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# New recurrences on Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, and Jacobsthal-Lucas numbers

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ABSTRACT

In this study, the Pell numbers are placed clockwise on the vertices of the polygons with a number corresponding to each vertex. Then, a relation among the numbers corresponding to a vertex is given. Furthermore, we obtain a formula which gives the $m$th term of the sequence formed at the $k$th vertex in an $n$-gon.

The same procedure is repeated with Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers.

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## 1. Introduction

Number sequences, especially Fibonacci number sequences, have attracted the attention of many scientists. Some studies on Fibonacci and Lucas number sequences are as follows: [1–5]

There are other number sequences that are obtained by using these sequences and that attract attention in the scientific world. Pell, Pell Lucas, Jacobsthal and Jacobsthal-Lucas number sequences obtained by changing the initial conditions are also of great importance [6–11].

Number sequences and their generalizations have found application in many branches of science [12–16].

One of the last studies in this area is [17] where they we gave some results of the triangles with coordinates of vertices.

In this work, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers are placed at the vertices of the polygons clockwise. Relations are found giving the $m$th term corresponding to the vertex $A_k$ in an $n$-gon, $0 \le k < n$. Also, a relation among the numbers corresponding to a vertex is given.

## 2. Materials and methods

**Definition 2.1.** The Fibonacci sequence is defined by

$$F_{n+2=}F_n + F_{n+1}, n \ge 0$$

with $F_0$=0 and $F_1$= 1.

The first few terms of this sequence are 0,1,1,2,3,5,8,13

The characteristic equation for the Fibonacci sequence is $x^2 - x - 1 = 0$.

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are obtained where $\alpha$ and $\beta$ are the roots of this equation.

**Definition 2.2.** The Lucas numbers are defined by the linear recurrence equation

$$L_n = L_{n-1} + L_{n-2}, n \ge 2 (n \in \mathbb{N})$$

with $L_0 = 2$ and $L_1 = 1$.

The Binet formula for this sequence is

$$L_n = \alpha^n + \beta^n$$

**Definition 2.3.** The Pell numbers are the sequence of integers $\{P_n\}_{n=0}^{\infty}$ defined by

$$P_n = 2P_{n-1} + P_{n-2}, n \ge 2 \ (n \in \mathbb{N})$$

with $P_0 = 0$ and $P_1 = 1$.

The first few terms of this sequence are 0, 1, 2, 5, 12, 29, 70, 169, 408, ...

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$$A_0 = \{0, 1, 2, 5, 12, 29, 70, \ldots\}$$

**Fig. 1.** Pell numbers placed over a dot.

We know that $P_{-n} = (-1)^{n+1}P_n$.

The characteristic equation becomes

$$x^2 - 2x - 1 = 0$$

which has roots $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

We know that $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$ and $\alpha\beta = -1$.

The Binet's formula for this sequence is

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

**Definition 2.4.** The Pell–Lucas numbers are defined by the recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 2 \ (n \in \mathbb{N})$$

with $Q_0 = 2$ and $Q_1 = 2$.

For some $n \in \mathbb{N}$, we get $\{Q_n\} = \{2, 2, 6, 14, 34, 82, 198, 478, 1154, \ldots\}$

We know that $Q_{-n} = (-1)^n Q_n$.

Binet's formula for the sequence is $Q_n = \alpha^n + \beta^n$.

**Definition 2.5.** The Jacobsthal numbers are defined by

$$J_n = \begin{cases} 0 \ if \ n = 0 \\ 1 \ if \ n = 1 \\ J_n = J_{n-1} + 2J_{n-2} \ if \ n > 1 \end{cases}.$$

**Definition 2.6.** The Jacobsthal-Lucas numbers are defined by

$$j_n = \begin{cases} 2 \ if \ n = 0 \\ 1 \ if \ n = 1 \\ j_n = j_{n-1} + 2j_{n-2} \ if \ n > 1 \end{cases}.$$

## 3. Results and discussion

### 3.1. New recurrence relations for the Pell and Pell-Lucas numbers

We will now place Pell numbers and Pell-Lucas numbers at the vertices of a polygon continuously clockwise. We will continue this process from one point to the $n$-gon. Let's name the vertex points of the polygon as $A_0, A_1, A_2, \ldots$ as seen in the figures. Our aim is to find a relation that gives any term of the new sequences occurring in each vertex $A_k$. We give this aim in Theorem 3.1.1 for Pell numbers and in Theorem 3.1.3 for Pell-Lucas numbers.

