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An application of Fibonacci sequences in groups

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Abstract

In this work, we have proved that, for the 3-step general Fibonacci recurrence and any finite p-group of exponent p and nilpotency class 2, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step general Fibonacci sequence in the field GF(p).
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1. Introduction

Let (s_i) denote the 3-step general recurrence defined by $s_i = ls_{i-1} + ms_{i-2} + ns_{i-3}$ for some $l, m, n \in \mathbb{N}$. We assume that p does not divide n, and then we get definition of 3-step general standard Fibonacci sequence as $(0, 0, 1, l, l^2 + m, l(l^2 + m) + lm + n, \ldots)$ in Z/pZ. If p were permitted to divide n then the sequence would be ultimately periodic, but would never return to the $0, 0, 1$. This sequence or loop must be periodic and we use the letter k to denote the fundamental period of (s_i) that is the shortest period of that sequence. Obviously k depends on p.

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Takahashi [8] gives a fast algorithm which is based on the product of Lucas number to compute large Fibonacci numbers. Fibonacci

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sequences have been an interesting subject in applied mathematics. West [11] has shown using transfer matrices that the number $|S_n(123, 3214)|$ of permutations avoiding the patterns 123 and 3214 is the Fibonacci number F_{2n}. The study of Fibonacci sequences in groups began with the earlier work of Wall [10] where the ordinary Fibonacci sequences in cyclic groups were investigated and then Vinson [9] who was particularly interested in ranks of apparition in ordinary Fibonacci sequences. In the mid 1980s, Wilcox [12] extended the problem to abelian groups. Prolific co-operation of Campbell et al. [3] expanded the theory to some finite simple groups. Ryba [7] constructed and analized a pair of sequences of representations of the symmetric groups. Aydın and Smith proved in [1] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and exponent a prime number p.

One of the latest works in this area is [2] where it is shown that the length of the 2-step general Fibonacci sequences are equal to the length of the 2-step general Fibonacci recurrences constructed by the two generating elements of finite nilpotent groups of nilpotency class 2 and exponent a prime number p.

We shall be interested in the shortest period of the 3-step general Fibonacci sequence the entries of which are taken in any finite p-group of exponent p and class 3. The aim of this work is to generalize [2] to cover the 3-step general Fibonacci sequences.

Defintion 1.1. Let H be normal in G, K be normal in G and $K \leqslant H$. If H/K is contained in the centre of G/K, then H/K is called a central factor of G. A group G is called nilpotent if it has finite series of normal subgroups

$$G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_r = 1$$

such that G_{i-1}/G_i is a central factor of G for each $i = 1, 2, \ldots, r$. The smallest possible r is called the nilpotency class of G.

Further details about nilpotent groups and related topics can be found in [4,5].

2. The main result

The fundamental period of a sequence satisfying a linear recurrence is sometimes called the Wall number of that sequence. The Wall number with respect to a particular linear recurrence $k(G) = k$ of a group G is the lowest common multiple of the fundamental period of all sequence satisfying the recurrence in the group G. As a different perspective of what it usually used to be, k denotes the fundamental period of the 3-step general Fibonacci sequences taken modulo a distinguished prime p.

Theorem 2.1. *Let G be a non-trivial finite p-group of exponent p and nilpotency class 2. Then we have $k(G) = k$. There are two assumptions which will insert*:

(i) $n \not\equiv 0 (\mathrm{mod}\ p)$,

(ii) $l + m + n - 1 \not\equiv 0 (\mathrm{mod}\ p)$.

3. The proof of Theorem 2.1 and some conjectures

Let G be a 3-generator relatively free group in the variety of nilpotent groups of class 2 and exponent p. Thus, G is free on a set of generators $\{x, y, z\}$. The group G has order p^6 and satisfies the relations $(y, x) = t, (z, x) = u$ and $(z, y) = v$. The subgroup $\langle t, u, v\rangle$ has order p^3, and is both the center and derived group of G. Every element of G has a unique representation as

$$x^a y^b z^c t^d u^e v^f,$$

where the exponents are elements of $\mathrm{GF}(p)$. We need an explicit formula for the element $(x^a y^b z^c t^d u^e v^f)^n$ which is clearly

$$(x^a y^b z^c t^d u^e v^f)^n = \left(x^{na} y^{nb} z^{nc} t^{nd + \binom{n}{2} ab} u^{ne + \binom{n}{2} ac} v^{nf + \binom{n}{2} bc} \right). \tag{1}$$

