minimal SU(5) model [15] and in literature it has not been studied except its supersymmetric type [54] is.

Using two 75_F fields at least two neutrinos become massive and third one stays massless. To make predictions for all three neutrinos a third 75_F can be added. This addition can also be used to construct PMNS matrix and prediction for the type of neutrino mass hierarcy. Our model found that neutrino mass ordering are in normal hierarchy [22-24].

In this model 45_S is added and it is known that this representation fix charged lepton and quark mass ratios [9]. It is expected to see the positive effect in our model as well. There are only contributions of two split multiplets of 45_S, (1,2) and (3,3), due to their negative contribution to B_{12}, which helps three coupling constants unify and achieving unification at higher energy scales. Thankfully, mass relations allow us to make masses of (1,2) and (3,3) small while putting others at or above the unification scale. Unluckily, it is not possible to have the same situation for fermions due to the upper mass limit for the right-handed neutrino, (1,1) in 75_S. If one of our considerations was not to explain neutrino mass naturally, we would have better unification scale with fewer intermediate scales as mentioned before in the first and second scenarios.

In this model contributions from loops are not counted. While constructing Lagrangian we only consider renormalizable terms. It is possible to use non-renormalizable mass terms for split multiplets [7]; however, that allows us to put masses anywhere possible unnaturally.

The SU(5) gauge theories are still viable even it has the same rank as the SM and restrictions from proton lifetime force its boundaries. The model we explained in this thesis lies at the edge of the miminal SU(5) gauge theories. Many studies are conducted on SU(5) gauge theories [55-56] and tries to explain experimental results. This study provides experimental and cosmological boundaries while preserving mass relations. Other studies [57] tries to do the similar explanation with groups with higher ranks. Comparing to these models SU(5) generally seems the most natural gauge theory to explain three fundamental interactions.

We can summarize the successes of the model in three categorizes; first one is the successful unification scale, another is appropriate neutrino mass, and finally a reasonable proton lifetime. Even though other models can provide those, non-supersymmetric SU(5) gauge theory which includes extra fields of 75_F and 45_S has not been studied while preserving mass ratios.

REFERENCES

[1]Georgi, H., Glashow, S.L. (1974), Unity of All Elementary Particle Forces.

PhysicalReview Letters, 32(8), 438-441.

[2]Glashow, S.L., Iliopoulos, J., Maiani, L. (1970). Weak Interactions with Lepton-Hadron Symmetry. Physical Review D, 2(7), 1285-1292.

[3]Bouchiat, C., Iliopoulos, J., Meyer, P. (1972). An Anomaly Free Version of Weinberg's Model. Physics Letters B, 38(7), 519-523.

[4]Murayama, H., Yanagida, T. (1991), A Viabla SU(5) GUT with Light Leptoquarks Bosons.Modern Physics Letters A, 7(1992), 147-152.

[5]Dorsner, I., Perez, P. F. (2005). Unification Without Supersymmetry:Neutrino Mass, Proton Decay and Light Leptoquarks, Nuclear Physics B, 723(1-2), 53-76.

[6]Dorsner, I., Perez, P.F., Felipe, R.G. (2006). Pthenomenologinal and Cosmological Aspects of a Minimal GUT Scenario. Nuclear Physics B, 747(1-2), 312-327.

[7]Bajc, B., Senjanovic, G. (2007). Seesaw at LHC. JTHEP , 08(2007), 0-14.

[8]Perez, P.F. (2007). Renormalizable Adjoint SU(5). Physics Letters B, 654(2007), 189-193.

[9]Georgi, H., Jarlskog, C. (1979). A New Lepton-Quarks Mass Relation in a Unified Theory. Physics Letters B, 86(3-4), 297-300.

[10]Perez, P. F., Murgui, C. (2016). Renormalizable SU(5) Unification. Physical Review D, 94(7), 7-14.

[11]Cox, P., Kusenko, A., Sumensari, O., Yanagida, T. (2017). SU(5) Unification with TeV-scale Leptoquarks. JTHEP, 03(2017), 0-35.

[12]Saad, S., 2019. Origin of a Two-loop Neutrino Mass from SU(5) Grand Unification. Physical Review D, 99(11), 1-16.

