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NEWS/VIEWS AND COMMENTS

Generalized Parastichy Fibonacci Numbers and Their Applications

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Abstract
In this study, we define the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers. Also, we find basic properties of these numbers, such as characteristic equations and properties. Then we give the Binet formulas, special generating function, and special summation formulas. We associate the terms of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers with matrices. In addition, we calculate the Simson formulas of these numbers. Moreover, we examine the relationships among these numbers. We relate Parastichy Fibonacci numbers with special number sequences such as Lucas numbers. Furthermore, we obtain important correlations between Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers. We explore the use of generalized Parastichy Fibonacci numbers in the context of hyperbolic quaternions. For these hyperbolic quaternions, we derive various properties, including generating functions and summation formulas. Finally, we establish a relationship between the terms of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers and their associated hyperbolic quaternion values.

Keywords Fibonacci numbers · Lucas numbers · Generating function · Binet formulas · Simson formulas

Mathematics Subject Classification 11B39 · 11B50 · 11B83 · 05A15 · 11R52

Introduction

Number sequences play a crucial role in understanding and solving problems that involve various patterns. A notable example of this is the Rabbit Problem in Leonardo Fibonacci's famous book Liber Abaci. Inspired by the rabbit problem, the Fibonacci sequence was developed, and the relationship between the terms of this sequence led to the discovery of the golden ratio. The Fibonacci and Lucas sequences are well-known number sequences that have intrigued scientists for centuries. These sequences have been applied in various fields such as algebraic coding theory [1], graph theory [2], biomathematics [3, 4], computer science [5], and more. Many generalizations of the Fibonacci sequence have also been introduced. Notable examples of such sequences include the Jacobsthal [6], Bronze Fibonacci [7], Padovan [8], Copper Fibonacci [9], k-Oresme [10], Pell [11], Narayana [12], Vieta Fibonacci [13] sequences.

For $n \in \mathbb{N}$, the Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \ and \ L_{n+2} = L_{n+1} + L_n$$

with the initial conditions $F_0 = 0$, $F_1 = 1$, an $L_0 = 2$, $L_1 = 1$.

For F_n and L_n, the Binet formulas are given by relations, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \text{ and } L_n = \alpha^n + \beta^n$$

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Table 1 Hyperbolic quaternions units [24]

	i_1	i_2	i_3	i_4
i_1	i_1	i_2	i_3	i_4
i_2	i_2	i_1	i_4	$-i_3$
i_3	i_3	$-i_4$	i_1	i_2
i_4	i_4	i_3	$-i_2$	i_1

Table 2 Parastichy Fibonacci numbers

n	F_nF_{n+1}	P_n	F_n^2
0	0×1	0	0
1	1×1	1	1
2	1×2	2	1
3	2×3	6	4
4	3×5	15	9
5	5×8	40	25
6	8×13	104	64
…	…	…	…

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. Here, the number α is the known golden ratio.

The relationship between parastichy numbers and the Fibonacci sequence forms the mathematical basis for the spiral patterns observed in many plants in nature. This relationship is profound, both in terms of geometric order and optimal placement.

Douady et al., in a book [14] that comments on some of the Fibonacci historical myths and completes some of the Fibonacci biological gaps, introduce the term "Parastichy numbers" to describe sequences associated with plant spirals which turn in two directions. The name seems to have its mathematical origin in the Bravais-Bravais theorem [14] where in a family containing n parastichies, on any parastichy, the numbers of each consecutive primordia differ by n. Numbers on each n-parastichy are congruent mod n. Each parastichy is considered a residue class. The word has lineage from the Greek para (against) and stichos (row).

Many applications of Fibonacci sequences have been made in different interdisciplinary fields, especially quaternions.

Quaternions were introduced by William Rowan Hamilton in 1843. They are widely utilized to control rotational movements in various fields, including chemistry [18], mechanics [19], 3D gaming [20], Eulerian angles [21], and kinematics [22]. In [23], Horadam introduced the concept of Complex Fibonacci and Fibonacci quaternions, uncovering a range of interesting properties.

The algebra of hyperbolic quaternions is an algebra that doesn't correspond to elements of the form over the real numbers.

$$q = xi_1 + yi_2 + zi_3 + ti_4, x, y, z, t \in \mathbb{R}$$

William Rowan Hamilton provided the characteristics of the q components outlined in Table 1.

Hyperbolic quaternions are a generalization of classical quaternions, used to better model certain physical and mathematical systems. They are especially useful in areas like special relativity and quantum mechanics, where classical quaternions may fall short. This structure arises by changing some basic assumptions about complex and quaternionic numbers.

In [24], Macfarlane did a lot of research on hyperbolic quaternions and their properties. An expression of the general form of hyperbolic quaternions is.

$s = s_1i_1 + s_2i_2 + s_3i_3 + s_4i_4 = (s_1, s_2, s_3, s_4)$.

Here, s_1, s_2, s_3, s_4 are the terms of the defined number and i_1, i_2, i_3, i_4 are hyperbolic quaternions. For recent studies on quaternions in relation to number sequences, refer to sources [25–29].

As shown above, numerous generalizations of the Fibonacci and Lucas sequences have been presented. In this study, we introduce new generalizations inspired by the Parastichy, Fibonacci, and Lucas sequences. We refer to these sequences as the Parastichy Fibonacci numbers and Parastichy Fibonacci-Lucas numbers, denoted as P_n and R_n, respectively.

Generalized Parastichy Fibonacci Numbers

In this section, we define the Parastichy Fibonacci numbers and Parastichy Fibonacci-Lucas numbers. Then we give some properties for these numbers.