In any $n$-gon, let $k$ and $m$ denote vertex number and order of any term of the new sequence occurring at any vertex, respectively, where $0 \leq k < n$ and $m \geq 1$.

For $n = 1$, let's write the Pell numbers consecutively over a dot as shown in Fig. 1.

For $n = 1$, we get $k = 0$. So, we have a recurrence relation

$$P_{(m-1)n+k} = P_{m-1} = 2P_{m-2} + P_{m-3}$$

with $P_0 = 0$ and $P_1 = 1$.

This is the well-known formula for the Pell numbers.

For $n = 2$, let's write the Pell numbers consecutively on the endpoints of a segment as shown in Fig. 2.

We have a recurrence relation

$$P_{(m-1)n+k} = P_{(m-1)2+k} = 6P_{(m-2)2+k} - P_{(m-3)2+k}$$

For $n = 3$, we place the Pell numbers on the vertices of a triangle clockwise as shown in Fig. 3.

We have a recurrence relation

$$P_{(m-1)n+k} = P_{(m-1)3+k} = 14P_{(m-2)3+k} + P_{(m-3)3+k}.$$

Now, for $n = 4$, we place the Pell numbers on the vertices of a tetragon clockwise as shown in Fig. 4.

We have a recurrence relation

$$P_{(m-1)n+k} = P_{(m-1)4+k} = 34P_{(m-2)4+k} - P_{(m-3)4+k}.$$

Now, we place the Pell numbers on the vertices of an $n$-gon clockwise as shown in Fig. 5.

For example, let's find the second term of the new sequence occurring at the vertex $A_1$ in a 4-gen.

That is, for $n = 4$, $m = 2$ and $k = 1$, This value corresponds to the 2nd term of the $A_1 = \{1, 29, 985, 33461, \ldots\}$ sequence in Fig. 4. This must be 29. Note that the enumeration of terms for the new sequence occurring at the vertex $A_k$ starts at 1 like $A_k = \{x_1, x_2, \ldots\}$.

Now, let's find this value from the given formula for $n = 4$.

$$P_{(m-1)n+k} = P_{(m-1)4+k} = 34P_{(m-2)4+k} - P_{(m-3)4+k}$$

$$P_{(2-1)4+1} = 34P_{(2-2)4+1} - P_{(2-3)4+1}$$

$$P_5 = 34P_1 - P_{-3}$$

$$29 = 34 - 5$$

In Fig. 5, let's consider the new sequence formed at the vertex $A_k$. $A_k = \{x_1, x_2, \ldots\} = \{P_k, P_{n+k}, P_{2n+k}, P_{3n+k}, \ldots\}$. Since the first term of this sequence is $P_k = P_{0n+k}$, the second term is $P_{n+k} = P_{1n+k}$, the third term is $P_{2n+k}$ and and if so on, it is seen that the $m$th term of the new sequence formed at the vertex $A_k$ in an $n$-gon is $P_{(m-1)n+k}$.

Also, the coefficient of $P_{(m-2)n+k}$ is Pell-Lucas number $Q_n$ and the coefficient of $P_{(m-3)n+k}$ is $(-1)^n$.

Let's give this case with the following theorem.

**Theorem 3.1.1.** *For $0 \leq k < n$, we have the following relation, which gives the mth term of the sequence corresponding to the vertex $A_k$*

$$P_{(m-1)n+k} = Q_n P_{(m-2)n+k} - (-1)^n P_{(m-3)n+k} \quad (1)$$

**Proof:** If we use Binet formulas of Pell and Pell-Lucas number, we get

$$P_{(m-2)n+k}Q_n - (-1)^n P_{(m-3)n+k} = \frac{\alpha^{(m-2)n+k} - \beta^{(m-2)n+k}}{\alpha - \beta}.(\alpha^n + \beta^n) - (-1)^n.\frac{\alpha^{(m-3)n+k} - \beta^{(m-3)n+k}}{\alpha - \beta}$$

$$= \frac{\alpha^{(m-1)n+k} - \beta^{(m-1)n+k}}{\alpha - \beta} + \frac{\alpha^{(m-2)n+k}.\beta^n}{\alpha - \beta} - \frac{\alpha^n.\beta^{(m-2)n+k}}{\alpha - \beta}$$

$$- \frac{(-1)^n.\alpha^{(m-3)n+k}}{\alpha - \beta} + \frac{(-1)^n.\beta^{(m-3)n+k}}{\alpha - \beta}$$

$$= \frac{\alpha^{(m-1)n+k} - \beta^{(m-1)n+k}}{\alpha - \beta} + \alpha^{(m-3)n+k}\left(\frac{\alpha^n\beta^n - (-1)^n}{\alpha - \beta}\right)$$

$$- \beta^{(m-3)n+k}\left(\frac{\beta^n\alpha^n - (-1)^n}{\alpha - \beta}\right).$$

