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Hyperbolic functions obtained from k-Pell sequences

Bahar Kuloğlu[1] · **Engin Özkan**[2]

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Abstract
In this paper, an expansion of the classical hyperbolic functions is presented and studied. Also, many features of the $k-$ Pell hyperbolic functions are given. Finally, some graph and curved surfaces related to the k $-$ Pell hyperbolic functions are introduced.

Keywords $k-$Pell sequence · $k-$Pell number · Pell hyperbolic function · Metallic Shofars

Mathematics subject classification 2020 · 11B37 · 11B39 · 11B83

1 Introduction

The simplest and most used sequences of numbers are Fibonacci sequences. It is a sequence obtained by adding the last two terms consecutively with the initial terms 0 and 1, usually denoted by F_n. That is, $\{F_n\} = \{0, 1, 1, 2, 3, 5, \dots\}$. Fibonacci numbers and their generalizations and polynomials have many exciting properties and applications in different branches of science, art, nature [8–10, 15, 19, 20]. $\phi = (1 + \sqrt{5})/2$, the convergence of the ratio of two consecutive Fibonacci numbers, called the Golden Mean, or Golden Section appears in many studies in many fields [4, 14]. Pell sequence are also studied by many researchers [1–3, 7, 11, 16, 18]. Many generalizations of the hyperbolic functions are given [5, 6, 17]. Also, Golden Section and its applications in many areas are given [13]. One of the most recent works in this area is [5] where new k-Fibonacci hyperbolic functions were presented and studied. In this study, we want to show that k-Pell sequences form a very close relationship between classical hyperbolic functions and k-Pell sequences by adapting many properties of k-Pell sequences to k-Pell hyperbolic functions while this relationship is exactly the same in some cases, in some cases also the coefficients are different, but still, they are similar.

✉ Engin Özkan
eozkanmath@gmail.com; eozkan@erzincan.edu.tr

Bahar Kuloğlu
bahar_kuloglu@hotmail.com

[1] Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Erzincan, Türkiye

[2] Faculty of Arts and Sciences, Erzincan Binali Yıldırım University, Erzincan, Türkiye

2 New k-Pell hyperbolic functions

Definition 1 For k $\geq$ 1 the k− Pell sequence is defined by:
$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$ for with $P_{k,0} = 0$ and $P_{k,1} = 1$.

The characteristic equation for k-Pell numbers is as follows:

$$r^2 - 2r - k = 0 \tag{1}$$

Binet's formula for the $k-$-Pell numbers is given by [2]:

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$$

where $r_1 = 1 + \sqrt{k+1}$ and $r_2 = 1 - \sqrt{k+1}$ are the roots of Eq. 1 and $r_1 > r_2$.

We know that, $r_1 + r_2 = 2$, $r_1.r_2 = -k$ and $r_1 - r_2 = 2\sqrt{k+1}$.

The Pell hyperbolic functions like in [5], in analogy with the known classical hyperbolic numbers, are defined as

$$sPhx = \frac{\alpha^{2x} - \alpha^{-2x}}{2\sqrt{2}},$$
$$cPhx = \frac{\alpha^{2x+1} + \alpha^{-(2x+1)}}{2\sqrt{2}},$$

where sPh and cPh are called, respectively, the Pell hyperbolic sine and cosine.

The k-Pell hyperbolic functions of above functions as follows:

$$sP_kh(x) = \frac{\alpha_k^{2x} - k^{2x}\alpha_k^{-2x}}{2\sqrt{1+k}},$$
$$cP_kh(x) = \frac{\alpha_k^{2x+1} + k^{2x+1}\alpha_k^{-(2x+1)}}{2\sqrt{1+k}},$$

where $\alpha_k = 1 + \sqrt{1+k}$. If $x = 2n$ then $sP_kh(x) = P_{k,4n}$, while if $x = 2n+1$ then $cP_kh(x) = P_{k,2n+1}$. k-Pell hyperbolic sine and cosine functions are defined as follows:

$$sP_kh(x) = \frac{\alpha_k^{x} - k^{x}\alpha_k^{-x}}{\alpha_k + k\alpha_k^{-1}},$$
$$cP_kh(x) = \frac{\alpha_k^{x} + k^{x}\alpha_k^{-x}}{\alpha_k + k\alpha_k^{-1}},$$

since $\alpha_k + k(\alpha_k)^{-1} = 2\sqrt{1+k}$.