[13]Dorsner, I., Saad, S. (2020). Towards a Minimal SU(5) GUT. Physical Review D, 101(1), 2-9.

[14]Dorner, I., Dzaferovic-Masic, E., Saad, S. (2021). Parameter Space Exploration of the Minimal S(5) Unification. Physics Review D, 104(1), 10-23.

[15]Klein, C., Lindner, M., Vogl, S., (2019) Radiative Neutrino Masses and Successful SU(5) Unification. Physical Review D, 100(7), 075024 .

[16]Georgi, H., (2000) Lie Algebras in Particle Physics: From Isospin to Unified Theories, 1st Edition, CRC Press, Boca Raton

[17]Griffiths, G., (2008) Introduction to Elementary Particles. 2nd Edition, WILEY-VCH Verlag GmbH&Co.KGaA, Weinheim.

[18]Peskin, M. E., Schroeder, D.V. (1995) An Introduction to Quantum Field Theory, 1st Edition, Addison-Wesley, Massachusetts.

[19]Zee, A. (2016) Group Theory in a Nutsthell for Physicists, 1st Edition, Princeton University Press, New Jersey.

[20]Zyla P.A. et. al. (2020). Review of Particle Physics. Progress of Theoretical and Experimental Physics, 2020(8), 083C01.

[21]Kounnas, C., Masiero, A., Nanopoulos, D.V., Olive, K.A., (1985) Grand Unification with and Without Supersymmetry and Cosmological Implications,1st Edition, World Scientific Publishing Co Pte Ltd, Singapore.

[22]Boucenna, S.M., Ohlsson, T., Pernov, M. (2019) A minimal Non-Supersymmetric SO(10) Model with Peccei-Quinn Symmetry. Physics Letters B, 292, 251-257.

[23]Altarelli, G., Feruccio, F. (2003) Theoretical Models of Neutrino Masses and Mixings. Springer Tracts in Modern Physics, 190, 169-208.

[24]Altarelli, G., Ferruccio, F. (1999). Neutrino Masses and Mixings : A Theoretical Perspective. Physics Reports, 320(1-6), 295-318.

[25]Zee, A. (1980). A Theory of Lepton Number Violation and Neutrino Majorana Masses. Physics Letters B, 93(4), 389-393.

[26]Cai, Y., Clarke, J.D., Schmidt, M.A., Volkas, R.R. (2015). Testing Radiative Neutrino Mass Models at the LHC. Journal of High Energy Physics, 2015(2), 1-40.

[27]Klein, C., Lindner, M. (2019). Minimal Radiative Neutrino Masses. Journal of High Energy Physics, 2019(3), 1-18.

[28]Hagedorn, C., Ohlsson, T., Riad, S., Schmidt, M.A. (2016). Unification of Gauge Couplings in Radiative Neutrino Mass Models. Journal of High Energy Physics, 2016(9), 111.

[29]Dorsner, I. (2017). One-Loop Neutrino Mass in SU(5). Proceedings of Science, CORFU2016, 067.

[30]Sierra, A., Hirsch, D., Kovalenko, S.,G. (2008). Leptoquarks:Neutrino Masses and Related Accelerator Signals. Physical Review D, 77(5),055011.

[31]Thelo, J.C., Hirsch, M., Ota, T. Pereira Dos Santos, F.A. (2015). Double Beta Decay and Neutrino Mass Models. Journal of High Energy Physics, 2015(5), 92.

[32]Arkani-Hamed, N., Dimopoulos, S., Dvali, G. March-Russel, J. (2001). Neutrino Masses from Large Extra Dimensions. Physical Review D, 65(2), 024032.

[33]Cordero-Carrion, I., Hirsch, M., Vicente, A. (2020). Master Majorana Neutrino Mass Parametrization. Journal of Physics:Conference Series, 1586, 12022.

[34]Cordero-Carrion, I., Hirsch, M., Vicente, A. (2020). General Parametrization of Majorana Neutrino Mass Models. Physical Review D, 101(7), 075032.

[35]Mohapatra, R.N. (2003). Understanding Neutrino Masses and Mixings in the Seesaw Framework. arXiv, 279-299.