Because of $\alpha.\beta = -1$, we have

$$= \frac{\alpha^{(m-1)n+k} - \beta^{(m-1)n+k}}{\alpha - \beta} + \alpha^{(m-3)n+k}\left(\frac{(-1)^n - (-1)^n}{\alpha - \beta}\right)$$

$$- \beta^{(m-3)n+k}\left(\frac{(-1)^n - (-1)^n}{\alpha - \beta}\right)$$

$$= \frac{\alpha^{(m-1)n+k} - \beta^{(m-1)n+k}}{\alpha - \beta}$$

$$= P_{(m-1)n+k}.$$

□




**Fig. 2.** Pell numbers placed consecutively at the endpoints of a segment.

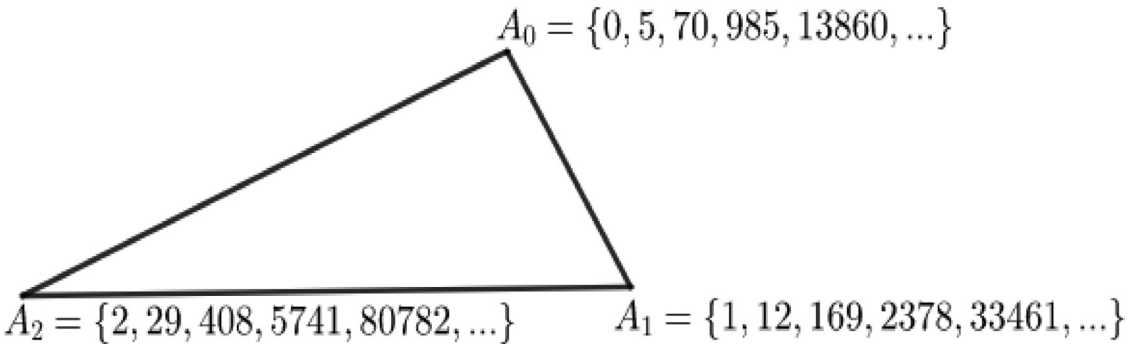



**Fig. 3.** Pell numbers placed clockwise at the vertices of a triangle.

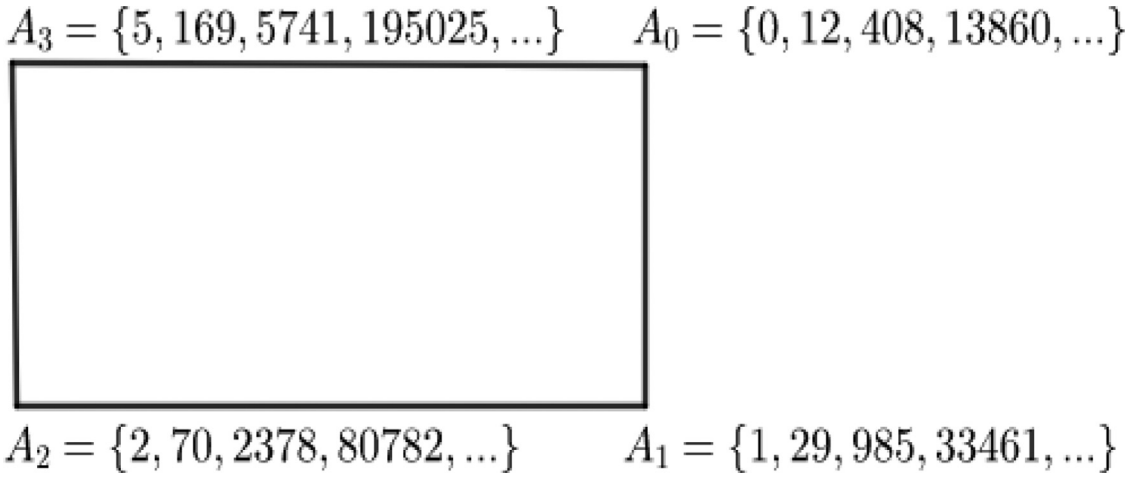



**Fig. 4.** Pell numbers placed clockwise at the vertices of a tetragon.

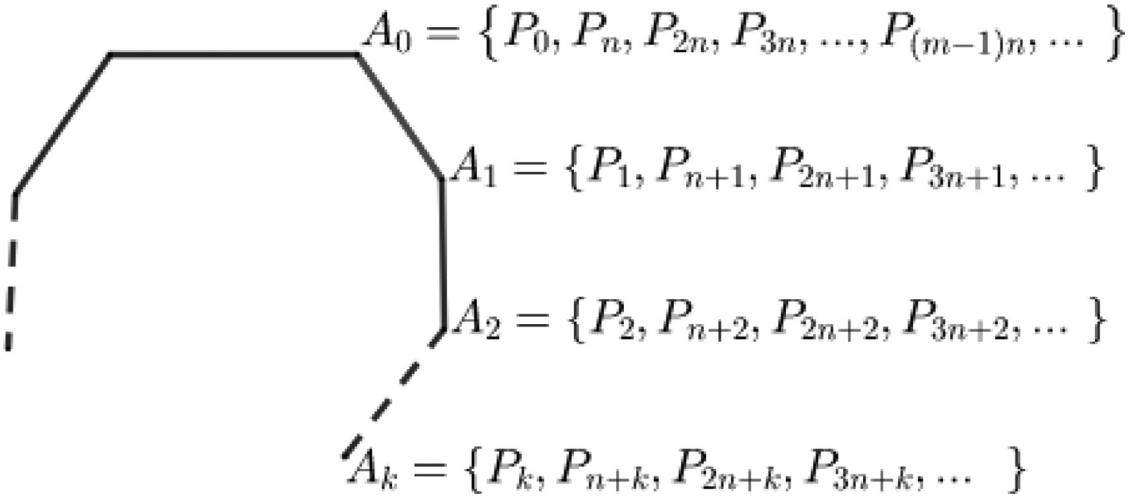



**Fig. 5.** Pell numbers placed clockwise at the vertices of an *n*-gon.

**Table 1**
The coefficients.

| Dot | Segment | Triangle | Tetragon | Pentagon | Hexagon | Heptagon |
|---|---|---|---|---|---|---|
| (2,1) | (6, -1) | (14,1) | (34, -1) | (82,1) | (198, -1) | (478,1) |

Let us give the coefficients corresponding to the above sequences in Table 1.

The first components in Table 1 are Pell-Lucas numbers, {2, 6, 14, 34, 82, 198, 478, ...}