This could be proved by induction on n using the group relations. We assume that $(x^{a_0} y^{b_0} z^{c_0} t^{d_0} u^{e_0} v^{f_0})$, $(x^{a_1} y^{b_1} z^{c_1} t^{d_1} u^{e_1} v^{f_1})$, $(x^{a_2} y^{b_2} z^{c_2} t^{d_2} u^{e_2} v^{f_2})$ be three elements of G. For our purpose, we need a formula to describe how the next term of the sequence should be. Therefore using by (1) and the group relations we can write

$$\begin{aligned}&(x^{a_0} y^{b_0} z^{c_0} t^{d_0} u^{e_0} v^{f_0})^n \cdot (x^{a_1} y^{b_1} z^{c_1} t^{d_1} u^{e_1} v^{f_1})^m \cdot (x^{a_2} y^{b_2} z^{c_2} t^{d_2} u^{e_2} v^{f_2})^l \\ &\quad = x^{a_3} y^{b_3} z^{c_3} t^{d_3} u^{e_3} v^{f_3},\end{aligned}$$

where

$$a_3 = na_0 + ma_1 + la_2,$$

$$b_3 = nb_0 + mb_1 + lb_2,$$

$$c_3 = nc_0 + mc_1 + lc_2,$$

$$\begin{aligned}d_3 = {} & nd_0 + md_1 + ld_2 + \binom{n}{2} a_0 b_0 + \binom{m}{2} a_1 b_1 + \binom{l}{2} a_2 b_2 + mna_1 b_0 \\ & + la_2(nb_0 + mb_1),\end{aligned}$$

$$e_3 = ne_0 + me_1 + le_2 + \binom{n}{2}a_0c_0 + \binom{m}{2}a_1c_1 + \binom{l}{2}a_2c_2 + mna_1c_0 + la_2(nc_0 + mc_1)$$

and

$$f_3 = nf_0 + mf_1 + lf_2 + \binom{n}{2}b_0c_0 + \binom{m}{2}b_1c_1 + \binom{l}{2}b_2c_2 + mnb_1c_0 + lb_2(nc_0 + mc_1).$$

We shall use vector notation to calculate the sequence and define a bi-infinite sequence $(r_i) = (a_i, b_i, c_i, d_i, e_i, f_i)$ via the 3-step general recurrence and initial data

$$r_0 = (1,0,0,0,0,0),$$
$$r_1 = (0,1,0,0,0,0)$$

and

$$r_2 = (0,0,1,0,0,0).$$

In the sequel, we define some auxiliary 3-step general Fibonacci sequences (s_i), (u_i) and (t_i) by using the following initial data:

$$(s_0, s_1, s_2) = (0,0,1),$$
$$(t_0, t_1, t_2) = (0,1,0)$$

and

$$(u_0, u_1, u_2) = (1,0,0),$$

so that $(a_i) = (u_i)$, $(b_i) = (t_i)$ and $(c_i) = (s_i)$. It is clear that (t_i) and (u_i) can be expressed in terms of (s_i) such that

$$t_i = s_{i+1} - ls_i \tag{2}$$

and

$$u_i = s_{i+2} - ls_{i+1} - ms_i. \tag{3}$$

A straightforward induction yields that for $\alpha \geqslant 0$

$$d_k = \binom{n}{2}\sum_{i=0}^{k-1} s_{k-i-1}u_it_i + \binom{m}{2}\sum_{i=0}^{k-1} s_{k-i-1}u_{i+1}t_{i+1} + \binom{l}{2}\sum_{i=0}^{k-1} s_{k-i-1}u_{i+2}t_{i+2} + mn\sum_{i=0}^{k-1} s_{k-i-1}u_{i+1}t_i + \ln\sum_{i=0}^{k-1} s_{k-i-1}u_{i+2}t_i + lm\sum_{i=0}^{k-1} s_{k-i-1}u_{i+2}t_{i+1},$$

$$e_k = \binom{n}{2}\sum_{i=0}^{k-1} s_{k-i-1}s_i u_i + \binom{m}{2}\sum_{i=0}^{k-1} s_{k-i-1}s_{i+1}u_{i+1} + \binom{l}{2}\sum_{i=0}^{k-1} s_{k-i-1}s_{i+2}u_{i+2} + mn\sum_{i=0}^{k-1} s_{k-i-1}s_i u_{i+1} + \ln\sum_{i=0}^{k-1} s_{k-i-1}s_i u_{i+2} + lm\sum_{i=0}^{k-1} s_{k-i-1}s_{i+1}u_{i+2}$$

and

$$f_k = \binom{n}{2}\sum_{i=0}^{k-1} s_{k-i-1}s_i t_i + \binom{m}{2}\sum_{i=0}^{k-1} s_{k-i-1}s_{i+1}t_{i+1} + \binom{l}{2}\sum_{i=0}^{k-1} s_{k-i-1}s_{i+2}t_{i+2} + mn\sum_{i=0}^{k-1} s_{k-i-1}s_i t_{i+1} + \ln\sum_{i=0}^{k-1} s_{k-i-1}s_i t_{i+2} + lm\sum_{i=0}^{k-1} s_{k-i-1}s_{i+1}t_{i+2}.$$

Each of these expressions can be written depending on just (s_i) using (2) and (3).

