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Research Article

On New Polynomial Sequences Constructed to Each Vertex in an n-Gon

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In this work, we bring to light the properties of newly formed polynomial sequences at each vertex of Pell polynomial sequences placed clockwise at each vertex in the n-gon. We compute the relation among the polynomials with such vertices. Moreover, in an n-gon, we generate a recurrence relation for a sequence giving the mth term formed at the kth. Similar to the situations we are talking about, we applied to Pell-Lucas polynomials, Jacobsthal polynomials, and Jacobsthal-Lucas polynomials and have obtained new relations recurrence.

1. Introduction

It is an interesting tool for many researchers to know about number sequences, especially thrust is given to Fibonacci numbers. In this direction, many researchers work on it and study its wings, and the Lucas numbers became a vital role for them to study as can be seen in [1–8]. While framing these sequences, other sequences came into existence such as Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas by replacing the initial conditions [9–15].

As in [16–23], in engineering, science, and technology, the structure of such sequences with generalizations plays a vital significance art and architecture. In studying such discipline, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials have shown their significance as in [24–27].

Recently, the study of triangles with coordinates of vertices have been studied in [14, 19].

We have made a new study for polynomials of Pell, Pell-Lucas, Jacobsthal, and Jacobsthal by reference to [19] articles in this study.

The motivation of this work is to study the clockwise nature of the Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials to be located on the vertices of the polygons.

We have computed some interesting relations in n-gon concerning the vertex B_k corresponding to mth term for $0 \le k < n$. Also, a relation corresponding to a vertex among the numbers will be determined.

2. Materials and Methods

Definition 1. The polynomials of the given by linear recurrence relation

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x), \tag{1}$$

with $P_0(x) = 0$ and $P_1(x) = 1$ called as Pell polynomials.

The negative Pell polynomials are defined by the recurrence relation,

$$P_{-n}(x) = (-1)^{n+1} P_n(x). \tag{2}$$

The first few terms of this sequence are $\{1, 2x, 4x^2 + 1, 8x^3 + 4x, 16x^4 + 12x^2 + 1, 32x^5 + 32x^3 + 6x, \ldots\}$.

For Fibonacci sequence, the characteristic equation is

$$r^2 - 2xr - 1 = 0. \tag{3}$$

The roots of this equation are $\alpha(x)$ and $\beta(v)$.

$$\alpha(x) = x + \sqrt{x^2 + 1} \text{ and } \beta(x) = x - \sqrt{x^2 + 1}. \tag{4}$$

The Binet formula of this sequence is

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}. \tag{5}$$

Definition 2. The linear reoccurrence relation for Pell-Lucas polynomials is

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x), \tag{6}$$

with $Q_0(x) = 2$ and $Q_1(x) = 2x$.

The first few terms of this sequence are $\{2, 2x, 4x^2 + 2, 8x^3 + 6x, 16x^4 + 16x^2 + 2, 32x^5 + 40x^3 + 10x, \ldots\}$.

The negative Pell-Lucas polynomials are defined by the recurrence relation,

$$Q_{-n}(x) = (-1)^n Q_n(x). \tag{7}$$

The Binet formula of this sequence is

$$Q_n(x) = \alpha^n(x) + \beta^n(x). \tag{8}$$

Definition 3. The recurrence relation for Jacobsthal polynomials is

$$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x), \tag{9}$$

with $J_0(x) = 0$ and $J_1(x) = 1$.

Note that

$$J_{-n}(x) = -(-1)^n (2x)^{-n} J_n(x). \tag{10}$$

The first few terms of this sequence are $\{0, 1, 1, 1 + 2x, 1 + 4x, 1 + 6x + 4x^2, 1 + 8x + 12x^2, \ldots\}$.

The characteristic equation for the Jacobsthal polynomials sequence is

$$k^2 - k - 2x = 0. \tag{11}$$

The roots of this equation are $\gamma(x)$ and $\delta(x)$,

$$\gamma(x) = \frac{1 + \sqrt{1 + 8x}}{2} \text{ and } \delta(x) = \frac{1 - \sqrt{1 + 8x}}{2}. \tag{12}$$

The Binet formula of this sequence is

$$J_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)}. \tag{13}$$

Definition 4. The recurrence relation for Jacobsthal-Lucas polynomials is

$$j_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x), \tag{14}$$

with $j_0(x) = 2$ and $j_1(x) = 1$.

Note that

$$j_{-n}(x) = (-1)^n (2x)^{-n} j_n(x). \tag{15}$$

The first few terms of this sequence are $\{2, 1, 1 + 4x, 1 + 8x + 8x^2, 1 + 10x + 20x^2, \ldots\}$.

The Binet formula of this sequence is

$$j_n(x) = \gamma^n(x) + \delta^n(x). \tag{16}$$

3. Results and Discussion

3.1. New Recurrence Relations for the Pell and Pell-Lucas Numbers. From here onwards, we shall consider a continuously clockwise sense approach to the polygon of the Pell polynomials and Pell-Lucas polynomials. We will continue to n-gon starting this placement from a point. For such a polygon, let us call each vertex point as $B_0, B_1, B_2, \ldots$ for $k = 0, 1, 2, \ldots$. Here, we aim at finding a relationship that locates any term of new B_k sequences to every vertex of the polygon. We establish such a relationship in Theorem 3.1.1 to Pell polynomials and in Theorem 3.1.3 to Pell-Lucas polynomials.