We can define Parastichy Fibonacci numbers P_n, by $P_n = F_nF_{n+1}$, so that we can obtain Table 2. Sloane [17] refers to these as golden rectangle numbers, which suggests generalizations, though parastichy retains the Fibonacci biological links; see also Barry [14–16].

We notice immediately that if

$$B_n = P_n - P_{n-1} \tag{1}$$

then.

$B_n = F_nF_{n+1} - F_{n-1}F_n = F_n(F_{n+1} - F_{n-1}) = F_n^2$.

Now we define the recurrence relations of Parastichy Fibonacci numbers and Parastichy Fibonacci-Lucas numbers.

Definition 2.1 For $n \in \mathbb{N}$, the Parastichy Fibonacci numbers P_n and Parastichy Fibonacci-Lucas numbers R_n are defined by, respectively,

$$P_{n+3} = 2P_{n+2} + 2P_{n+1} - P_n \tag{2}$$

with $P_0 = 0$, $P_1 = 1$, and $P_2 = 2$

,

$$R_{n+3} = 2R_{n+2} + 2R_{n+1} - R_n, \tag{3}$$

with $R_0 = 3$, $R_1 = 2$, and $R_2 = 8$.

Then let's give some information about the equations of these numbers.

The characteristic equation of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers is

$$x^3 - 2x^2 - 2x + 1 = 0. \tag{4}$$

The roots of the characteristic equation are as follows;

$\lambda = \frac{3+\sqrt{5}}{2}$, $\psi = \frac{3-\sqrt{5}}{2}$, and $\delta = -1$.

Here, the λ number is the square of the known golden ratio (α) number.

Next, we give the relationships between these roots below:

$\lambda + \psi = 3$, $\lambda\psi = 2$, $\lambda + \psi + \delta = 2$, $\lambda\psi + \lambda\delta + \psi\delta = -2$, and $\lambda\psi\delta = -1$.

The first few values of Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers are

$$0, 1, 2, 6, 15, 40, 104, 273, 714, 1870, \ldots$$

$$3, 2, 8, 17, 48, 122, 323, 842, 2208, 5777, \ldots$$

In the following theorem, we express the Binet formulas of the Parastichy Fibonacci P_n and Parastichy Fibonacci-Lucas R_n numbers.

Theorem 2.1 *Let $n \in \mathbb{N}$. We obtain.*

i. $P_n = \frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^{n+1} + \frac{1}{(\psi-\lambda)(\psi-\delta)}\psi^{n+1} + \frac{1}{(\delta-\lambda)(\delta-\psi)}\delta^{n+1}$,

ii. $R_n = \lambda^n + \psi^n + \delta^n$.

Proof i The Binet form of a sequence is as follows.

$$P_n = x\lambda^n + y\psi^n + z\delta^n. \tag{5}$$

For these n values, we obtain;

$P_0 = x + y + z$,

$P_1 = x\lambda + y\psi + z\delta$,

$P_2 = x\lambda^2 + y\psi^2 + z\delta^2$.

We find.

$x = \frac{\lambda}{(\lambda-\psi)(\lambda-\delta)}$, $y = \frac{\psi}{(\psi-\lambda)(\psi-\delta)}$, and $z = \frac{\delta}{(\delta-\lambda)(\delta-\psi)}$.

Thus, we obtain.

$P_n = \frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^{n+1} + \frac{1}{(\psi-\lambda)(\psi-\delta)}\psi^{n+1} + \frac{1}{(\delta-\lambda)(\delta-\psi)}\delta^{n+1}$.

The other proof is similar. □

In the following theorem, a study is made on the convergence (ratio of the larger of two consecutive terms to the smaller one) of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers.

Theorem 2.2 *Let $n \in \mathbb{N}$. The following equations are satisfied:*

i. $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \lambda$,

ii. $\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = \lambda$.

Proof. i. If the Binet formula is used for proofs, we obtain.

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \frac{\frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^{n+2} + \frac{1}{(\psi-\lambda)(\psi-\delta)}\psi^{n+2} + \frac{1}{(\delta-\lambda)(\delta-\psi)}\delta^{n+2}}{\frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^{n+1} + \frac{1}{(\psi-\lambda)(\psi-\delta)}\psi^{n+1} + \frac{1}{(\delta-\lambda)(\delta-\psi)}\delta^{n+1}}.$$

If we use the properties of the characteristic equation of limit and Parastichy Fibonacci numbers, we get

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \frac{\frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^{n+2}}{\frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^{n+1}}$$

$= \lambda$.

The other proof is similar. □

Theorem 2.3 *The special generating functions for Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers are given as follows, respectively,*

i. $p(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-2x^2+x^3}$,

ii. $r(x) = \sum_{n=0}^{\infty} R_n x^n = \frac{3-4x-2x^2}{1-2x-2x^2+x^3}$,

iii. $\wp(x) = \sum_{n=0}^{\infty} \frac{P_n}{n!} x^n = \frac{\lambda}{(\lambda-\psi)(\lambda-\delta)} e^{\lambda x} + \frac{\psi}{(\psi-\lambda)(\psi-\delta)} e^{\psi x} + \frac{\delta}{(\delta-\lambda)(\delta-\psi)} e^{\delta x}$,

iv. $Re(x) = \sum_{n=0}^{\infty} \frac{R_n}{n!} x^n = e^{\lambda x} + e^{\psi x} + e^{\delta x}$.