The graphics for new k-Pell hyperbolic functions are shown in Fig. 1a, b

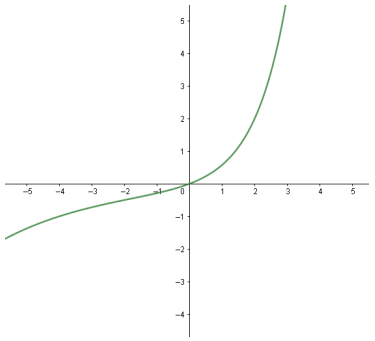
From last definitions, it holds that $sP_kh(2n) = P_{k,2n}$ and $cP_kh(2n+1) = P_{k,2n+1}$, as it can be easily confirmed by using $\alpha_k^{-2n} = (\frac{-1}{\alpha_k})^{2n}$ and $\alpha_k^{-(2n+1)} = -(\frac{-1}{\alpha_k})^{2n+1}$.

After that, we will substitute α for α_k to simplify.

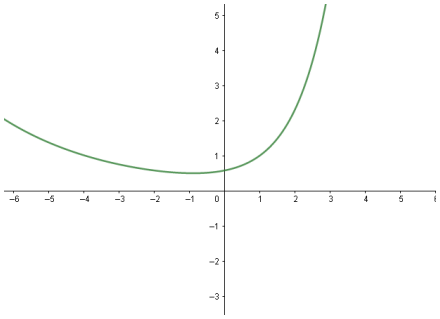
2.1 Properties of the $sPhx$ and $cPhx$ functions

Proposition 1 *The basic feature of these functions, which can be called a version of the Pythagorean trigonometric identity, is*

$$(cP_kh(x))^2 - (sP_kh(x))^2 = \frac{k^x}{1+k}. \tag{2}$$



(a) sine k-Pell hyperbolic functions $sPh(x)$



(b) cosine k-Pell hyperbolic functions $cPh(x)$

Fig. 1 k-Pell hyperbolic functions

Proposition 2 *We have the following sum and difference identity*

$$cP_kh(x+y) = \sqrt{k+1}(cP_kh(x).cP_kh(y) + sP_kh(x).sP_kh(y),$$

$$cP_kh(x-y) = \sqrt{k+1}(cP_kh(x).cP_kh(y) - sP_kh(x).sP_kh(y),$$

$$sP_kh(x+y) = \sqrt{k+1}(sP_kh(x).cP_kh(y) + cP_kh(x).sP_kh(y),$$

$$sP_kh(x-y) = \sqrt{k^2+8}(sP_kh(x).cP_kh(y) - cP_kh(x).sP_kh(y).$$

Proof We will now prove the first equality. The proof of the others is similar to the proof of the first equality.

$$\begin{aligned} cP_kh(x).cP_kh(y) &+ sP_kh(x).sP_kh(y) \\ &= \frac{(\alpha_k^x + k^x\alpha_k^{-x})(\alpha_k^y + k^y\alpha_k^{-y}) + (\alpha_k^x - k^x\alpha_k^{-x})(\alpha_k^y - k^y\alpha_k^{-y})}{\alpha_k + k\alpha_k^{-1}} \\ &= \frac{1}{\sqrt{k+1}} cP_kh(x+y). \end{aligned}$$

By making $y = x$ in the previous first and third formulas, we get the following result. □

Corollary 3 *Double variable formulas*

$$cP_kh(2x) = \sqrt{k+1}[(cP_kh(x))^2 + (sP_kh(x))^2] \tag{3}$$

$$sP_kh(2x) = 2\sqrt{k+1}sP_kh(x)cP_kh(x) \tag{4}$$

Corollary 4 *Half variable formulas*

$$(cP_kh(x))^2 = \frac{1}{2\sqrt{k+1}}\left[cP_kh(2x) + \frac{k^x}{\sqrt{k+1}}\right],$$

$$(sP_kh(x))^2 = \frac{1}{\sqrt{k+1}}\left[cP_kh(2x) - \frac{k^x}{\sqrt{k+1}}\right].$$