Lemma 3.1. *For all integers α and β, we have*

$$s_{\alpha+\beta} = s_\alpha s_{\beta+2} + (s_{\alpha+1} - s_\alpha)s_{\beta+1} + (s_{\alpha+2} - s_{\alpha+1} - s_\alpha)s_\beta. \tag{4}$$

Proof. Consider the rotation of the standard 3-step Fibonacci sequence

$$s^{+\alpha} = (s_\alpha, s_{\alpha+1}, s_{\alpha+2}, \ldots).$$

We can express this sequence as a linear combination of s, s^+ $(= s^{+1})$ and s^{++} $(= s^{+2})$ by

$$s^\alpha = s_\alpha s^{++} + (s_{\alpha+1} - s_\alpha)s^+ + (s_{\alpha+2} - s_{\alpha+1} - s_\alpha)s.$$

To see this, just observe that the sequences on each side of this equation satisfy the 3-step Fibonacci recurrence, and agree in positions 0, 1 and 2. Now take the entry in position β on each side to obtain the desired formula (4), and we are done. □

Lemma 3.2. *The quantities $c_{a,b}$ satisfy the equation*

$$c_{a,b+2} + c_{a+1,b+1} - c_{a,b+1} + c_{a+2,b} - c_{a+1,b} - c_{a,b} = 0$$

for all integers a and b.

Proof. Put $\alpha = i + a$ and $\beta = -i + b$ in (4). Multiply throughout by s_i and sum over the range $0 \leqslant i < k$ to obtain

$$s_{a+b}\sum s_i = c_{a,b+2} + c_{a+1,b+1} - c_{a,b+1} + c_{a+2,b} - c_{a+1,b} - c_{a,b}.$$

Now we know that $\sum s_i$ vanishes by using the definition of 3-step Fibonacci recurrences and so

$$c_{a,b+2} + c_{a+1,b+1} - c_{a,b+1} + c_{a+2,b} - c_{a+1,b} - c_{a,b} = 0$$

as required. □

Lemma 3.3. *For all integers α and β, we have*

$$\sum_{i=0}^{k-1} s_i s_{i+\alpha} s_{-i+\beta} = 0.$$

Proof. Let

$$c_{\alpha,\beta} = \sum_{i=0}^{k-1} s_i s_{i+\alpha} s_{-i+\beta},$$

where s_i is defined by the recurrence $s_i = ls_{i-1} + ms_{i-2} + ns_{i-3}$ and initial data $(s_0, s_1, s_2) = (0, 0, 1)$. Now we claim that the quantities $c_{\alpha,\beta}$ satisfy the following what we call are template equations:

$$c_{\alpha,\beta} = lc_{\alpha-1,\beta} + mc_{\alpha-2,\beta} + nc_{\alpha-3,\beta},$$

$$c_{\alpha,\beta} = lc_{\alpha,\beta-1} + mc_{\alpha,\beta-2} + nc_{\alpha,\beta-3},$$

$$c_{\alpha,\beta} = lc_{\alpha+1,\beta-1} + mc_{\alpha+2,\beta-2} + nc_{\alpha+3,\beta-3}$$

and

$$c_{\alpha,\beta+2} + c_{\alpha+1,\beta+1} - lc_{\alpha,\beta+1} + c_{\alpha+2,\beta} - lc_{\alpha+1,\beta} - mc_{\alpha,\beta} = 0. \qquad \square$$

It can be immediately verified that the first three equations are easy consequences of the 3-step general recurrence relation. The fourth is obtained by Lemma 3.2. This collection of four systems of equations is enough to force each $c_{\alpha,\beta}$ to vanish except for finitely many bad primes. The Computer Algebra System AXIOM [6] enabled us to see that the template equations are enough to force each $c_{\alpha,\beta}$ to vanish. Computer programs and their results available on a request. Moreover, we have lots of data for various values of l, m and n but we give obtained results in the Table 1 for some values of l, m and n.