Let the vertex number be represented by k and m represents the order of any term of the new sequence occurring at any vertex in any n-gon, with $0 \le k < n$ and $m \ge 1$.

Taking $n = 1$, the Pell polynomials in succession over a dot are given, as depicted in Figure 1.

For $n = 1$ yields $k = 0$ and giving a recurrence relation,

$$P_{(m-1)n+k}(x) = P_{m-1}(x) = 2xP_{m-2}(x) + P_{m-3}(x), \tag{17}$$

with $P_0(x) = 0$ and $P_1(x) = 1$ of the form of Pell polynomials.

For $n = 2$, choose the Pell polynomials in succession at the endpoints of a line, as depicted in Figure 2.

So, it determines a recurrence relation of the form

$$P_{(m-1)n+k}(x) = P_{(m-1)2+k}(x) = \left(4x^2 + 2\right)P_{(m-2)2+k}(x) - P_{(m-3)2+k}(x). \tag{18}$$

For $n = 3$, the vertices of the triangle will be labeled by Pell polynomials clockwise, as given in Figure 3.

So, it determines a recurrence relation of the form

$$P_{(m-1)n+k}(x) = P_{(m-1)3+k}(x) = \left(8x^3 + 6x\right)P_{(m-2)3+k}(x) + P_{(m-3)3+k}(x). \tag{19}$$

Now, for $n = 4$, the Pell numbers are taken clockwise on the vertices of a tetragon, as given in Figure 4.

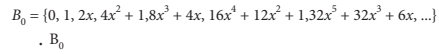


Figure 1: Pell polynomials placed over a dot.

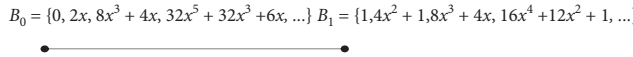


Figure 2: Pell polynomials kept at the endpoints of a line.

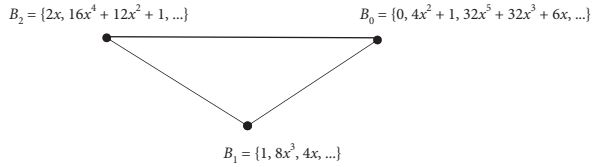


Figure 3: Pell polynomials kept in clockwise to vertices of a triangle.

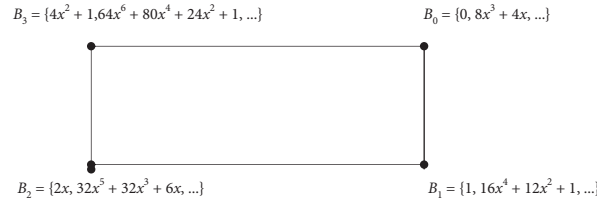


Figure 4: On tetragon representing Pell polynomials which are kept clockwise at its vertices.

So, it determines a recurrence relation of the form

$$P_{(m-1)n+k}(x) = P_{(m-1)4+k}(x) = \left(16x^4 + 16x^2 + 2\right)P_{(m-2)4+k}(x) - P_{(m-3)4+k}(x). \tag{20}$$

Similarly, for an n-gon, we take clockwise the Pell numbers on its vertices, as given in Figure 5.

For example, in a new sequence to determine the second term of it occurring at the vertex B_1 in a 3-gon.

Thus, the value for $n = 3$, $m = 3$, and $k = 1$ corresponds to the 3rd term of $B_1 = \{1, 8x^3 + 4x, 64x^6 + 80x^4 + 24x^2 + 1, \ldots\}$ sequence in Figure 3.

This term must be $64x^6 + 80x^4 + 24x^2 + 1$.

Now, to determine this value from the recurrence relation with $n = 3$ from

$$P_{(m-1)n+k}(x) = P_{(m-1)3+k}(x) = \left(8x^3 + 6x\right)P_{(m-2)3+k}(x) + P_{(m-3)3+k}(x), \tag{21}$$

so that

$$\begin{aligned} P_{(3-1)3+1} &= \left(8x^3 + 6x\right)P_{(3-2)3+1}(x) + P_{(3-3)3+1}(x), \\ P_7(x) &= \left(8x^3 + 6x\right)P_4(x) + P_1(x), \\ 64x^6 + 80x^4 + 24x^2 + 1 &= \left(8x^3 + 6x\right)\left(8x^3 + 4x\right) + 1 \\ &= 64x^6 + 32x^4 + 48x^4 + 24x^2 + 1. \end{aligned} \tag{22}$$

In Figure 5, we define a new sequence created at the vertex B_k.

$B_k = \{x_1, x_2, \ldots\} = \{P_k, P_{n+k}, P_{2n+k}, P_{3n+k}, \ldots\}$. As $P_k = P_{0n+k}$ is the first term to this sequence, the second term is $P_{n+k}(x) = P_{1n+k}(x)$ and the third term is $P_{2n+k}(x)$, and if it continues like this, it is observed that the mth term will be $P_{(m-1)n+k}(x)$ of the new sequence formed in an n-gon.