2.2 The $sPhx$ and $cPhx$ functions and the $k-$ Pell numbers

Proposition 5 We have the recursive relation

$$sP_kh(x+1) = 2cP_kh(x) + ksP_kh(x-1).$$

Proof

$$\begin{aligned} 2cP_kh(x) + ksP_kh(x-1) &= 2\frac{\left(\alpha^x + k^x\alpha^{-x}\right)}{\alpha + k\alpha^{-1}} + k\frac{\left(\alpha^{x-1} - k^{x-1}\alpha^{-x+1}\right)}{\alpha + k\alpha^{-1}} \\ &= \frac{1}{\alpha + k\alpha^{-1}}\left(\alpha^x\left(2 + \frac{k}{\alpha}\right) + k^x\alpha^{-x}(2-\alpha)\right) \\ &= \frac{1}{\alpha + k\alpha^{-1}}(\alpha^{x+1} - k^{x+1}\alpha^{-x-1}) \end{aligned}$$

Since $\alpha = 2 + \frac{k}{\alpha}$ and $2 - \alpha = \frac{-k}{\alpha}$, we have $\frac{\alpha^{x+1} - k^{x+1}\alpha^{-x-1}}{\alpha + k\alpha^{-1}} = sP_kh(x+1)$ □

Proposition 6 Catalan's identity for the $k-$ Pell hyperbolic functions

$$cP_kh(x-s).cP_kh(x+s) - (cP_kh(x))^2 = k^{x-s}(sP_kh(s))^2.$$

Proof By definition $cP_kh(x)$,

$$\begin{aligned} &cP_kh(x-s).cP_kh(x+s) - (cP_kh(x))^2 \\ &= \frac{(\alpha^{x-s} + k^{x-s}\alpha^{-x+s})(\alpha^{x+s} + k^{x+s}\alpha^{-x-s}) - (\alpha^x + k^x\alpha^{-x})^2}{(\alpha + k\alpha^{-1})^2} \\ &= \frac{k^{x+s}\alpha^{-2s} + k^{x-s}\alpha^{2s} - k^{x+1}}{(\alpha + k\alpha^{-1})^2} \\ &= \frac{k^{x-s}(\alpha^{2s} + \alpha^{-2s}k^{2s} - 2k^s)}{(\alpha + k\alpha^{-1})^2} \\ &= k^{x-s}(sP_kh(s))^2. \end{aligned}$$

□

Corollary 7 $cP_kh(x-s)cP_kh(x+s) - (sP_kh(x))^2 = k^{x-s}(cP_kh(s))^2,$

$$sP_kh(x-s)sP_kh(x+s) - (sP_kh(x))^2 = -k^{x-s}(sP_kh(s))^2,$$

$$sP_kh(x-s)sP_kh(x+s) - (cP_kh(x))^2 = -k^{x-s}(cP_kh(s))^2.$$

If we put $s = 1$ into classical Catalan's identity, then we obtain the Cassini's identity for the $k-$ Pell numbers [2]:

$$P_{k,n-1}P_{k,n+1} - P_{k,n}^2 = (-1)^n k^{n-1} P_{k,1}$$

The suitable identity for the $k-$ Pell hyperbolic functions are given below.

Proposition 8 d'Ocagne's identity for the $k-$ Pell numbers [2] is $P_{k,m}P_{k,n+1} - P_{k,m+1}P_{k,n} = (-1)^n k^n P_{k,m-n}$ if $m > n$.

For the $k-$ Pell hyperbolic functions, Cassini's identity are given as follows:

$$cP_kh(x-1)cP_kh(x+1) - (sP_kh(x))^2 = k^{x-1},$$

$$sP_kh(x-1)sP_kh(x+1) - (sP_kh(x))^2 = -k^{x-1}.$$

Proposition 9 For the $k-$ Pell hyperbolic functions, d'Ocagne's identity is

$$cP_kh(x)cP_kh(y+s) - sP_kh(x+s)sP_kh(y) = k^y cP_kh(s)cP_kh(x-y) \tag{5}$$

$$cP_kh(x)sP_kh(y+s) - cP_kh(x+s)sP_kh(y) = k^y sP_kh(s)cP_kh(x-y) \tag{6}$$

Proof From Eq. 5 we have:

$$\begin{aligned} &cP_kh(x)cP_kh(y+s) - sP_kh(x+s)sP_kh(y) \\ &= \frac{(\alpha^x + k^x\alpha^{-x})(\alpha^{y+s} + k^{y+s}\alpha^{-y-s}) - (\alpha^{x+s} - k^{x+s}\alpha^{-x-s})(\alpha^y - k^y\alpha^{-y})}{(\alpha + k\alpha^{-1})^2} \\ &= \frac{k^{y+s}\alpha^{x-y-s} + k^x\alpha^{-x+y+s} + k^y\alpha^{x-y+s} + k^{x+s}\alpha^{-x+y-s}}{(\alpha + k\alpha^{-1})^2} \\ &= \frac{\alpha^s + k^s\alpha^{-s}}{\alpha + k\alpha^{-1}} . \frac{k^y\alpha^{x-y} + k^x\alpha^{-x+y}}{\alpha + k\alpha^{-1}} = k^y cP_kh(s)cP_kh(x-y) \end{aligned}$$

Equation 6 can be obtained as in Eq. 5. □

3 The quasi-sine k-Pell functions and the k-Pell numbers

Binet's formula for the $k-$ Pell sequence has the form $P_{k,n} = \frac{\alpha^n - (-k)^n\alpha^{-n}}{\alpha + k\alpha^{-1}}$. Also, we have given the $k-$ Pell hyperbolic sine function as $sP_kh(x) = \frac{\alpha^x - k^x\alpha^{-x}}{\alpha + k\alpha^{-1}}$. So, if x is an even integer number we have $sP_kh(x) = P_{k,n}$. Therefore, we give the following definition.

Definition 2 The quasi-sine $k-$ Pell function is denoted by $PP_k(x)$ and

$$PP_k(x) = \frac{\alpha^x - k^x\cos(\pi x)\alpha^{-x}}{\alpha + k\alpha^{-1}}.$$

Notice that $PP_k(n) = P_{k,n}$ for $n \in \mathbf{Z}$. So the points of the form $(n,\ P_{k,n})$ concern to the graphics of these functions. For $k = 1,\ 2,\ 3,\ 26.8$ the graphics of $PP_k(x)$ are given in Fig. 2. Here for $k = 26.8$ and $x \geq 0$ $k-$ Pell hyperbolic function has the first extremum point.

While functions $PP_k(n)$ and $P_{k,n}$ are similar in many properties, they differ considerably in the values taken by the relative extremum points and the number of these extremes. For instance, if x takes positive values, then it is seen that $PP_{26.8}(x)$ present an only maximum at $x = 1.549695\ldots$ and an only minimum at $x = 1.649169\ldots$ while for values of $k < 26.8$ function $PP_k(x)$ does not have relative extremes for $x > 0$. Also, for $x > 0$, the number of maximum and minimum stay steady when k increases. For $k = 0.01$ in Fig. 3 shows the graphic of $PP_k(x)$.

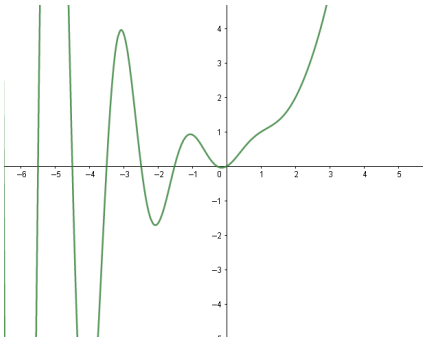
For $x < 0$, the number of maximum and minimum the infinite. For instance, the first minimum for $x < 0$ and $k = 1$ is at $x = -0.1531744\ldots$ and for $k = 3$ is at $x = -0.10129103\ldots$.

In Fig. 4, for k = 1.2 it is possible to see the tangent curves obtained along the values taken by the $PP_k(x)$ function. Attention that the tangent points have the form $(n,\ P_{k,n})$ for $n \in \mathbf{Z}$.

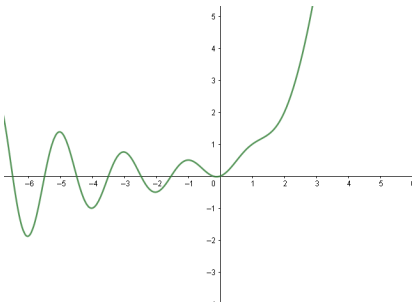
For these quasi-sine k—Pell functions, there are several identities that are versions of the analogues identities of k—Pell numbers [2]. Some of them are revealed here.