Proof. i For the Parastichy Fibonacci numbers, we get.

$$p(x) = \sum_{n=0}^{\infty} P_n x^n = x + 2x^2 + \sum_{n=3}^{\infty} P_n x^n$$

$$= x + 2x^2 + 2\sum_{n=3}^{\infty} P_{n-1}x^n + 2\sum_{n=3}^{\infty} P_{n-2}x^n - \sum_{n=3}^{\infty} P_{n-3}x^n$$

$$= x + 2x^2 + 2x\sum_{n=2}^{\infty} Px^n + 2x^2\sum_{n=1}^{\infty} P_n x^n - x^3\sum_{n=0}^{\infty} P_n x^n.$$

Thus, we obtain.

$$p(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-2x^2+x^3}.$$

The other proofs are similar. □

Next, we obtain special summation formulas of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers.

Theorem 2.4 *Let $n \in \mathbb{N}$. We obtain.*

i. $\sum_{s=0}^{n} P_s = \frac{3P_n + P_{n-1} - P_{n-2} - 1}{2}$,
ii. $\sum_{s=0}^{n} R_s = \frac{3R_n + R_{n-1} - R_{n-2} + 3}{2}$.

Proof. ii From the definition of the Parastichy Fibonacci-Lucas sequence, we obtain.

$$R_3 = 2R_2 + 2R_1 - R_0,$$
$$R_4 = 2R_3 + 2R_2 - R_1,$$

$$R_n = 2R_{n-1} + 2R_{n-2} - R_{n-3}.$$

So, we have.

$$-13 + \sum_{s=0}^{n} R_s = 2\sum_{s=2}^{n-1} R_s + 2\sum_{s=1}^{n-2} R_s - \sum_{s=0}^{n-3} R_s,$$

$$-13 + \sum_{s=0}^{n} R_s = 2\left(-R_n - 5 + \sum_{s=0}^{n} R_s\right) + 2\left(-R_n - R_{n-1} - 3 + \sum_{s=0}^{n} R_s\right)$$

$$-\left(-R_n - R_{n-1} - R_{n-2} + \sum_{s=0}^{n} R_s\right)$$

Thus, we obtain.

$$\sum_{s=0}^{n} R_s = \frac{3R_n + R_{n-1} - R_{n-2} + 3}{2}.$$

The other proof is similar. □

In the following theorem, we associate the terms of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers with matrices. In addition, we calculate the Simson formulas of these numbers.

Theorem 2.5 *Let $n \in \mathbb{N}$. The following equations are true:*

i. *For the Parastichy Fibonacci numbers,*

1. $\begin{bmatrix} P_{n+2} \\ P_{n+1} \\ P_n \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$,

2. $\begin{bmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$,

3. $\begin{bmatrix} P_{n+2} \\ P_{n+1} \\ P_n \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{n+1} \\ P_n \\ P_{n-1} \end{bmatrix}$.

ii. *For the Parastichy Fibonacci-Lucas numbers,*

1. $\begin{bmatrix} R_{n+2} \\ R_{n+1} \\ R_n \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix}$,

2. $\begin{bmatrix} R_n \\ R_{n+1} \\ R_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix}^n \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}$,

3. $\begin{bmatrix} R_{n+2} \\ R_{n+1} \\ R_n \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_{n+1} \\ R_n \\ R_{n-1} \end{bmatrix}$.

Proof. i. 1 Let us show the proof by induction over n. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P_{n+1} \\ P_n \\ P_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} P_{n+2} \\ P_{n+1} \\ P_n \end{bmatrix}.$$

From the last equation, for n, it can be seen that the equality is true.

The other proofs are similar. □

Theorem 2.6 *Let $n \in \mathbb{N}$. The following equation is satisfied:*

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} P_{n+1} & 2P_n - P_{n-1} & -P_n \\ P_n & 2P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & 2P_{n-2} - P_{n-3} & -P_{n-2} \end{bmatrix}.$$

Proof Let us show the proof by induction over n. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain.

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_n & 2P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & 2P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & 2P_{n-3} - P_{n-4} & -P_{n-3} \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_{n+1} & 2P_n - P_{n-1} & -P_n \\ P_n & 2P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & 2P_{n-2} - P_{n-3} & -P_{n-2} \end{bmatrix}.$$

From the last equation, for n, it can be seen that the equality is true. □

Theorem 2.7. *(Simson Formulas)*

Let $n \in \mathbb{N}$. The following equations are satisfied:

i. $$det \begin{bmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \end{bmatrix} = (-1)^n,$$ ***ii.*** $$det \begin{bmatrix} R_{n+2} & R_{n+1} & R_n \\ R_{n+1} & R_n & R_{n-1} \\ R_n & R_{n-1} & R_{n-2} \end{bmatrix} = (-1)^n\, 125.$$

Proof. i Let us show the proof by induction over n. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain.

$$det \begin{bmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \end{bmatrix} = det \begin{bmatrix} 2P_{n+1} + 2P_n - P_{n-1} & P_{n+1} & P_n \\ 2P_n + 2P_{n-1} - P_{n-2} & P_n & P_{n-1} \\ 2P_{n-1} + 2P_{n-2} - P_{n-3} & P_{n-1} & P_{n-2} \end{bmatrix}$$

$$= det \begin{bmatrix} 2P_{n+1} & P_{n+1} & P_n \\ 2P_n & P_n & P_{n-1} \\ 2P_{n-1} & P_{n-1} & P_{n-2} \end{bmatrix} +$$

$$det \begin{bmatrix} 2P_n & P_{n+1} & P_n \\ 2P_{n-1} & P_n & P_{n-1} \\ 2P_{n-2} & P_{n-1} & P_{n-2} \end{bmatrix} +$$

$$det \begin{bmatrix} -P_{n-1} & P_{n+1} & P_n \\ -P_{n-2} & P_n & P_{n-1} \\ -P_{n-3} & P_{n-1} & P_{n-2} \end{bmatrix}$$

$$= (-1)\, det \begin{bmatrix} P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \\ P_{n-1} & P_{n-2} & P_{n-3} \end{bmatrix}$$

$$= (-1)^n.$$

From the last equation, for n, it can be seen that the equality is true.