The second components in Table 1 are {1,-1, 1, -1, 1, -1, 1, -1, ...}. {1, -1, 1, -1, 1, -1, 1, -1, ...} is numbered as sequences A033999 in The On-Line Encyclopedia of Integer Sequences [18].

**Theorem 3.1.2.** *For $0 \le k < n$, we have the following relation between the numbers corresponding to the vertex $A_k$*

$$P_{mn+k} = P_{n+k}\left(\sum_{t=0}^{\frac{2m-3-(-1)^m}{4}} (-1)^{(n+1)t}\binom{m-t-1}{t} Q_n^{m-2t-1}\right) + P_k\left(\sum_{t=0}^{\frac{2m-5-(-1)^{m-1}}{4}} (-1)^{(n+1)(t+1)}\binom{m-t-2}{t} Q_n^{m-2t-2}\right).$$

**Proof:** Let us prove the theorem by induction on $m$.

For $m = 1$, we get

$$P_{n+k} = P_{n+k}\left(\sum_{t=0}^{0} (-1)^{(n+1)t}\binom{-t}{t} Q_n^{-2t}\right) + P_k\left(\sum_{t=0}^{-1} (-1)^{(n+1)(t+1)}\binom{-t-1}{t} Q_n^{-2t-2}\right)$$
$$= P_{n+k}\left((-1)^0\binom{0}{0}(Q_n)^0\right) + P_n\left((-1)^{n+1}\binom{-1}{0}(Q_n)^{-2}\right)$$

$$= P_{n+k}$$

Suppose the claim is true for $m = s$. That is, we have

$$P_{sn+k} = P_{n+k}\left(\sum_{t=0}^{\frac{2k-3-(-1)^s}{4}} (-1)^{(n+1)t}\binom{s-t-1}{t} Q_n^{s-2t-1}\right) + P_k\left(\sum_{t=0}^{\frac{2k-5-(-1)^{s-1}}{4}} (-1)^{(n+1)(t+1)}\binom{s-t-2}{t} Q_n^{s-2t-2}\right). \tag{2}$$