In addition, the coefficient of $P_{(m-2)n+k}(x)$ is Pell-Lucas polynomials $Q_n(x)$ and $(-1)^n$ is the coefficient of $P_{(m-3)n+k}(x)$.

So, we have the following result.

Theorem 1. *For $0 \le k < n$, the relation*

$$P_{(m-1)n+k}(x) = Q_n(x)P_{(m-2)n+k}(x) - (-1)^n P_{(m-3)n+k}(x) \tag{23}$$

is satisfied, giving mth term of the sequence with respect to vertex B_k.

Proof. Using Binet formulas of Pell and Pell-Lucas polynomials, we see

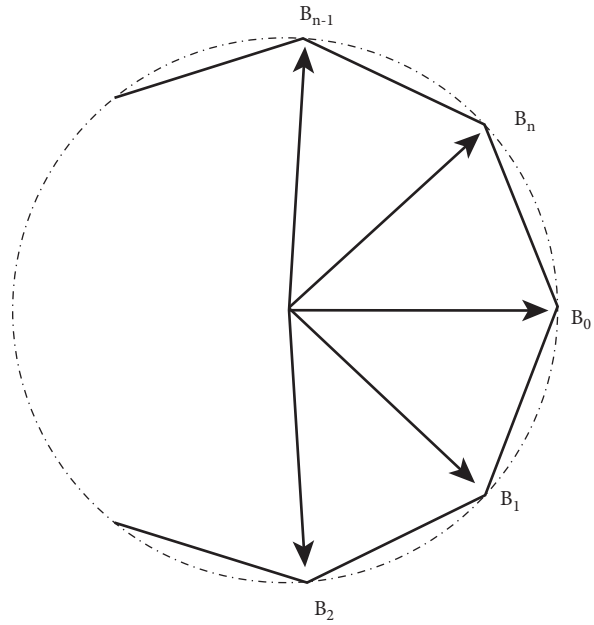


Figure 5: On n-gon representation of Pell polynomials kept clockwise at its vertices.

$$
\begin{aligned}
&Q_n(x)P_{(m-2)n+k}(x) - (-1)^n P_{(m-3)n+k}(x)\\
&= \left(\alpha^n(x) + \beta^n(x)\right)\left(\frac{\alpha^{(m-2)n+k}(x) - \beta^{(m-2)n+k}(x)}{\alpha(x) - \beta(x)}\right) - (-1)^n\left(\frac{\alpha^{(m-3)n+k}(x) - \beta^{(m-3)n+k}(x)}{\alpha(x) - \beta(x)}\right)\\
&= \frac{\alpha^{(m-1)n+k}(x) - \beta^{(m-1)n+k}(x)}{\alpha(x) - \beta(x)} - \frac{\alpha^n(x)\beta^{(m-2)n+k}(x)}{\alpha(x) - \beta(x)} + \frac{\beta^n(x)\alpha^{(m-2)n+k}(x)}{\alpha(x) - \beta(x)}\\
&\quad - \frac{(-1)^n.\alpha^{(m-3)n+k}(x)}{\alpha(x) - \beta(x)} + \frac{(-1)^n.\beta^{(m-3)n+k}(x)}{\alpha(x) - \beta(x)}\\
&= \frac{\alpha^{(m-1)n+k}(x) - \beta^{(m-1)n+k}(x)}{\alpha(x) - \beta(x)} + \alpha^{(m-3)n+k}(x)\left(\frac{\beta^n(x)\alpha^n(x) - (-1)^n}{\alpha(x) - \beta(x)}\right)\\
&\quad - \beta^{(m-3)n+k}(x)\left(\frac{\alpha^n(x)\beta^n(x) - (-1)^n}{\alpha(x) - \beta(x)}\right).
\end{aligned}
\tag{24}
$$

Because of $\alpha(x)\beta(x) = -1$, we have

$$
\begin{aligned}
&= \frac{\alpha^{(m-1)n+k}(x) - \beta^{(m-1)n+k}(x)}{\alpha(x) - \beta(x)} + \alpha^{(m-3)n+k}(x)\left(\frac{(-1)^n - (-1)^n}{\alpha(x) - \beta(x)}\right)\\
&\quad - \beta^{(m-3)n+k}(x)\left(\frac{(-1)^n - (-1)^n}{\alpha - \beta}\right)\\
&= \frac{\alpha^{(m-1)n+k} - \beta^{(m-1)n+k}}{\alpha - \beta}\\
&= P_{(m-1)n+k}(x).
\end{aligned}
\tag{25}
$$

Let us indicate the coefficients corresponding to this sequence in Table 1.

Table 1 clearly indicates that the first components are Pell-Lucas polynomials, $\{2x, 4x^2 + 1, 8x^3 + 6x, 16x^4 + 16x^2 + 2, 32x^5 + 40x^3 + 10x, 64x^6 + 96x^4 + 36x^2 + 2, 128x^7 + 224x^5 + 112x^3 + 14x, \ldots\}$.