The other proofs are similar. □

Special Relations

In this section, we examine the relationships between the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers. In addition, we associate Parastichy Fibonacci numbers with special number sequences.

In the following theorem, we examine the relations between the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers.

Theorem 3.1 *Let $n \in \mathbb{N}$. The following equations are true:*

i. $$P_n = \tfrac{1}{25}\left(R_{n+2} + 2R_{n+1} - 4R_n\right),$$ ***ii.*** $$R_n = 2P_{n+2} - P_{n+1} - 8P_n.$$

Proof. i The following relation is used for proofs;

$$P_n = a \times R_{n+2} + b \times R_{n+1} + c \times R_n. \tag{6}$$

For these n values, we obtain;

$P_0 = a \times R_2 + b \times R_1 + c \times R_0,$
$P_1 = a \times R_3 + b \times R_2 + c \times R_1,$
$P_2 = a \times R_4 + b \times R_3 + c \times R_2.$
We find.
$a = \frac{1}{25}$, $b = \frac{2}{25}$, and $c = -\frac{4}{25}$.
Thus, we obtain.
$P_n = \frac{1}{25}\left(R_{n+2} + 2R_{n+1} - 4R_n\right).$
The other proofs are similar. □

Theorem 3.2 *Let $n \in \mathbb{N}$. The following equation is satisfied:*

$$P_n R_n = P_{2n} - (-1)^n P_{-n}.$$

Proof Let $P_n = \frac{\lambda}{(\lambda-\psi)(\lambda-\delta)}\lambda^n + \frac{\psi}{(\psi-\lambda)(\psi-\delta)}\psi^n + \frac{\delta}{(\delta-\lambda)(\delta-\psi)}\delta^n = X\lambda^n + Y\psi^n + Z\delta^n.$

If Binet formulas are used for proofs, we obtain

$$(-1)^n P_{-n} = (\lambda^n \psi^n \delta^n)\left(X\lambda^{-n} + Y\psi^{-n} + Z\delta^{-n}\right)$$

$$= X\psi^n\delta^n + Y\lambda^n\delta^n + Z\lambda^n\psi^n$$

and.

$\frac{1}{2}\left(R_n^2 - R_{2n}\right) = \frac{1}{2}$
$\left[\left(\lambda^n + \psi^n + \delta^n\right)^2 - \left(\lambda^{2n} + \psi^{2n} + \delta^{2n}\right)\right]$

$= \lambda^n\psi^n + \lambda^n\delta^n + \psi^n\delta^n.$

Thus,

$$P_nR_n = \left(X\lambda^n + Y\psi^n + Z\delta^n\right)\left(\lambda^n + \psi^n + \delta^n\right)$$

$$= X\lambda^{2n} + Y\psi^{2n} + Z\delta^{2n} + \left(X + Y + Z\right)\lambda^n\psi^n$$

$$+ \left(X + Y + Z\right)\lambda^n\delta^n + \left(X + Y + Z\right)$$

$$\psi^n\delta^n - \left(X\lambda^n\delta^n + Y\lambda^n\delta^n + Z\lambda^n\psi^n\right)$$

$$= X\lambda^{2n} + Y\psi^{2n} + Z\delta^{2n} +$$

$$\left(X + Y + Z\right)\left(\lambda^n\psi^n + \lambda^n\delta^n + \psi^n\delta^n\right)$$

$$- \left(X\psi^n\delta^n + Y\lambda^n\delta^n + Z\lambda^n\psi^n\right).$$

Since $X + Y + Z = P_0 = 0$. So, we obtain.

$P_nR_n = P_{2n} - (-1)^n P_{-n}$. □

Theorem 3.3 *Let $m, n \in \mathbb{N}$. The following equations are satisfied:*

i
$$P_{n+m} = P_nP_{m+1} + P_{n+1}\left(2P_m - P_{m-1}\right) - P_{n-2}P_m,$$

i ***i***
$$R_{n+m} = R_nP_{m+1} + R_{n-1}\left(2P_m - P_{m-1}\right) - R_{n-2}P_m.$$

Proof. i Let us show the proof by induction over m. For $m = 1$, the following equations are obtained:

$$P_{n+1} = P_nP_{2+1} + P_{n+1}\left(2P_1 - P_{1-1}\right) - P_{n-2}P_1$$

$$= 2P_n + 2P_{n-1} - P_{n-2}.$$

Then by assumption, for $m = k$, $m = k + 1$, and $m = k + 2$, we get, respectively,

$P_{n+k} = P_nP_{k+1} + P_{n+1}\left(2P_k - P_{k-1}\right) - P_{n-2}P_k,$

$P_{n+k+1} = P_nP_{k+1+1} + P_{n+1}$
$\left(2P_{k+1} - P_{k+1-1}\right) - P_{n-2}P_{k+1},$

$P_{n+k+2} = P_nP_{k+2+1} + P_{n+1}$
$\left(2P_{k+2} - P_{k+2-1}\right) - P_{n-2}P_{k+2}.$