For $m = s + 1$,
If we use Eq. (1) then we obtain

$$P_{(s+1)n+k} = P_{sn+k}Q_n - (-1)^n P_{(s-1)n+k}$$

By using Eq. (2), we have

$$P_{(s+1)n+k} = P_{n+k}\left(\sum_{t=0}^{\frac{2k-3-(-1)^s}{4}} (-1)^{(n+1)t}\binom{s-t-1}{t} Q_n^{s-2t-1}\right)Q_n + P_k\left(\sum_{t=0}^{\frac{2k-5-(-1)^{s-1}}{4}} (-1)^{(n+1)(t+1)}\binom{s-t-2}{t} Q_n^{s-2t-2}\right)Q_n$$
$$- (-1)^n P_{n+k}\left(\sum_{t=0}^{\frac{2k-5-(-1)^{s-1}}{4}} (-1)^{(n+1)t}\binom{s-t-2}{t} Q_n^{s-2t-2}\right) - (-1)^n P_k\left(\sum_{t=0}^{\frac{2k-7-(-1)^{s-2}}{4}} (-1)^{(n+1)(t+1)}\binom{s-t-3}{t} Q_n^{s-2t-3}\right).$$

When $s = 2r$, we have

$$P_{(s+1)n+k} = P_{n+k}\left(\sum_{t=0}^{r-1} (-1)^{(n+1)t}\binom{2r-t-1}{t} Q_n^{2r-2t}\right)$$
$$+ P_{n+k}\left(\sum_{t=0}^{r-1} (-1)^{(n+1)(t+1)}\binom{2r-t-2}{t} Q_n^{2r-2t-2}\right) + P_k\left(\sum_{t=0}^{r-1} (-1)^{(n+1)(t+1)}\binom{2r-t-2}{t} Q_n^{2r-2t-1}\right)$$
$$+ P_k\left(\sum_{t=0}^{r-2} (-1)^{(n+1)(t+2)}\binom{2r-t-3}{t} Q_n^{2r-2t-3}\right)$$
$$= P_{n+k}\left[\binom{2r-1}{0}Q_n^{2r} + (-1)^{n+1}\binom{2r-2}{1}Q_n^{2r-2} + \binom{2r-3}{2}Q_n^{2r-4} + \cdots + (-1)^{(n+1)(r-1)}\binom{r}{r-1}Q_n^2\right.$$
$$\left. + (-1)^{n+1}\binom{2r-2}{0}Q_n^{2r-2} + \binom{2r-3}{1}Q_n^{2r-4} + \ldots + (-1)^{(n+1)(r-1)}\binom{r}{r-2}Q_n^2 + (-1)^{(n+1)r}\binom{r-1}{r-1}\right]$$

$$
\begin{aligned}
&+ P_k\Bigg[(-1)^{n+1}\binom{2r-2}{0}Q_n^{2r-1} + \binom{2r-3}{1}Q_n^{2r-3} + (-1)^{n+1}\binom{2r-4}{2}Q_n^{2r-5} + \ldots + (-1)^{(n+1)r}\binom{r-1}{r-1}Q_n \\
&+ \binom{2r-3}{0}Q_n^{2r-3} + (-1)^{n+1}\binom{2r-4}{1}Q_n^{2r-5} + \binom{2r-5}{2}Q_n^{2r-7} + \ldots + (-1)^{(n+1)r}\binom{r-1}{r-2}Q_n\Bigg] \\
&= P_{n+k}\left(Q_n^{2r} + (-1)^{n+1}\binom{2r-1}{1}Q_n^{2r-2} + \binom{2r-2}{2}Q_n^{2r-4} + \ldots + (-1)^{(n+1)(r-1)}\binom{r+1}{r-1}Q_n^2 + (-1)^{(n+1)r}\right) \\
&+ P_k\left((-1)^{n+1}Q_n^{2r-1} + \binom{2r-2}{1}Q_n^{2r-3} + (-1)^{n+1}\binom{2r-3}{2}Q_n^{2r-5} + \ldots + (-1)^{(n+1)r}\binom{r}{r-1}Q_n\right) \\
&= P_{n+k}\left(\sum_{t=0}^{r}(-1)^{(n+1)t}\binom{2r-t}{t}Q_n^{2r-2t}\right) + P_k\left(\sum_{t=0}^{r-1}(-1)^{(n+1)(t+1)}\binom{2r-t-1}{t}Q_n^{2r-2t-1}\right) \\
&= P_{n+k}\left(\sum_{t=0}^{\frac{2k-1-(-1)^{s+1}}{4}}(-1)^{(n+1)t}\binom{s-t}{t}Q_n^{s-2t}\right) + P_k\left(\sum_{t=0}^{\frac{2k-3-(-1)^{s}}{4}}(-1)^{(n+1)(t+1)}\binom{s-t-1}{t}Q_n^{s-2t-1}\right)
\end{aligned}
$$