[36]King, S.F. (2016). Neutrino Mass and Mixing in the Seesaw Playground. Nuclear Physics B, 908, 456-466.

[37]Bari, P.D. (2005). See-saw Geometry and Leptogenesis. Nuclear Physics B, 727(1-2), 318-354.

[38]Kajita, T. (2016). Discovery of Athmosptheric Neutrino Oscillations. International Journal of Modern Physics A, 31(27), 16300477.

[39]Davis, R. (1964). Solar Neutrinos. Ⅱ. Experimental. Physical Review Letters, 12(11), 303-305.

[40]Bachall, J.N. (1964). Solar Neutrinos.I. Theoretical. Physical Review Letters, 12(11), 300-302.

[41] Jiajun, L. (2014). Phenomenological Studies of Neutrino Physics.PhD Thesis, Iowa State University,Iowa, ABD,18-22.

[42] Giganti, C., Lavignac, S., Zito, M. (2018).Neutrino Oscillations: The Rise of the PMNS Paradigm. Progress in Particle and Nuclear Physics, 98, 1-54.

[43] Minkowski, P.C. (1977). $\mu \rightarrow e\gamma$ At a Rate of One out of 109 Muon Decays?. Physics Letters B, 67(4), 421-428.

[44]Brdar, V., Helmboldt, A. J., Iwamoto, S., Schmitz, K. (2019). Type Ⅰ Seesaw Mechanism as the Common Origin of Neutrino Mass, Baryon Asymmetry, and the Electroweak Scale. Physical Review D, 100(7), 075029.

[45] Aghanim, N. et. al. (2020). Planck 2018 Results Ⅰ. Overview and the Cosmological Legacy of Planck. Astronomy & Astrophysics, 641(A1), 56.

[46] Karkkainen, T., (2018)Neutrino Masses and Oscillations : in Theories beyond the Standard Model. PhD Thesis, University of Helsinki, Helsinki, Finland, 48-67.

[47] Foot, R., Lew, H., The, X.G., Joshi, G.C. (1989). See-saw Neutrino Masses Induced by a Triplet of Leptons. Zeitschrift für Physik C Particles and Fields, 44, 441-444.

[48] Perez, P.F., Wise, M. B. (2009) On the Origin Of Neutrino Masses. Physical Review D, 80(5), 053006.

[49]Gresham, M. I., Wise, M.B. (2007). Color Octet Scalar Production at the CERN LHC. Physical Review D, 76(7), 075003.

[50]Babu, K.S., Ma, E. (1984). Supression of Proton Decay in SU(5) Grand Unification. Physics Letters B, 144(5-6), 381-385.

[51] Branco, G.C., Ferreira, P.M., Lavoura, L., Rebelo, M.N., Sther, M., Silva, J.P. (2012). Theory and Phenomenology of two-Higgs-doublet Models. Physics Reports, 516(1-2), 1-102.

[52]Crivellin, A., Greub, S., Müller, D., Saturnino, F. (2021). Scalar Leptoquarks in Leptonic Processes. Journal of High Energy Physics, 2021(2), 182.

[53]Dorsner, I., Fajfer, S., Greljo, A., Kamenik, J.F., Kosnik, N. (2016). Physics of Leptoquarks in Precision Experiments and at Particle Colliders. Physics Reports, 641, 1-68.

[54]Murayama, H., Okada, Y., Yanagida, T. (1992). The Georgi-Jarlskog Mass Relation in a Supersymmetric Grand Unified Model. Progress of Theoretical Physics, 88(4), 791-795.

[55]Romao, M.C. (2011) SU(5) Grand Unification Theory Revisited. MS Thesis, Lisboa Technic University Higther Techical Institute, Lisboa, Portuguese, 3-83.

[56]Klein, C.K.M. (2019) Minimal Radiative Neutrino Mass. MS Thesis, University of Theidelberg, Theidelberg, Germany, 11-86.

[57]Pernow, M. (2019) Phenomenology of SO(10) Grand Unified Theories. PhD Thesis, KHT Royal Unstitute of Technology, Stockholm, Sweden, 5-62.

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