Theorem 10 (Recursive relation)

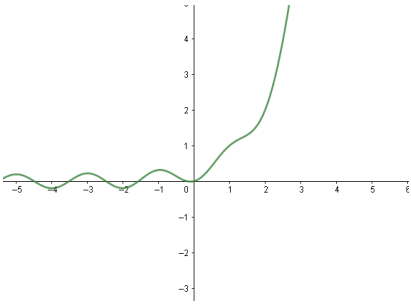
$$PP_k(x+2) = 2PP_k(x+1) + kPP_k(x).$$



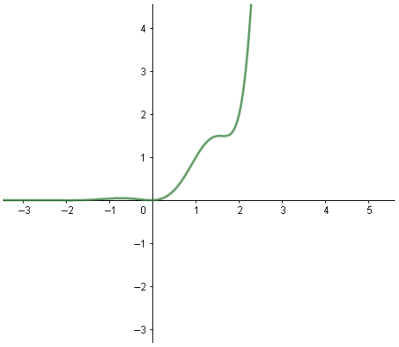
(a) $y = PP_1(x)$



(b) $y = PP_2(x)$



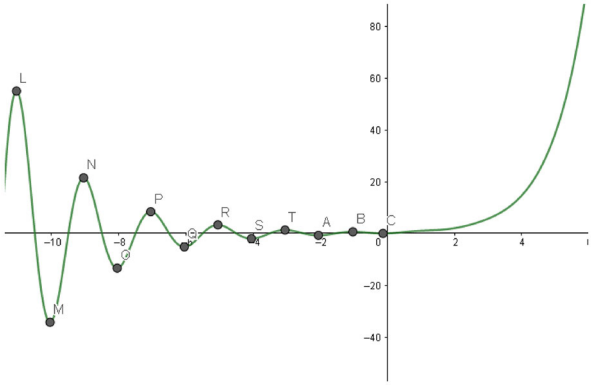
(c) $y = PP_3(x)$

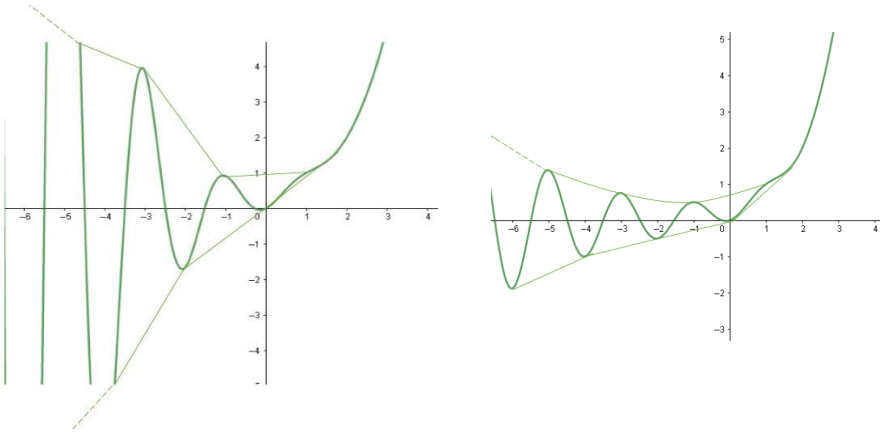


(d) $y = PP_{26.8}(x)$

Fig. 2 The quasi-sine $k-$ Pell functions, $PP_k(x)$ for $k = 1, 2, 3, 26.8$

Fig. 3 The quasi-sine 0.01-Pell function





(a) $y = PP_1(x)$ (b) $y = PP_2(x)$

Fig. 4 The quasi-sine k-Pell functions for $k = 1, 2$ with their improving tangent curves

Proof

$$\begin{aligned}
&2PP_k(x+1) + kPP_k(x)\\
&= 2\frac{\alpha^{x+1} - k^{x+1}cos(\pi(x+1))\alpha^{-x-1}}{\alpha + k\alpha^{-1}} + k\frac{\alpha^x - k^x cos(\pi x)\alpha^{-x}}{\alpha + k\alpha^{-1}}\\
&= \frac{\alpha^x(2\alpha + k) + cos(\pi x)\alpha^{-x-2}k^{x+1}(2\alpha - \alpha^2)}{\alpha + k\alpha^{-1}}\\
&= \frac{\alpha^{x+2} - cos(\pi x)\alpha^{-x-2}k^{x+2}}{\alpha + k\alpha^{-1}}\\
&= PP_k(x+2)
\end{aligned}$$