when $s = 2r-1$, we get

$$
\begin{aligned}
&= P_{n+k}\left(\sum_{t=0}^{r-1}(-1)^{(n+1)t}\binom{2r-t-2}{t}Q_n^{2r-2t-1}\right) + P_{n+k}\left(\sum_{t=0}^{r-2}(-1)^{(n+1)(t+1)}\binom{2r-t-3}{t}Q_n^{2r-2t-3}\right) \\
&+ P_k\left(\sum_{t=0}^{r-2}(-1)^{(n+1)(t+1)}\binom{2r-t-3}{t}Q_n^{2r-2t-2}\right) + P_k\left(\sum_{t=0}^{r-2}(-1)^{(n+1)(t+2)}\binom{2r-t-4}{t}Q_n^{2r-2t-4}\right) \\
&= P_{n+k}\Bigg(\binom{2r-2}{0}Q_n^{2r-1} + (-1)^{n+1}\binom{2r-3}{1}Q_n^{2r-3} + \binom{2r-4}{2}Q_n^{2r-5} + \ldots + (-1)^{(n+1)(r-1)}\binom{r-1}{r-1}Q_n \\
&+ (-1)^{n+1}\binom{2r-3}{0}Q_n^{2r-3} + \binom{2r-4}{1}Q_n^{2r-5} + \ldots + (-1)^{(n+1)(r-1)}\binom{r-1}{r-2}Q_n\Bigg) \\
&+ P_k\Bigg((-1)^{n+1}\binom{2r-3}{0}Q_n^{2r-2} + \binom{2r-4}{1}Q_n^{2r-4} + (-1)^{n+1}\binom{2r-5}{2}Q_n^{2r-6} + \ldots + (-1)^{(n+1)(r-1)}\binom{r-1}{r-2}Q_n^2 \\
&+ \binom{2r-4}{0}Q_n^{2r-4} + (-1)^{n+1}\binom{2r-5}{1}Q_n^{2r-6} + \ldots + (-1)^{(n+1)(r-1)}\binom{r-1}{r-3}Q_n^2 + +(-1)^{(n+1)r}\binom{r-2}{r-2}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= P_{n+k}\left[Q_n^{2r-1} + (-1)^{n+1}\binom{2r-2}{1}Q_n^{2r-3} + \binom{2r-3}{2}Q_n^{2r-5} + \ldots + (-1)^{(n+1)(r-1)}\binom{r}{r-1}Q_n\right] \\
&+ P_k\left[(-1)^{n+1}Q_n^{2r-2} + \binom{2r-3}{1}Q_n^{2r-4} + (-1)^{n+1}\binom{2r-4}{2}Q_n^{2r-6} + \ldots + (-1)^{(n+1)r}\right] \\
&= P_{n+k}\left(\sum_{t=0}^{r-1}(-1)^{(n+1)t}\binom{2r-t-1}{t}Q_n^{2r-2t-1}\right) + P_k\left(\sum_{t=0}^{r-1}(-1)^{(n+1)(t+1)}\binom{2r-t-2}{t}Q_n^{2r-2t-2}\right) \\
&= P_{n+k}\left(\sum_{t=0}^{\frac{2s-1-(-1)^{s+1}}{4}}(-1)^{(n+1)t}\binom{s-t}{t}Q_n^{s-2t}\right) + P_k\left(\sum_{t=0}^{\frac{2s-3-(-1)^{s}}{4}}(-1)^{(n+1)(t+1)}\binom{s-t-1}{t}Q_n^{s-2t-1}\right).
\end{aligned}
$$

□

Now, we place the Pell-Lucas numbers on the vertices of an $n$-gon clockwise.
In an $n$-gon, the $m$th term of the new sequence formed at the vertex $A_k$ is $Q_{(m-1)n+k}$.
So, we can write the following theorem.