Table 1 clearly indicates that the second are {1, – 1, 1, –1, 1, –1, 1, –1, . . .}. {1, – 1, 1, –1, 1, –1, 1, –1, . . .} is numbered as sequences A033999 in the On-Line Encyclopedia of Integer Sequences [25]. □

Theorem 2. *The relation for $0 \le k < n$ between the numbers with respect to the vertex B_k is*

$$P_{mn+k}(x) = P_{n+k}(x)\left(\sum_{t=0}^{2m-3-(-1)^m/4} (-1)^{(n+1)t}\binom{m-v-1}{v} Q_n^{m-2t-1}(x)\right) + P_k(x)\left(\sum_{t=0}^{2m-5-(-1)^{m-1}/4} (-1)^{(n+1)(t+1)}\binom{m-v-2}{v} Q_n^{m-2t-2}(x)\right). \tag{26}$$

Proof. To establish this result, we induct on m.

Thus, when $m = 1$, we see

$$\begin{aligned} P_{n+k}(x) &= P_{n+k}(x)\left(\sum_{v=0}^{0} (-1)^{(n+1)v}\binom{-v}{v} Q_n^{-2v}(x)\right) + P_k(x)\left(\sum_{v=0}^{-1} (-1)^{(n+1)(v+1)}\binom{-v-1}{v} Q_n^{-2v-2}(x)\right) \\ &= P_{n+k}(x)\left((-1)^0\binom{0}{0}(Q_n)^0\right) + P_n(x)\left((-1)^{n+1}\binom{-1}{0} Q_n{}^{-2}(x)\right) \\ &= P_{n+k}(x). \end{aligned} \tag{27}$$

For $m = s$, we assume the claim is true, that is,

$$P_{sn+k}(x) = P_{n+k}(x)\left(\sum_{v=0}^{\frac{2s-3-(-1)^s}{4}} (-1)^{(n+1)v}\binom{s-v-1}{v} Q_n^{s-2v-1}(x)\right) + P_k(x)\left(\sum_{v=0}^{\frac{2s-5-(-1)^{s-1}}{4}} (-1)^{(n+1)(v+1)}\binom{s-v-2}{v} Q_n^{s-2v-2}(x)\right). \tag{28}$$

For $m = s + 1$, let us show it is also true.
For this, if we use (23), then we see

$$P_{(s+1)n+k}(x) = P_{sn+k}(x)Q_n(x) - (-1)^n P_{(s-1)n+k}(x). \tag{29}$$

By using (28), we have

Table 1: The coefficients.

Dot	Segment	Triangle	Tetragon	Pentagon	Hexagon	Heptagon
$(2x, 1)$	$(4x^2+1,-1)$	$(8x^3+6x, 1)$	$(16x^4+16x^2+2, -1)$	$(32x^5+40x^3+10x, 1)$	$(64x^6+96x^4+36x^2+2, -1)$	$(128x^7+224x^5+112x^3+14x)$

$$\begin{aligned}
P_{(s+1)n+k} &= P_{n+k}(x)\left(\sum_{v=0}^{2m-3-(-1)^s/4}(-1)^{(n+1)v}\binom{s-v-1}{v}Q_n^{s-2v-1}(x)\right)Q_n(x)\\
&+ P_k(x)\left(\sum_{v=0}^{2m-5-(-1)^{s-1}/4}(-1)^{(n+1)(v+1)}\binom{s-v-2}{v}Q_n^{s-2v-2}(x)\right)Q_n(x)\\
&- (-1)^n P_{n+k}(x)\left(\sum_{v=0}^{2m-5-(-1)^{s-1}/4}(-1)^{(n+1)v}\binom{s-v-2}{v}Q_n^{s-2v-2}(x)\right)\\
&- (-1)^n P_k(x)\left(\sum_{v=0}^{2m-7-(-1)^{s-2}/4}(-1)^{(n+1)(v+1)}\binom{s-v-3}{v}Q_n^{s-2v-3}(x)\right).
\end{aligned} \tag{30}$$