Because $2/\alpha - 1 = -k/\alpha^2$. □

Theorem 11 (Catalan's identity) If $s \in \boldsymbol{Z}$, then:

$$PP_k(x+s).PP_k(x-s) - (PP_k(x))^2 = k^{x-s}(-1)^{s+1}\cos(\pi x)(PP_k(s))^2.$$

Proof

$$\begin{aligned}
&PP_k(x+s)\ .PP_k(x-s) - (PP_k(x))^2\\
&= \left(\frac{\alpha^{x+s} - k^{x+s}cos(\pi(x+s))\alpha^{-x-s}}{\alpha + k\alpha^{-1}}\right)\\
&\left(\frac{\alpha^{x-s} - k^{x-s}cos(\pi(x-s))\alpha^{-x+s}}{\alpha + k\alpha^{-1}}\right) - \left(\frac{\alpha^x - k^x cos(\pi x)\alpha^{-x}}{\alpha + k\alpha^{-1}}\right)^2\\
&= k^{x-s}\left(\frac{(-1)^{s+1}cos(\pi x)\alpha^{2s} + k^{2s}\alpha^{-2s}(-1)^{s+1}cos(\pi x) + k^{s+1}cos(\pi x)}{(\alpha + k\alpha^{-1})^2}\right)\\
&= (-1)^{s+1}cos(\pi x)k^{x-s}\left(\frac{\alpha^{2s} + k^{2s}\alpha^{-2s} - k^{s+1}cos(\pi s)}{(\alpha + k\alpha^{-1})^2}\right)\\
&= (-1)^{s+1}cos(\pi x)k^{x-s}\left(\frac{\alpha^s - k^s cos(\pi s)\alpha^{-s}}{\alpha + k\alpha^{-1}}\right)^2\\
&= (-1)^{s+1}cos(\pi x)k^{x-s}(PP_k(s))^2
\end{aligned}$$

□

If x is an integer n in the previous result then Catalan's identity for the $k-$-Pell numbers appears. Moreover, if $s = 1$, then we get: $PP_k(x+1).PP_k(x-1)-(PP_k(x))^2 = cos(\pi x)k^{x-1}$.

If we take $x = n$, then we find Cassini's identity [?] for the $k-$ Pell numbers:

Theorem 12 (Asymptotic quotient). *We have*
$\lim_{x\to\infty} \frac{PP_k(x+s)}{PP_k(x)} = \alpha^s$ where s is any integer.

Proof For $\alpha > 1$, then:

$$lim_{x\to\infty} \frac{PP_k(x+s)}{PP_k(x)} = lim_{x\to\infty} \frac{\alpha^s - k^{x+s}cos(\pi(x+s))\alpha^{-2x-s}}{1 - k^x cos(\pi(x))\alpha^{-2x}} = \alpha^s$$

Specialy, if $k = 1$, then $\alpha = r_1$ and $\lim_{x\to\infty} \frac{PP_k(x+1)}{PP_k(x)} = r_1$. □

4 3D curved surfaces for the $sPhx$ and $cPhx$ functions

Definition 3 The complex valued function given as follows is denoted by $CPP_k(x)$ and named the three-dimensional k− Pell spirals:

$$CPP_k(x) = \frac{\alpha^x - k^x \cos(\pi x)\alpha^{-x}}{\alpha + k\alpha^{-1}} + i\frac{k^x \sin(\pi x)\alpha^{-x}}{\alpha + k\alpha^{-1}}.$$

$\Re(CPP_k(x)) = PP_k(x)$ is the quasi-sine $k-$ Pell function. The shape of the three-$k-$ Pell spiral recalls a three-dimensional spiral. Function $CPP_k(x)$ can be given as:

$$CPP_k(x) = \frac{\alpha^x + k^x i e^{i\pi(1/2-x)}\alpha^{-x}}{\alpha + k\alpha^{-1}}$$

We give some of the properties for $CPP_k(x)$ as follow:

• If $k = 1$, then we obtain the three-dimensional Pell spiral.$CPP_1(x) = CPP_k(x) = \frac{r^x + k^x i e^{i\pi(1/2-x)}r^{-x}}{2\sqrt{2}}$

• If x is an integer then, the imaginery part of the three-dimensional $k-$ Pell spiral is zero, and hence $CPP_k(n)$ results in Binet's formula for the general term of the $k-$ Pell sequence [2].

• Lastly, for every k, it is $CPP_k(0) = 0 \text{and} CPP_k(1) = 1$.