**Theorem 3.1.3.** *Let the Pell-Lucas numbers be placed on the vertices of an n-gon clockwise. For $0 \le k < n$, we have the following relation, which gives the mth term of the sequence corresponding to the vertex $A_k$*

$$Q_{(m-1)n+k} = Q_{(m-2)n+k}Q_n - (-1)^n Q_{(m-3)n+k}.$$

**Proof:** By using Binet formula for Pell-Lucas numbers, we obtain

$$Q_{(m-2)n+k}Q_n-(-1)^nQ_{(m-3)n+k}=\left(\alpha^{(m-2)n+k}+\beta^{(m-2)n+k}\right)(\alpha^n+\beta^n)$$

$$-(-1)^n\left(\alpha^{(m-3)n+k}+\beta^{(m-3)n+k}\right)$$

$$=\alpha^{(m-1)n+k}+\beta^{(m-1)n+k}+\alpha^{(m-2)n+k}\beta^n+\alpha^n\beta^{(m-2)n+k}-(-1)^n\left(\alpha^{(m-3)n+k}+\beta^{(m-3)n+k}\right)$$

$$=\alpha^{(m-1)n+k}+\beta^{(m-1)n+k}+(\alpha\beta)^n\left(\alpha^{(m-3)n+k}+\beta^{(m-3)n+k}\right)-(-1)^n\left(\alpha^{(m-3)n+k}+\beta^{(m-3)n+k}\right)$$

□

Because of $\alpha\beta=-1$, we have

$$=\alpha^{(m-1)n+k}+\beta^{(m-1)n+k}+(-1)^n\left(\alpha^{(m-3)n+k}+\beta^{(m-3)n+k}\right)-(-1)^n\left(\alpha^{(m-3)n+k}+\beta^{(m-3)n+k}\right)$$

$$=\alpha^{(m-1)n+k}+\beta^{(m-1)n+k}$$

$$=Q_{(m-1)n+k}$$

□

**Theorem 3.1.4.** *Let the Pell-Lucas numbers be placed on the vertices of an n-gon clockwise. For $0\le k<n$, we have the following relation between the numbers corresponding to the vertex $A_k$*

$$Q_{mn+k}=Q_{n+k}\left(\sum_{t=0}^{\frac{2m-3-(-1)^m}{4}}(-1)^{(n+1)t}\binom{m-t-1}{t}Q_n^{m-2t-1}\right)+Q_k\left(\sum_{t=0}^{\frac{2m-5-(-1)^{m-1}}{4}}(-1)^{(n+1)(t+1)}\binom{m-t-2}{t}Q_n^{m-2t-2}\right)$$

**Proof:** The proof is similar to that of Theorem 3.1.2.

### *3.2. New recurrence relations for the Jacobsthal and Jacobsthal Lucas numbers*

Now, as in Section 3.1, let's place the Jacobsthal numbers clockwise, respectively, on a point, at the end points of a segment, at the vertices of a triangle, and finally at the vertices of an $n$-gon.

In an $n$-gon, the $m$th term of the new sequence formed at the vertex $A_k$ is $J_{(m-1)n+k}$. Also, the coefficient of $J_{(m-2)n+k}$ is Jacobsthal-Lucas number $j_n$ and the coefficient of $P_{(m-3)n+k}$ is $(-2)^n$.

Now, we can write the following theorem.

**Theorem 3.2.1.** *Let Jacobsthal numbers be placed on the vertices of an n-gon clockwise. For $0\le k<n$, we have the following relation that gives the* m*th term of the sequence corresponding to the vertex $A_k$*

$$J_{(m-1)n+k}=j_nJ_{(m-2)n+k}-(-2)^nJ_{(m-3)n+k}.$$

**Proof:** If we use Binet Formula for Jacobsthal and Jacobsthal-Lucas numbers, we obtain

$$j_nJ_{(m-2)n+k}-(-2)^nJ_{(m-3)n+k}=\left(2^n+(-1)^n\right)\left(\frac{2^{(m-2)n+k}-(-1)^{(m-2)n+k}}{3}\right)$$

$$-(-2)^n\left(\frac{2^{(m-3)n+k}-(-1)^{(m-3)n+k}}{3}\right)$$

$$=\frac{2^{(m-1)n+k}-(-1)^{(m-1)n+k}}{3}+\frac{(-1)^n2^{(m-2)n+k}-2^n(-1)^{(m-2)n+k}}{3}$$

$$-\frac{(-1)^n2^{(m-2)n+k}-2^n(-1)^{(m-2)n+k}}{3}$$

$$=\frac{2^{(m-1)n+k}-(-1)^{(m-1)n+k}}{3}$$

$$=J_{(m-1)n+k}.$$

□

**Table 2**
The coefficients.

| Dot | Segment | Triangle | Tetragon | Pentagon | Hexagon | Heptagon |
|---|---|---|---|---|---|---|
| (1,2) | $(5, -2^2)$ | $(7, 2^3)$ | $(17, -2^4)$ | $(31, 2^5)$ | $(65,-2^6)$ | $(127, 2^7)$ |

Let us give the coefficients corresponding to the above sequences in Table 2.