When $s = 2r$, we have

$$\begin{aligned}
P_{(s+1)n+k} &= P_{n+k}(x)\left(\sum_{v=0}^{r-1}(-1)^{(n+1)v}\binom{2r-v-1}{v}Q_n^{2r-2v}(x)\right)\\
&+ P_{n+k}(x)\left(\sum_{v=0}^{r-1}(-1)^{(n+1)(v+1)}\binom{2r-v-2}{v}Q_n^{2r-2v-2}(x)\right)\\
&+ P_k(x)\left(\sum_{v=0}^{r-1}(-1)^{(n+1)(v+1)}\binom{2r-v-2}{v}Q_n^{2r-2v-1}(x)\right)\\
&+ P_k(x)\left(\sum_{v=0}^{r-2}(-1)^{(n+1)(v+2)}\binom{2r-v-3}{v}Q_n^{2r-2v-3}(x)\right)\\
&= P_{n+k}(x)\begin{bmatrix}\binom{2r-1}{0}Q_n^{2r}(x)+(-1)^{n+1}\binom{2r-2}{1}Q_n^{2r-2}(x)+\binom{2r-3}{2}Q_n^{2r-4}(x)+\ldots+(-1)^{(n+1)(r-1)}\binom{r}{r-1}Q_n^2(x)\\ (-1)^{n+1}\binom{2r-2}{0}Q_n^{2r-2}(x)+(-1)^{(n+1)2}\binom{2r-3}{1}Q_n^{2r-4}(x)+\ldots+(-1)^{(n+1)r}\binom{r-1}{r-1}\end{bmatrix}\\
&+ P_k\begin{bmatrix}(-1)^{n+1}\binom{2r-2}{0}Q_n^{2r-2}(x)+(-1)^{2(n+1)}\binom{2r-3}{1}Q_n^{2r-3}(x)+\ldots+(-1)^{(n+1)r}\binom{r-1}{r-1}Q_n(x)+(-1)^{2(n+1)}\binom{2r-3}{0}Q_n^{2r-3}(x)\\ +(-1)^{3(n+1)}\binom{2r-4}{1}Q_n^{2r-5}(x)+\ldots+(-1)^{(n+1)r}\binom{r-1}{r-2}Q_n(x)\end{bmatrix}\\
&= P_{n+k}(x)\left(Q_n^{2r}(x)+(-1)^{n+1}\binom{2r-1}{0}Q_n^{2r-2}(x)+(-1)^{2(n+1)}\binom{2r-2}{2}Q_n^{2r-4}(x)+\ldots+(-1)^{(n+1)(r-1)}\binom{r+1}{r-2}Q_n^2(x)+(-1)^{(n+1)r}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ P_k\left((-1)^{n+1} Q_n^{2r-1}(x) + (-1)^{2(n+1)} \binom{2r-2}{0} Q_n^{2r-3}(x) + \ldots + (-1)^{(n+1)r} \binom{r}{r-1} Q_n(x) \right) \\
&= P_{n+k}(x) \left(\sum_{v=0}^{r} (-1)^{(n+1)v} \binom{2r-v}{v} Q_n^{2r-2v}(x) \right) \\
&\quad + P_k(x) \left(\sum_{v=0}^{r-1} (-1)^{(n+1)(v+1)} \binom{2r-v-1}{v} Q_n^{2r-2v-1}(x) \right) \\
&= P_{n+k} \left(\frac{2s-1-(-1)^{s+1}}{\sum_{v=0}^{4}} (-1)^{(n+1)v} \binom{s-v}{v} Q_n^{s-2v}(x) \right) \\
&\quad + P_k \left(\frac{2s-3-(-1)^{s}}{\sum_{v=0}^{4}} (-1)^{(n+1)(v+1)} \binom{s-v-1}{v} Q_n^{s-2v-1}(x) \right).
\end{aligned} \tag{31}
$$

The equality has been shown for $s = 2r$.

Now, when $s = 2r-1$, we get

$$
\begin{aligned}
&= P_{n+k}(x) \left(\sum_{v=0}^{r-1} (-1)^{(n+1)v} \binom{2r-v-2}{v} Q_n^{2r-2v-1}(x) \right) \\
&\quad + P_{n+k}(x) \left(\sum_{t=0}^{r-2} (-1)^{(n+1)(v+1)} \binom{2r-v-3}{v} Q_n^{2r-2v-3}(x) \right) \\
&\quad + P_k \left(\sum_{v=0}^{r-2} (-1)^{(n+1)(v+1)} \binom{2r-v-3}{v} Q_n^{2r-2v-2}(x) \right) \\
&\quad + P_k \left(\sum_{v=0}^{r-2} (-1)^{(n+1)(v+2)} \binom{2r-v-4}{v} Q_n^{2r-2v-4}(x) \right) \\
&= P_{n+k}(x) \left(\binom{2r-2}{0} Q_n^{2r-1}(x) + (-1)^{n+1} \binom{2r-3}{1} Q_n^{2r-3}(x) + (-1)^{2(n+1)} \binom{2r-4}{2} Q_n^{2r-5}(x) + \ldots + (-1)^{(n+1)(r-1)} \binom{r-1}{r-1} \right. \\
&\quad \left. Q_n(x) + (-1)^{n+1} \binom{2r-3}{0} Q_n^{2r-3}(x) + (-1)^{2(n+1)} \binom{2r-4}{1} Q_n^{2r-5}(x) + \ldots + (-1)^{(n+1)(r-1)} \binom{r-1}{r-2} Q_n(x) \right) \\
&\quad + P_k \left(\begin{array}{c} (-1)^{n+1} \binom{2r-3}{0} Q_n^{2r-2}(x) + (-1)^{2(n+1)} \binom{2r-4}{1} Q_n^{2r-4}(x) + (-1)^{3(n+1)} \binom{2r-5}{2} Q_n^{2r-6}(x) + \ldots + (-1)^{(n+1)(r-1)} \binom{r-1}{r-1} Q_n^2(x) + (-1)^{2(n+1)} \binom{2r-4}{0} Q_n^{2r-4}(x) \\ (-1)^{3(n+1)} \binom{2r-5}{1} Q_n^{2r-6}(x) + \ldots + (-1)^{(n+1)r} \binom{r-1}{r-1} \end{array} \right) \\
&= P_{n+k}(x) \left[Q_n^{2r-1}(x) + (-1)^{n+1} \binom{2r-2}{1} Q_n^{2r-3}(x) + (-1)^{2(n+1)} \binom{2r-3}{2} Q_n^{2r-5}(x) + \ldots + (-1)^{(n+1)(r-1)} \binom{r}{r-2} Q_n(x) \right] \\
&\quad + P_k \left[(-1)^{n+1} Q_n^{2r-2} + (-1)^{2(n+1)} \binom{2r-3}{1} Q_n^{2r-4}(x) + (-1)^{3(n+1)} \binom{2r-4}{2} Q_n^{2r-6}(x) + \ldots + (-1)^{(n+1)r} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= P_{n+k}(x)\left(\sum_{v=0}^{r-1}(-1)^{(n+1)t}\binom{2r-v-1}{v}Q_n^{2r-2v-1}(x)\right)+P_k(x)\left(\sum_{v=0}^{r-1}(-1)^{(n+1)(v+1)}\binom{2r-v-2}{v}Q_n^{2r-2v-2}(x)\right)\\
&= P_{n+k}(x)\left(\sum_{v=0}^{\frac{2s-1-(-1)^{s+1}}{4}}(-1)^{(n+1)v}\binom{s-v}{v}Q_n^{s-2v}(x)\right)\\
&\quad + P_k(x)\left(\sum_{v=0}^{\frac{23-3-(-1)^{s}}{4}}(-1)^{(n+1)(v+1)}\binom{s-v-1}{v}Q_n^{s-2v-1}\right).
\end{aligned}
\tag{32}
$$