Now, after multiplying both sides of these three equations by 2; 2; -1; respectively, we get

$$2P_{n+k+2} + 2P_{n+k+1} - P_{n+k}$$

$$= 2\left(P_nP_{k+2+1} + P_{n+1}\left(2P_{k+2} - P_{k+2-1}\right) - P_{n-2}P_{k+2}\right)$$

$$+2\left(P_nP_{k+1+1} + P_{n+1}\left(2P_{k+1} - P_{k+1-1}\right) - P_{n-2}P_{k+1}\right)$$

$$- \left(P_nP_{k+1} + P_{n+1}\left(2P_k - P_{k-1}\right) - P_{n-2}P_k\right)$$

$$= P_n\left(2P_{k+3} + 2P_{k+2} - P_{k+1}\right) + P_{n+1}\left(4P_{k+2} + 4P_{k+1} - 2P_k - \left(2P_{k+1} + 2P_k - P_{k-1}\right)\right)$$

$$-P_{n-2}\left(2P_{k+2} + 2P_{k+1} - P_k\right).$$

$$= P_nP_{k+4} + P_{n+1}\left(2P_{k+3} - 2P_{k+2}\right) - P_{n-2}P_{k+3}$$

$$= P_{n+k+3}.$$

From the last equality, it is seen that the relation is also valid for $n = k + 3$, and the proof is finished.

The other proof is similar. □

Theorem 3.4 *Let $n \in \mathbb{N}$. Parastichy Fibonacci numbers are obtained with the help of Fibonacci and Lucas numbers with the following relations.*

i. $P_n = \frac{1}{5}\left(L_{2n+1} + (-1)^n\right),$
ii. $P_n = \frac{1}{5}\left(F_{2n}\left(2 - F_{2n}\right) + F_{2n+1}\left(F_{2n-1} + 1\right)\right).$

Proof. i The following equations can be written for the proof:

$$P_n = F_nF_{n+1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} =$$

$$\frac{\alpha^{2n+1} + \beta^{2n+1} - \alpha^n\beta^{n+1} - \alpha^{n+1}\beta^n}{\alpha^{2n+1} + \beta^{(2n+1)}\left(\alpha - \beta\right)}$$

$$= \frac{-\alpha^n\beta^n\left(\alpha + \beta\right)}{(\alpha-\beta)(\alpha-\beta)}.$$

Since, $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$, $\alpha + \beta = 1$, and $L_n = \alpha^n + \beta^n$. We obtain.

$P_n = \frac{1}{5}\left(L_{2n+1} + (-1)^n\right)$.

The other proof is similar. □

Application

In this section, we present the application of generalized Parastichy Fibonacci numbers to hyperbolic quaternions. Then we define hyperbolic Parastichy Fibonacci, and Parastichy Fibonacci-Lucas quaternions. For these quaternions, we give many properties such as generating functions and Binet formulas. Finally, the terms of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers are associated with their hyperbolic quaternion values.

Definition 4.1 For $n \in \mathbb{N}$, the hyperbolic Parastichy Fibonacci quaternions $\check{H}P_n$, and hyperbolic

Parastichy Fibonacci-Lucas quaternions $\check{H}R_n$ are defined by, respectively,

$$\begin{aligned}\check{H}P_n &= P_n i_1 + P_{n+1} i_2 + P_{n+2} i_3 \\ &\quad + P_{n+3} i_4 = (P_n, P_{n+1}, P_{n+2}, P_{n+3}),\end{aligned} \tag{7}$$

$$\begin{aligned}\check{H}R_n &= R_n i_1 + R_{n+1} i_2 + R_{n+2} i_3 \\ &\quad + R_{n+3} i_4 = (R_n, R_{n+1}, R_{n+2}, R_{n+3}).\end{aligned} \tag{8}$$

where P_n is n^{th} Parastichy Fibonacci numbers, R_n is n^{th} Parastichy Fibonacci-Lucas numbers, and i_1, i_2, i_3, i_4 are the hyperbolic quaternion units in Table 1.Let us now give the first four terms of the $\check{H}P_n$ and $\check{H}R_n$, respectively,

- $\check{H}P_0 = i_2 + 2i_3 + 6i_4$,
- $\check{H}P_1 = i_1 + 2i_2 + 6i_3 + 15i_4$,
- $\check{H}P_2 = 2i_1 + 6i_2 + 15i_3 + 40i_4$,
- $\check{H}P_3 = 6i_1 + 15i_2 + 40i_3 + 104i_4$,
- $\check{H}R_0 = 3i_1 + 2i_2 + 8i_3 + 17i_4$,
- $\check{H}R_1 = 2i_1 + 8i_2 + 17i_3 + 48i_4$,
- $\check{H}R_2 = 8i_1 + 17i_2 + 48i_3 + 122i_4$,
- $\check{H}R_3 = 17i_1 + 48i_2 + 122i_3 + 323i_4$.

Definition 4.2 For $n \in \mathbb{N}$, the conjugate of hyperbolic Parastichy Fibonacci quaternions $\check{H}P_n^*$, and conjugate hyperbolic Parastichy Fibonacci-Lucas quaternions $\check{H}R_n^*$ are defined by, respectively,

$$\begin{aligned}\check{H}P_n^* &= P_n i_1 - P_{n+1} i_2 - P_{n+2} i_3 \\ &\quad - P_{n+3} i_4 = (P_n, -P_{n+1}, -P_{n+2}, -P_{n+3}),\end{aligned} \tag{9}$$

and

$$\begin{aligned}\check{H}R_n^* &= R_n i_1 - R_{n+1} i_2 - R_{n+2} i_3 \\ &\quad - R_{n+3} i_4 = (R_n, -R_{n+1}, -R_{n+2}, -R_{n+3}).\end{aligned} \tag{10}$$

Definition 4.3 For $n \in \mathbb{N}$, the norms of the hyperbolic Parastichy Fibonacci quaternions $\check{H}P_n$, and norms of the hyperbolic Parastichy Fibonacci-Lucas quaternions $\check{H}R_n$ are defined by, respectively,

$$\check{H}P_n = \sqrt{P_n^2 + P_{n+1}^2 + P_{n+2}^2 + P_{n+3}^2}, \tag{11}$$

and

$$\check{H}R_n = \sqrt{R_n^2 + R_{n+1}^2 + R_{n+2}^2 + R_{n+3}^2}. \tag{12}$$

In the following theorem, the Binet formulas of the $\check{H}P_n$, $\check{H}P_n^*$, $\check{H}R_n$, and $\check{H}R_n^*$ quaternions are expressed.