Theorem 13 The three-dimensional k− Pell spiral confirm the following Pell-type recurrence relation:

$$CPP_k(x+2) = 2CPP_k(x+1) + kCPP_k(x).$$

Proof Since $\alpha > 1$ and $2\alpha + k = \alpha^2$, so $\alpha - 2 = k/\alpha$, we have:

$$\begin{aligned} &2CPP_k(x+1) + kCPP_k(x) \\ &= 2\frac{\alpha^{x+1} + k^{x+1} i e^{i\pi(1/2-x-1)}\alpha^{-x-1}}{\alpha + k\alpha^{-1}} + k\frac{\alpha^x + k^x i e^{i\pi(1/2-x)}\alpha^{-x}}{\alpha + k\alpha^{-1}} \\ &= \frac{\alpha^{x+2} + ie^{-i\pi x}\alpha^{-x-1}k^{x+1}(-k+\alpha)i}{\alpha + k\alpha^{-1}} \\ &= \frac{\alpha^{x+2} + k^{x+2} i e^{i\pi(1/2-x-2)}\alpha^{-x-2}}{\alpha + k\alpha^{-1}} = CPP_k(x) \end{aligned}$$

□

Since we have showed, we can give Catalan's identity for the three-dimensional $k-$ Pell spiral in the next theorem.

Theorem 14 Catalan's identity for the three-dimensional k− Pell spiral is as follows:

$$CPP_k(x-s).CPP_k(x+s)-(CPP_k(x))^2=(-1)^{x+s+1}k^{x-s}(CPP_k(s))^2.$$

Proof Since $e^{i\pi(1/2-x)}=ie^{-i\pi x}$ and $e^{i\pi s}=(-1)^s$, $e^{i(\pi/2\pm\pi s)}=i(-1)^s$ and s is an integer. Therefore,

$$\begin{aligned}
&CPP_k(x-s).CPP_k(x+s)-(CPP_k(x))^2\\
&=\frac{\alpha^{x+s}+k^{x+s}ie^{i\pi(1/2-x-s)}\alpha^{-x-s}}{\alpha+k\alpha^{-1}}.\frac{\alpha^{x-s}+k^{x-s}ie^{i\pi(1/2-x+s)}\alpha^{-x+s}}{\alpha+k\alpha^{-1}}\\
&\quad-\left(\frac{\alpha^x+k^xie^{i\pi(1/2-x)}\alpha^{-x}}{\alpha+k\alpha^{-1}}\right)^2\\
&=\frac{\alpha^{2s}k^{x-s}ie^{i\pi(1/2-x+s)}+\alpha^{-2s}k^{x+s}ie^{i\pi(1/2-x-s)}-k^{s+1}ie^{i\pi(1/2-x)}}{(\alpha+k\alpha^{-1})^2}\\
&=\frac{k^{x-s}ie^{i\pi(1/2-x+s)}(\alpha^{2s}+\alpha^{-2s}k^{2s}e^{-2is\pi}-k^{s+1}e^{-is\pi})}{(\alpha+k\alpha^{-1})^2}\\
&=k^{x-s}ie^{i\pi(1/2-x+s)}CPP_k(s))^2=(-1)^{x+s+1}k^{x-s}(CPP_k(s))^2
\end{aligned}$$

where if $s=1$, then we obtain Cassini's identity:

$$CPP_k(x-1).CPP_k(x+1)-(CPP_k(x))^2=(-k)^{x-1}.$$

□

4.1 The shrinking incomplete metallic shofars

The three-dimensional $k-$ Pell spiral has real and imaginary parts as follow,

$$\Re(CPP_k(x))=\frac{\alpha^x-k^xcos(\pi x)\alpha^{-x}}{\alpha+k\alpha^{-1}}=y(x),$$
$$\Im(CPP_k(x))=\frac{k^xsin(\pi x)\alpha^{-x}}{\alpha+k\alpha^{-1}}=z(x).$$