Thus, the proof is obtained.

Let us give example as an application. For $n = 3, k = 2$, and $m = 2$, we have

$$
\begin{aligned}
P_8(x) &= P_5(x)\left(\sum_{v=0}^{0}(-1)^{(4)v}\binom{1-v}{v}Q_3^{1-2v}(x)\right)+P_2(x)\left(\sum_{v=0}^{0}(-1)^{(4)(v+1)}\binom{-v}{v}Q_3^{-2v}(x)\right),\\
P_8(x) &= P_5(x)Q_3(x)+P_2(x)1,\\
128x^7+192x^5+80x^3+8x &= \left(16x^4+12x^2+1\right)\left(8x^3+6x\right)+2x\\
&= 128x^7+96x^5+96x^5+72x^3+8x^3+6x+2x\\
&= 128x^7+192x^5+80x^3+8x.
\end{aligned}
\tag{33}
$$

Now, let us place the Pell-Lucas polynomials like the Pell polynomials continuously clockwise at the vertices of a regular polygon.

For this new sequence, the mth term with respect to the vertex B_k of an n-gon is $Q_{(m-1)n+k}(x)$.

So, we have the following theorem about this situation. □

Theorem 3. *If on the vertices of an n-gon, the Pell-Lucas polynomials are placed clockwise, then for $0 \le k < n$, the mth term of sequence with respect to the vertex B_k is*

$$
Q_{(m-1)n+k}(x) = Q_{(m-2)n+k}(x)Q_n(x)-(-1)^nQ_{(m-3)n+k}(x).
\tag{34}
$$

Proof. To prove the result, by employing Binet formula for Pell-Lucas polynomials, we see

$$
\begin{aligned}
&Q_{(m-2)n+k}(x)Q_n(x)-(-1)^nQ_{(m-3)n+k}(x)\\
&=\left(\alpha^{(m-2)n+k}(x)+\beta^{(m-2)n+k}(x)\right)\left(\alpha^n(x)+\beta^n(x)\right)-(-1)^n\left(\alpha^{(m-3)n+k}(x)+\beta^{(m-3)n+k}(x)\right)\\
&=\alpha^{(m-1)n+k}(x)+\beta^{(m-1)n+k}(x)+\alpha^{(m-2)n+k}(x)\beta^n(x)+\alpha^n(x)\beta^{(m-2)n+k}(x)\\
&\quad-(-1)^n\alpha^{(m-3)n+k}(x)-(-1)^n\beta^{(m-3)n+k}(x)\\
&=\alpha^{(m-1)n+k}+\beta^{(m-1)n+k}-\alpha^{(m-3)n+k}(x)\left((-1)^n-\alpha^n(x)\beta^n(x)\right)-\beta^{(m-3)n+k}(x)\\
&\quad\left((-1)^n-\alpha^n(x)\beta^n(x)\right).
\end{aligned}
\tag{35}
$$

Because of $\alpha\beta = -1$, we have

$$= \alpha^{(m-1)n+k}(x) + \beta^{(m-1)n+k}(x), \\ = Q_{(m-1)n+k}(x). \tag{36}$$

□

between the numbers with respect to the vertex B_k.

Theorem 4. *For an n-gon, if Pell-Lucas polynomials are kept on the vertices clockwise sense. Then, for $0 \le k < n$, there exists a relation of the form*

$$Q_{mn+k}(x) = Q_{n+k}(x)\left(\sum_{v=0}^{2m-3-(-1)^m/4} (-1)^{(n+1)v}\binom{m-v-1}{v} Q_n^{m-2v-1}(x)\right) \\ + Q_k(x)\left(\sum_{v=0}^{2m-5-(-1)^{m-1}/4} (-1)^{(n+1)(v+1)}\binom{m-v-2}{v} Q_n^{m-2v-2}(x)\right), \tag{37}$$

Proof. The result can be proved in a similar fashion as Theorem 3.1.2.