Theorem 4.1 *Let $n \in \mathbb{N}$. We obtain.*

$$\begin{aligned}\check{H}P_n &= \frac{1}{(\lambda-\psi)(\lambda-\delta)}\bar{\lambda}\lambda^{n+1} \\ \textbf{\textit{i.}} \qquad &+ \frac{1}{(\psi-\lambda)(\psi-\delta)}\bar{\psi}\psi^{n+1}, \\ &+ \frac{1}{(\delta-\lambda)(\delta-\psi)}\bar{\delta}\delta^{n+1}, \\ \check{H}P_n^* &= \frac{1}{(\lambda-\psi)(\lambda-\delta)}\bar{\lambda}^*\lambda^{n+1} \\ \textbf{\textit{ii.}} \qquad &+ \frac{1}{(\psi-\lambda)(\psi-\delta)}\bar{\psi}^*\psi^{n+1}, \\ &+ \frac{1}{(\delta-\lambda)(\delta-\psi)}\bar{\delta}^*\delta^{n+1},\end{aligned}$$

iii. $\check{H}R_n = \bar{\lambda}\lambda^n + \bar{\psi}\psi^n + \bar{\delta}\delta^n$,
iv. $\check{H}R_n^* = \bar{\lambda}^*\lambda^n + \bar{\psi}^*\psi^n + \bar{\delta}^*\delta^n$,
where.
$\bar{\lambda} = i_1 + \lambda i_2 + \lambda^2 i_3 + \lambda^3 i_4 = (1, \alpha, \alpha^2, \alpha^3)$, $\bar{\lambda}^* = (1, -\lambda, -\lambda^2, -\lambda^3)$,
$\bar{\psi} = i_1 + \psi i_2 + \psi^2 i_3 + \psi^3 i_4 = (1, \psi, \psi^2, \psi^3)$, $\bar{\psi}^* = (1, -\psi, -\psi^2, -\psi^3)$.

$\bar{\Delta} = i_1 + \delta i_2 + \delta^2 i_3 + \delta^3 i_4 = (1, \delta, \delta^2, \delta^3)$, *and*

$\bar{\delta}^* = (1, -\delta, -\delta^2, -\delta^3)$.

Proof. i The Binet form of the hyperbolic Parastichy Fibonacci quaternions is.

$\check{H}P_n = a\frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^{n+1} + b\frac{1}{(\psi-\lambda)(\psi-\delta)}\psi^{n+1} + c\frac{1}{(\delta-\lambda)(\delta-\psi)}\delta^{n+1}$.

With the initial conditions, the following equations are obtained.

$$\check{H}P_0 = i_2 + 2i_3 + 6i_4 = a\frac{1}{(\lambda-\psi)(\lambda-\delta)}$$

$$\lambda + b\frac{1}{(\psi-\lambda)(\psi-\delta)}\psi + c\frac{1}{(\delta-\lambda)(\delta-\psi)}\delta$$

$\check{H}P_1 = i_1 + 2i_2 + 6i_3 + 15i_4 = a\frac{1}{(\lambda-\psi)(\lambda-\delta)}\lambda^2 + b\frac{1}{(\psi-\lambda)(\psi-\delta)}\psi^2 + c\frac{1}{(\delta-\lambda)(\delta-\psi)}\delta^2$,

Thus, we have

$$a = i_1 + \lambda i_2 + \lambda^2 i_3 + \lambda^3 i_4 = \bar{\lambda}$$

$$b = i_1 + \psi i_2 + \psi^2 i_3 + \psi^3 i_4 = \bar{\psi},$$

$c = i_1 + \delta i_2 + \delta^2 i_3 + \delta^3 i_4 = \bar{\delta}$.

So, we obtain.

$H\,P_n = \frac{1}{(\lambda-\psi)(\lambda-\delta)}\bar{\lambda}\lambda^{n+1} + \frac{1}{(\psi-\lambda)(\psi-\delta)}\bar{\psi}\psi^{n+1} + \frac{1}{(\delta-\lambda)(\delta-\psi)}\bar{\delta}\delta^{n+1}$.

The other proofs are similar. □

In the following theorems, we examine the relations between the $\check{H}P_n$, $\check{H}P_n^*$, $\check{H}R_n$, and $\check{H}R_n^*$ quaternions.

Theorem 4.2 *Let $n \in \mathbb{N}$. The following equations are true:*

i. $\check{H}P_{n+3} = 2\check{H}P_{n+2} + 2\check{H}P_{n+1} - \check{H}P_n$,
ii. $\check{H}P^*_{n+3} = 2\check{H}P^*_{n+2} + 2\check{H}P^*_{n+1} - \check{H}P^*_n$,
iii. $\check{H}R_{n+3} = 2\check{H}R_{n+2} + 2\check{H}R_{n+1} - \check{H}R_n$,
iv. $\check{H}R^*_{n+2} = 2\check{H}R^*_{n+2} + 2\check{H}R^*_{n+1} - \check{H}R^*_n$.