The first components in Table 2 are Jacobsthal-Lucas numbers are {1, 5, 17, 17 31, 65, 127, ...} .

The second components in Table 2 are {1, $-2^2$, $2^3$, $-2^4$, $2^5$, $-2^6$, $2^7$, ...}. {1, -1, 1, -1, 1, -1, 1, -1 ...} is numbered as sequences A084633 in The On-Line Encyclopedia of Integer Sequences [18].

**Theorem 3.2.2.** *Let the Pell-Lucas numbers be placed on the vertices of an n-gon clockwise. For $0 \le k < n$, we have the following relation between the numbers corresponding to the vertex $A_k$*

$$J_{mn+k} = J_{n+k}\left(\sum_{t=0}^{\frac{2m-3-(-1)^m}{4}} (-1)^{(n+1)t}2^{nt}\binom{m-t-1}{t} j_n^{m-2t-1}\right) + J_k\left(\sum_{t=0}^{\frac{2m-5-(-1)^{m-1}}{4}} (-1)^{(n+1)(t+1)}2^{n(t+1)}\binom{m-t-2}{t} j_n^{m-2t-2}\right)$$

**Proof:** The proof is similar to that of Theorem 3.1.2.

Now, we place the Jacobsthal-Lucas numbers on the vertices of an $n$-gon clockwise.

In an $n$-gon, the $m$th term of new the sequence formed at the vertex $A_k$ is $j_{(m-1)n+k}$.

Then we can write the following theorem. □

**Theorem 3.2.3.** *Let the Jacobsthal-Lucas numbers be placed on the vertices of an n-gon clockwise. For $0 \le k < n$, we have the following relation that gives the mth term corresponding to the vertex $A_k$*

$$j_{(m-1)n+k} = j_{(m-2)n+k}j_n - (-2)^n j_{(m-3)n+k}.$$

**Proof:** By using Binet formula for Jacobsthal-Lucas numbers, we obtain

$$\begin{aligned}
&j_{(m-2)n+k}j_n - (-2)^n j_{(m-3)n+k} \\
&= \left(2^{(m-2)n+k} + (-1)^{(m-2)n+k}\right)\left(2^n + (-1)^n\right) \\
&-(-2)^n\left(2^{(m-3)n+k} + (-1)^{(m-3)n+k}\right) \\
&= 2^{(m-1)n+k} + (-1)^{(m-1)n+k} + (-1)^{(m-2)n+k}2^n + (-1)^n 2^{(m-2)n+k} \\
&-(-1)^{(m-2)n+k}2^n - (-1)^n 2^{(m-2)n+k} \\
&= 2^{(m-1)n+k} + (-1)^{(m-1)n+k} \\
&= j_{(m-1)n+k}
\end{aligned}$$

□

**Theorem 3.2.4.** *Let the Jacobsthal-Lucas numbers be placed on the vertices of an n-gon clockwise. For $0 \le k < n$, we have the following relation between the numbers corresponding to the vertex $A_k$*

$$j_{mn+k} = j_{n+k}\left(\sum_{t=0}^{\frac{2m-3-(-1)^m}{4}} (-1)^{(n+1)t}2^{nt}\binom{m-t-1}{t} j_n^{m-2t-1}\right) + j_k\left(\sum_{t=0}^{\frac{2m-5-(-1)^{m-1}}{4}} (-1)^{(n+1)(t+1)}2^{n(t+1)}\binom{m-t-2}{t} j_n^{m-2t-2}\right)$$

**Proof:** The proof is like that of Theorem 3.1.2. □

## 4. Conclusion

Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers are placed clockwise at the vertices of the polygons. Relationships giving the m'th term for numbers corresponding to $A_k$ vertex in an $n$-gon are found. The relationship between $P_{mn}$ numbers is given using the relation obtained by arranging the Pell numbers around the polygons. As suggested, new correlations are found between the terms corresponding to each vertex and new results are found using these relationships. This paper can be extended to Horadam and Narayana numbers.

## Declaration of Competing Interest

The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## CRediT authorship contribution statement

**Songül Çelik:** Conceptualization, Formal analysis, Investigation. **İnan Durukan:** Formal analysis, Investigation. **Engin Özkan:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft.

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