For example, when $n = 4, k = 1$ and $m = 2$, we have

$$Q_9(x) = Q_5(x)\left(\sum_{v=0}^{0} (-1)^{(5)v}\binom{1-v}{v} Q_4^{1-2v}(x)\right) + Q_1(x)\left(\sum_{v=0}^{0} (-1)^{(5)(v+1)}\binom{-v}{v} Q_4^{-2v}(x)\right)$$

$$Q_9(x) = Q_5(x)Q_4(x) - Q_1(x) \\ \cdot 512x^9 + 1152x^7 + 864x^5 + 240x^3 + 18x \\ = \left(32x^5 + 40x^3 + 10x\right)\left(16x^4 + 16x^2 + 2\right) - 2x \\ = 512x^9 + 512x^7 + 64x^5 + 640x^7 + 640x^5 + 80x^3 + 160x^5 + 160x^3 + 20x - 2x \\ = 512x^9 + 1152x^7 + 864x^5 + 240x^3 + 18x. \tag{38}$$

□

3.2. Recurrences Notion Corresponding to Jacobsthal and Jacobsthal-Lucas Polynomials. In this direction, we involve the clockwise sense of the Jacobsthal polynomials on the point, at extreme points, at vertices of the triangle, and lastly at the vertices of an n-gon.

Since for this new sequence, the mth term computed at vertex B_k is $J_{(m-1)n+k}(x)$ for an n-gon. Also, the coefficient of $J_{(m-2)n+k}(x)$ is Jacobsthal-Lucas polynomials $j_n(x)$ and the coefficient of $J_{(m-3)n+k}(x)$ is $(-2x)^n$.

In this case, we have the following result.

Theorem 5. *On an n-gon, suppose Jacobsthal numbers are kept on its vertices clockwise. Then, for $0 \le k < n$, we have*

$$J_{(m-1)n+k}(x) = j_n(x)J_{(m-2)n+k}(x) - (-2x)^n J_{(m-3)n+k}(x), \tag{39}$$

and it determines the mth term of the sequence with respect to vertex B_k.

Proof. By using the Binet formulas of Jacobsthal and Jacobsthal-Lucas polynomials, we have the following:

$$
\begin{aligned}
&j_n(x)J_{(m-2)n+k}(x)-(-2x)^nJ_{(m-3)n+k}(x)\\
&=\left(\gamma^n(x)+\delta^n(x)\right)\left(\frac{\gamma^{(m-2)n+k}(x)-\delta^{(m-2)n+k}(x)}{\gamma(x)-\delta(x)}\right)\\
&\quad-(-2x)^n\left(\frac{\gamma^{(m-3)n+k}(x)-\delta^{(m-3)n+k}(x)}{\gamma(x)-\delta(x)}\right)\\
&=\frac{\gamma^{(m-1)n+k}(x)-\delta^{(m-1)n+k}(x)}{\gamma(x)-z(x)}+\frac{\delta^n(x)\gamma^{(m-2)n+k}(x)-\gamma^n(x)\delta^{(m-2)n+k}(x)}{\gamma(x)-\delta(x)}\\
&\quad-\frac{(-2x)^n\gamma^{(m-3)n+k}(x)}{\gamma(x)-\delta(x)}+\frac{(-2x)^n\delta^{(m-3)n+k}(x)}{\gamma(x)-\delta(x)}\\
&=\frac{\gamma^{(m-1)n+k}(x)-\delta^{(m-1)n+k}(x)}{\gamma(x)-\delta(x)}+\alpha^{(m-3)n+k}(x)\left(\frac{\delta^n(x)\gamma^n(x)-(-2x)^n}{\gamma(x)-\delta(x)}\right)\\
&\quad-\gamma^{(m-3)n+k}(x)\left(\frac{\gamma^n(x)\delta^n(x)-(-2x)^n}{\gamma(x)-\delta(x)}\right).
\end{aligned}
\tag{40}
$$

Since $\gamma(x)\delta(x)=-2x$, $\gamma^n(x)\delta^n(x)-(-2x)^n=0$, thus

$$
j_n(x)J_{(m-2)n+k}(x)-(-2x)^nJ_{(m-3)n+k}(x)=J_{(m-1)n+k}(x).
\tag{41}
$$

In Table 2, we write the coefficients concerning the above sequences.

So, clearly in Table 2, the Jacobsthal-Lucas polynomials $\{1, 1+4x, 1+6x, 1+8x+8x^2,$
$1+10x+20x^2, 1+12x+36x^2+16x^3,$
$1+14x+56x^2+56x^3\}$ are first components.

The second components in Table 2 are $\{2x, -(2x)^2, (2x)^3, -(2x)^4, 2^5, -(2x)^6, (2x)^7, \ldots\}$. {1, − 1, 1, –1, 1, –1, 1, –1, . . .} is enumerate as sequences A084633 in the On-Line Encyclopedia of Integer Sequences [25]. □

Theorem 6. *On an n-gon, let the Pell-Lucas polynomials be kept on its vertices clockwise. Then, for $0 \le k < n$, there exists a relation corresponding to numbers with respect to the vertex B_k as*

$$
\begin{aligned}
J_{mn+k}(x)&=J_{n+k}(x)\left(\frac{2m-3-(-1)^m}{\sum_{t=0}^{4}(-1)^{(n+1)t}(2x)^{nt}\binom{m-t-1}{t}j_n^{m-2t-1}(x)}\right)\\
&\quad+J_k(x)\left(\frac{2m-5-(-1)^{m-1}}{\sum_{t=0}^{4}(-1)^{(n+1)(t+1)}(2x)^{n(t+1)}\binom{m-t-2}{t}j_n^{m-2t-2}(x)}\right).
\end{aligned}
\tag{42}
$$

Proof. As in Theorem 2, the proof can be performed.