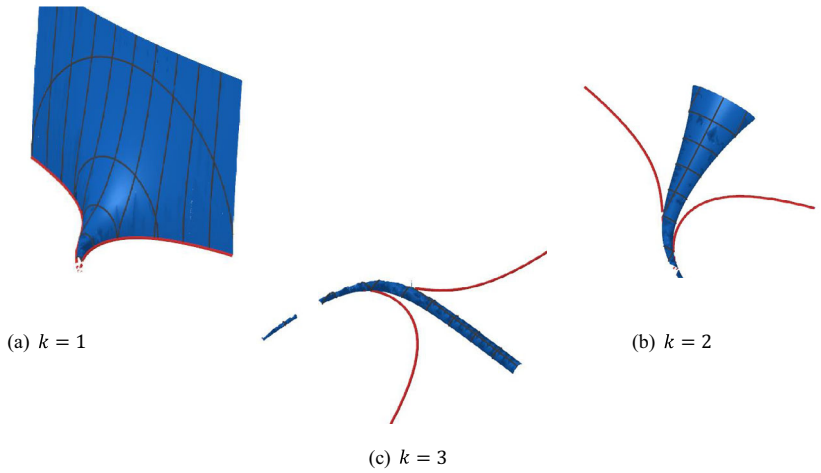
If we accept Y axis as real axis and Z axis as imaginary axis then we can give the following equations:

$$y(x)-\frac{\alpha^x}{\alpha+k\alpha^{-1}}=\frac{-k^xcos(\pi x)\alpha^{-x}}{\alpha+k\alpha^{-1}},$$
$$z(x)=\frac{k^xsin(\pi x)\alpha^{-x}}{\alpha+k\alpha^{-1}}.$$

By adding the squares of previous expressions and taking y and z as independent variables, the following equation is obtained:

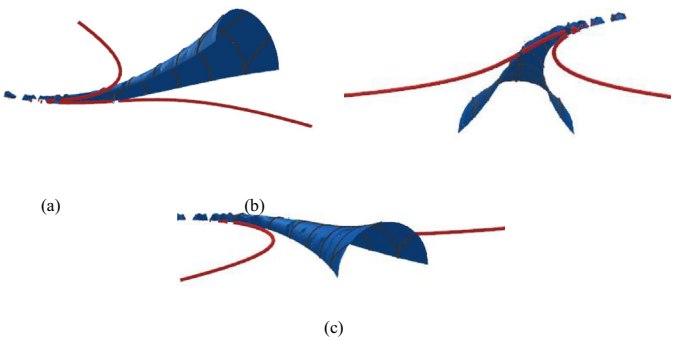
$$\left(y-\frac{\alpha^x}{\alpha+k\alpha^{-1}}\right)^2+z^2=\left(\frac{k^x\alpha^{-x}}{\alpha+k\alpha^{-1}}\right)^2 \tag{7}$$

For $k = 1$ Eq. 7 corresponds to a surface called the *Shrinking Incomplete Golden Shofar* with equation $(y - \frac{\alpha^x}{2\sqrt{2}})^2 + z^2 = (\frac{1}{8\alpha^{2x}})$. For $k = 2$, we find the *Shrinking Incomplete Silver Shofar* with equation $(y - \frac{\alpha^x}{2\sqrt{3}})^2 + z^2 = (\frac{2^{2x}}{12\alpha^{2x}})$. Finally, for $k = 3$, we get the *Shrinking Incomplete Bronze Shofar* with equation $(y - \frac{\alpha^x}{4})^2 + z^2 = (\frac{3^{2x}}{16\alpha^{2x}})$.We show the Shrinking Incomplete Metallic Shofar for $k = 1, 2, 3$ in Fig. 5 and the Shrinking incomplete Silver Shofar and its reflections in the coordinate planes in Fig. 6.



(a) $k = 1$ (b) $k = 2$ (c) $k = 3$

Fig. 5 The Shrinking Incomplete Metallic shofars for $k = 1, 2, 3$



(a) (b) (c)

Fig. 6 The Shrinking Incomplete Bronze Shofar and its reflections on the coordinate planes

5 Conclusions

In this paper, we have introduced the new $k-$ Pell hyperbolic functions connected with the $k-$ Pell sequences. Various properties for these functions have been derived and associated with similar identities for $k-$ Pell numbers. Moreover, the $k-$ Pell hyperbolic functions are used to present the alleged Shrinking Incomplete Metallic Shofars which expand the Shrinking Incomplete Silver Shofar.

Authors' contributions Both authors contributed to the study's conception and design. Material preparation, data collection and analysis were performed by Engin ÖZKAN. The first draft of the manuscript was written by Bahar KULOĞLU and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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Declarations

Conflict of interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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