Proof. i If the definition is used, we have.

$$2\check{H}P_{n+2} + 2\check{H}P_{n+1} - \check{H}P_n$$

$$= 2\left(P_{n+2}i_1 + P_{n+3}i_2 + P_{n+4}i_3 + P_{n+5}i_4\right)$$

$$+2\left(P_{n+1}i_1 + P_{n+2}i_2 + P_{n+3}i_3 + P_{n+4}i_4\right)$$

$$-\left(P_n i_1 + P_{n+1}i_2 + P_{n+2}i_3 + P_{n+3}i_4\right)$$

$$= \left(2P_{n+2} + 2P_{n+1} - P_n\right)i_1 + \left(2P_{n+3} + 2P_{n+2}\right.$$

$$\left.-P_{n+1}\right)i_2 + \left(2P_{n+4} + 2P_{n+3} - P_{n+2}\right)i_3$$

$$+\left(2P_{n+5} + 2P_{n+4} - P_{n+3}\right)i_4.$$

Since $P_{n+3} = 2P_{n+2} + 2P_{n+1} - P_n$. Thus, we obtain.

$\check{H}P_{n+3} = 2\check{H}P_{n+2} + 2\check{H}P_{n+1} - \check{H}P_n$.

The other proofs are similar. □

Theorem 4.3 *Let $n \in \mathbb{N}$. The following equations are satisfied:*

i. $\check{H}P_n = \frac{1}{25}(H\,R_{n+2} + 2\check{H}R_{n+1} - 4\,H\,R_n)$,
ii. $\check{H}R_n = 2\check{H}P_{n+2} - \check{H}P_{n+1} - 8\check{H}P_n$,
iii. $\check{H}P^*_n = \frac{1}{25}\left(\check{H}R^*_{n+2} + 2\check{H}R^*_{n+1} - 4\check{H}R^*_n\right)$,
iv. $\check{H}R^*_n = 2\check{H}P^*_{n+2} - \check{H}P^*_{n+1} - 8\check{H}P^*_n$.

Proof. i The following relation is used for proofs;

$$\check{H}P_n = m \times \check{H}R_{n+2} + n \times \check{H}R_{n+1} + p \times \check{H}R_n. \tag{13}$$

For these n values, we obtain;

$\check{H}P_0 = m \times \check{H}R_2 + n \times \check{H}R_1 + p \times \check{H}R_0$,
$\check{H}P_1 = m \times \check{H}R_3 + n \times \check{H}R_2 + p \times \check{H}R_1$,
$\check{H}P_2 = m \times \check{H}R_4 + n \times \check{H}R_3 + p \times \check{H}R_2$.

We find.

$m = \frac{1}{25}$, $n = \frac{2}{25}$, and $p = -\frac{4}{25}$.

Thus, we obtain.

$\check{H}P_n = \frac{1}{25}\left(\check{H}R_{n+2} + 2\check{H}R_{n+1} - 4\check{H}R_n\right)$.

The other proofs are similar. □

In the following theorems, we associate the terms of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers with their hyperbolic quaternion values.

Theorem 4.4 *We obtain.*

i. $\check{H}P_n + \check{H}P^*_n = 2P_n i_1$,
ii. $\check{H}P_n - \check{H}P^*_n = 2P_{n+1}i_2 + 2P_{n+2}i_3 + 2P_{n+3}i_4$,
iii. $\check{H}P^2_n = 2P_n\check{H}P_n + \left(\check{H}P^2_n - 2P^2_n\right)i_1$,
iv. $\check{H}R^2_n = 2R_n\check{H}R_n + \left(\check{H}R^2_n - 2R^2_n\right)i_1$,
v. $\check{H}R_n + \check{H}R^*_n = 2R_n i_1$,
vi. $\check{H}R_n - \check{H}R^*_n = 2R_{n+1}i_2 + 2R_{n+2}i_3 + 2R_{n+3}i_4$.

Proof. ii If the definition is used, we have.

$$\check{H}P^2_n = \left(P_n i_1 + P_{n+1}i_2 + P_{n+2}i_3 + P_{n+3}i_3\right)$$

$$\left(P_n i_1 + P_{n+1}i_2 + P_{n+2}i_3 + P_{n+3}i_4\right)$$

$$= \left(P^2_n + P^2_{n+1} + P^2_{n+2} + P^2_{n+3}\right)i_1$$

$$+\left(2P_nP_{n+1}\right)i_2 + \left(2P_nP_{n+2}\right)i_3 + \left(2P_nP_{n+3}\right)i_4$$

$$= 2P_n\left(P_n i_1 + P_{n+1}i_2 + P_{n+2}i_3 + P_{n+3}i_4\right)$$
$$+\left(-P^2_n + P^2_{n+1} + P^2_{n+2} + P^2_{n+3}\right)i_1.$$

Thus, we obtain.

$\check{H}P^2_n = 2P_n\,HP_n + \left(\check{H}P^2_n - 2P^2_n\right)i_1$.

The other proofs are similar. □

Theorem 4.5 *We obtain.*

i. $i_1\check{H}P_n - i_2\check{H}P_{n+1} - i_3\check{H}P_{n+2} - i_4\check{H}P_{n+3} = \left(P_n - P_{n+2} - P_{n+4} + P_{n+6}\right)i_1$.

ii. $i_1\check{H}R_n - i_2\check{H}R_{n+1} - i_3\check{H}R_{n+2} - i_4\check{H}R_{n+3} = \left(R_n - R_{n+2} - R_{n+3} + R_{n+4} + 2R_{n+5}\right)i_1$.