Now, for an n-gon, taking the Jacobsthal-Lucas polynomials on the vertices clockwise and in this n-gon, the mth term formed to this new sequence at the vertex B_k is $j_{(m-1)n+k}(x)$.

We have the following result in this direction. □

Theorem 7. *On an n-gon, if the Jacobsthal-Lucas polynomials are kept on their vertices clockwise. Then, for $0 \le k < n$, the mth term with respect to the vertex B_k is*

$$
j_{(m-1)n+k}(x)=j_n(x)j_{(m-2)n+k}(x)-(-2x)^nj_{(m-3)n+k}(x).
\tag{43}
$$

Table 2: The coefficients.

Dot	Segment	Triangle	Tetragon	Pentagon	Hexagon	Heptagon
$(1, 2x)$	$(1 + 4x, -4x^2)$	$(1 + 6x, 8x^3)$	$(1 + 8x + 8x^2, -16x^4)$	$(1 + 10x + 20x^2, 32x^5)$	$(1 + 12x + 36x^2 + 16x^3, -64x^6)$	$(1 + 14x + 56x^2 + 56x^3, 128x^7)$

Proof. To establish this result, Binet formula for Jacobsthal-Lucas numbers is employed and gives

$$\begin{aligned}
&j_n(x)j_{(m-2)n+k}(x) - (-2x)^n j_{(m-3)n+k}(x)\\
&= \left(\gamma^n(x) + \delta^n(x)\right)\left(\gamma^{(m-2)n+k}(x) + \delta^{(m-2)n+k}(x)\right)\\
&\quad - (-2x)^n\left(\gamma^{(m-3)n+k}(x) + \gamma^{(m-3)n+k}(x)\right)\\
&= \gamma^{(m-1)n+k}(x) + \delta^{(m-1)n+k}(x) + \gamma^n(x)\delta^{(m-2)n+k}(x) + \delta^n(x)\gamma^{(m-2)n+k}(x)\\
&\quad - (-2x)^n\gamma^{(m-3)n+k}(x) - (-2x)^n\delta^{(m-3)n+k}(x)\\
&= \gamma^{(m-1)n+k}(x) + \delta^{(m-1)n+k}(x) + \gamma^{(m-3)n+k}(x)\left(\delta^n(x)\alpha^n(x) - (-2x)^n\right)\\
&\quad + \delta^{(m-3)n+k}(x)\left(\gamma^n(x)\delta^n(x) - (-2x)^n\right).
\end{aligned} \tag{44}$$

Since $\gamma(x)\delta(x) = -2x$, $\gamma^n(x)\delta^n(x) - (-2x)^n = 0$, thus

$$j_n(x)j_{(m-2)n+k}(x) - (-2x)^n j_{(m-3)n+k}(x) = j_{(m-1)n+k}(x). \tag{45}$$

□

Theorem 8. *On an n-gon, let the Jacobsthal-Lucas numbers be kept on its vertices clockwise. Then, for $0 \le k < n$, the numbers with respect to the vertex A_k are*

$$\begin{aligned}
j_{mn+k}(x) = j_{n+k}(x)&\left(\frac{2m-3-(-1)^m}{\sum_{v=0}^{4}}(-1)^{(n+1)v}(2x)^{nv}\binom{m-v-1}{v}j_n^{m-2v-1}(x)\right)\\
&+ j_k(x)\left(\frac{2m-5-(-1)^{m-1}}{\sum_{v=0}^{4}}(-1)^{(n+1)(v+1)}(2x)^{n(v+1)}\binom{m-v-2}{v}j_n^{m-2v-2}c\right).
\end{aligned} \tag{46}$$

Proof. The result can be proved in a similar fashion as Theorem 2.

For example, when $n = 2, k = 1$ and $m = 2$, we get

$$\begin{aligned}
j_5(x) &= j_3(x)\left(\sum_{v=0}^{0}(-1)^{(3)v}(2x)^{2v}\binom{1-v}{v}j_2^{1-2v}(x)\right)\\
&\quad + j_1(x)\left(\sum_{v=0}^{0}(-1)^{(3)(v+1)}(2x)^{2(v+1)}\binom{-v}{v}j_2^{-2v}(x)\right)\\
j_5(x) &= j_3(x)j_2(x) - j_1(x)\left(4x^2\right)\\
20x^2 + 10x + 1 &= (1+6x)(1+4x) - 4x^2\\
&= 1 + 4x + 6x + 24x^2 - 4x^2\\
&= 20x^2 + 10x + 1.
\end{aligned} \tag{47}$$

4. Conclusions

In this paper, we have described the clockwise sense of Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials on the vertices of polygons. We have computed the recurrence relations for $P_{mn+k}(x)$ numbers by sorting the Pell polynomials around the polygons. Also, we have determined new correlations between the terms with respect to each vertex and generated new findings by employing these results. The work can be further extended to k–Jacobsthal and Jacobsthal-Lucas numbers and k–Fibonacci and Lucas numbers. [20-27]

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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