Lemma 3.4. *For all integers α, β and γ, we have*

$$\sum_{i=0}^{k-1} s_{i+\alpha} s_{-i+\beta} s_{i+\gamma} = 0.$$

Proof. This is simply the upshot of Lemma 3.3. □

In order to prove Theorem 2.1, we are faced with the task of showing that

$$d_k = d_{k+1} = d_{k+2} = e_k = e_{k+1} = e_{k+2} = f_k = f_{k+1} = f_{k+2} = 0,$$

Table 1

l, m, n	Primes	Wall number	
		Nilp. class 1	Nilp. class 2
0,1,2	2,13	156	156
	5,7,11	124,171,120	124,171,120
1,0,5	2,5,17	16	16
	7,11,13	24,665,168	24,665,168
1,1,4	2,5,7	10,48	10,48
	11,13,17	60,168,288	60,168,288
1,2,2	2,37	36	36
	5,7,11	12,171,120	12,171,120
1,3,2	2,5,7,83	8,48,82	40,336,82
	11,13,17	190,2196,48	190,2196,48
1,3,4	2,3,5,7,149,	24,12,74	120,84,74
	11,13,17	665,1098,96	665,1098,96
1,4,3	2,3,7,11,131	6,15,130	42,165,130
	5,13,17	124,549,144	124,549,144
2,1,3	2,3,5	4,24	4,120
	7,11,13	48,120,84	48,120,84
2,2,2	2,5,7,31	8,48,30	40,336,30
	11,13,17	120,1098,1228	120,1098,122
2,3,4	2,3,5,23	4,264	20,6072
	7,11,13	342,665,549	342,665,549
2,4,4	2,3,29	28	28
	7,11,13	48,665,1098	48,665,1098
2,5,2	2,3,43	14	14
	7,11,13	48,30,12	48,30,12
3,1,1	2,11	55	55
	5,7,13	31,24,84	31,24,84
3,2,1	2,3,5	5	25
	7,11,13	42,133,56	42,133,56
3,3,1	2,3,5	8	8
	7,11,13	19,40,61	19,40,61
3,3,3	2,3		
	5,7,11	124,342,665	124,342,665
3,4,4	2,5,227	12,226	60,226
	7,11,13	171,665,168	171,665,168
4,1,1	2,5,19	10,18	10,18
	7,11,13	57,120,12	57,120,12

(*continued on next page*)

Table 1 (*continued*)

l, *m*, *n*	Primes	Wall number	
		Nilp. class 1	Nilp. class 2
4,4,4	2,5,11	4,10	4,110
	7,13,17	171,1098,96	171,1098,96
4,5,5	2,5,13,53	56,52	728,52
	7,11,17	342,665,4912	342,665,4912
5,2,3	2,3,107		
	7,11,13	24,665,24	24,665,24
5,4,2	2,5,17,2131	16	272
	7,11,13	171,10,168	171,10,168
5,4,5	2,5,13,179	12	156
	7,11,17	42,665,4912	42,665,4912

that is, these individual sums actually vanish for all but a finite number of what we call are bad primes for which we do not know whether $d_k = e_k = f_k = 0$ or not.

We are back to task at hand. The fact that $d_k = e_k = f_k = 0$ is now an easy consequences of Lemmas 3.3 and 3.4. It is a matter of algebraic manipulation to show that $d_{k+1} = d_{k+2} = e_{k+1} = e_{k+2} = f_{k+1} = f_{k+2} = 0$ provided that d_k, e_k and f_k are vanish.

For the various values of l, m and n, Table 1 shows some bad primes. However, for some l, m and n, we do know all primes satisfying $d_k = e_k = f_k = 0$. The first and second columns of the table contain the value of l, m and n and corresponding bad primes, respectively.

For the same l, m and n, some good primes are given in the next row, where $d_k = e_k = f_k = 0$. Different Wall numbers for the nilpotency class 1 and nilpotency class 2 mean that d_k, e_k and f_k are not zero for both the certain values of l, m and n and the prime number p. Wall numbers separated by commas are corresponding values for the prime numbers separated by commas, respectively, for some l, m and n. Since we work on modulo prime number p, Wall numbers are not given when one of the l, m and n is bigger than the prime number p.

To conclude, we have proved the following except for the exclusive primes given in Table 1.

Theorem 3.5. *Take any three generators x, y, z of the* 3*-generator relatively free group G in the variety of nilpotent groups of class* 2 *and exponent p. Form a* 3*-step general Fibonacci sequence using these generators as initial data. The fundamental period of this sequence is k.*

Any 3-step general Fibonacci sequence in any finite p-group P of exponent p lives in a 3-generator subgroup H of P, and there is a homomorphism from the

free group to this group H sending initial data to initial data. Hence the proof of Theorem 2.1. is completed.

Conjecture. The step number can be increased to four. It seems that more complex sums than those of this document are needed to be vanished. Thus, it is likely to put more restrictions on prime numbers in question.

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