Proof. i If the definition is used, we have.

$$i_1\check{H}P_n - i_2\check{H}P_{n+1} - i_3\check{H}P_{n+2} - i_4\check{H}P_{n+3} = \left(P_n - P_{n+2} - P_{n+4} + P_{n+6}\right)i_1$$

$$+\left(P_{n+1} - P_{n+1} - P_{n+5} + P_{n+5}\right)i_2 + \left(P_{n+2} + P_{n+4} - P_{n+2} - P_{n+4}\right)i_3$$

$$+\left(P_{n+3}-P_{n+3}+P_{n+3}-P_{n+3}\right)i_4$$

$$=\left(P_n-P_{n+2}-P_{n+4}+P_{n+6}\right)i_1$$

The other proof is similar. □

Theorem 4.6 *The special generating functions for hyperbolic Parastichy Fibonacci and Parastichy Fibonacci-Lucas quaternions are given as follows, respectively,*

i. $P(x) = \sum_{n=0}^{\infty} \check{H}P_n x^n =$

$\frac{1}{1-2x-2x^2+x^3}\left(x, 1, 2+2x+x^2, 6+3x+2x^2\right)$,

ii. $R(x) = \sum_{n=0}^{\infty} R_n x^n.$

$= \frac{1}{1-2x-2x^2+x^3}$

$\left(3-4x-2x^2, 2+4x-3x^2, 8+x-2x^2, 17+4x-8x^2\right)$

Proof. i For the hyperbolic Parastichy Fibonacci quaternions, we get.

$$P(x) = \sum_{n=0}^{\infty} \check{H}P_n x^n$$

$$= \check{H}P_0 + \check{H}P_1 x + \check{H}P_2 x^2 + \sum_{n=3}^{\infty} \check{H}P_n x^n$$

$$= \check{H}P_0 + \check{H}P_1 x + \check{H}P_2 x^2 + 2\sum_{n=3}^{\infty} \check{H}P_{n-1}x^n + 2\sum_{n=3}^{\infty} \check{H}P_{n-2}x^n - \sum_{n=3}^{\infty} \check{H}P_{n-3}x^n$$

$$= \check{H}P_0 + \check{H}P_1 x + \check{H}P_2 x^2$$
$$+2x\sum_{n=2}^{\infty} \check{H}P x^n + 2x^2 \sum_{n=1}^{\infty} \check{H}P_n x^n - x^3 \sum_{n=0}^{\infty} \check{H}P_n x^n.$$

Thus, we obtain.

$$P(x) = \sum_{n=0}^{\infty} \check{H}P_n x^n$$

$$= \frac{1}{1-2x-2x^2+x^3}\left(x, 1, 2+2x+x^2, 6+3x+2x^2\right).$$

The other proof is similar. □

In the following theorem, we obtain special summation formulas of the hyperbolic Parastichy Fibonacci and Parastichy Fibonacci-Lucas quaternions.

Theorem 4.7 *Let $n \in \mathbb{N}$. We obtain.*

i. $\sum_{s=0}^{n} \check{H}P_s = \frac{3\check{H}P_n + \check{H}P_{n-1} - \check{H}P_{n-2} - (1,1,3,7)}{2}$,

ii. $\sum_{s=0}^{n} \check{H}R_s = \frac{3R_n + R_{n-1} - R_{n-2} + (3,-3,-7,-23)}{2}$.

Proof. ii From the definition of the hyperbolic Parastichy Fibonacci-Lucas quaternions, we obtain.

$$\check{H}R_3 = 2\check{H}R_2 + 2\check{H}R_1 - \check{H}R_0,$$
$$\check{H}R_4 = 2\check{H}R_3 + 2\check{H}R_2 - \check{H}R_1,$$

$$\check{H}R_n = 2\check{H}R_{n-1} + 2\check{H}R_{n-2} - \check{H}R_{n-3}.$$

So, we have.

$$-\check{H}R_0 - \check{H}R_1 - \check{H}R_2 + \sum_{s=0}^{n} \check{H}R_s$$
$$= 2\sum_{s=2}^{n-1} \check{H}R_s + 2\sum_{s=1}^{n-2} \check{H}R_s - \sum_{s=0}^{n-3} \check{H}R_s,$$

Thus, we obtain.

$$\sum_{s=0}^{n} \check{H}R_s = \frac{3R_n + R_{n-1} - R_{n-2} + (3,-3,-7,-23)}{2}.$$

The other proof is similar. □

Conclusion

In this study, we introduce the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers and explore their fundamental properties, including their characteristic equations. We derive the Binet formulas, special generating functions, and specific summation formulas for these numbers. Additionally, we establish a connection between the terms of these numbers and matrices. We also compute the Simson formulas for these numbers. Furthermore, we investigate the relationships between the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers and their connections to other notable number sequences, such as the Lucas numbers. We highlight significant correlations between these numbers. We gave a presentation on the use of generalized Parastichy Fibonacci numbers in the context of hyperbolic quaternions. In this presentation, we outlined several properties of these hyperbolic quaternions. Finally, we explored the terms of the Parastichy Fibonacci and Parastichy Fibonacci-Lucas numbers related to their corresponding hyperbolic quaternion values. With the results and applications of this study, the Parastichy Fibonacci numbers of different number sequences can be applied, such as Parastichy Jacobsthal numbers and Parastichy Pell numbers. Furthermore, interesting results can be obtained by using different polynomial properties [30–32].

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