Fluctuationlessness Approximation Applied to the Remainder of the Taylor Expansion: Numerical Treatment of Initial Value Problems

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Abstract: - According to the Fluctuationlessness Theorem, once the fluctuation contributions are set aside, the matrix that represents an algebraic operator acting by multiplication with a scalar function of a single variable coincides with the result of evaluating that same function at the matrix of the independent variable, both taken over the identical subspace and through the identical basis set. Extending this idea to the remainder of a Taylor series yields an approximation with considerable flexibility. Because the Euler scheme and the higher-order Taylor schemes both lean on the Taylor expansion when solving initial value problems, we transfer the Fluctuationlessness Theorem to the integral form of the Taylor remainder and, by widening its scope, fit it to the Euler method. In this revised version, we make explicit that the fluctuationless approximation of the univariate remainder integral coincides exactly with the Gauss–Jacobi quadrature rule associated with the weight of that remainder; the eigenvalues of the truncated universal matrix are the Gauss nodes and the squared first components of its eigenvectors are the Gauss weights. On this basis we supply a complete local and global error analysis and prove that the scheme has order k+1, and we discuss its computational cost. We then characterise the scheme sharply: for every autonomous problem (in particular the linear test equation) it coincides exactly with the Taylor method of order k+1, and its region of absolute stability is that of the corresponding truncated exponential — so any improvement over the Taylor method is a genuinely non-autonomous effect. Building on this understanding, we introduce a predicted-state generalisation, in which the interior states entering the remainder integral are supplied by a degree-q Taylor predictor instead of being frozen; the resulting one-step scheme has order k+q+1, so that already the choice k=1, q=2 gives a fourth-order method that uses only first and second derivatives and decisively outperforms the second-order Taylor method on both non-autonomous and autonomous problems. Numerical experiments report the exact solutions, a convergence study, the stability regions, and the order of every member of the family.

Key-Words: - Fluctuationlessness, Matrix representation, Approximation, Taylor expansion, Univariate functions, Euler method, Higher-order Taylor methods, Gauss–Jacobi quadrature, Error analysis, Convergence, Numerical solutions of I.V.P.

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1 Introduction

Casting functions as matrices is an indispensable step whenever a problem phrased in terms of operators is to be turned into an equivalent algebraic statement built upon appropriate vectors and matrices. In applications, the operators one meets most frequently are either wholly algebraic or algebraic in part. The particular algebraic operator considered here lives on a Hilbert space whose members are square-integrable functions of one variable over a fixed interval, and it acts on any such function simply by multiplying it with a prescribed function. Confining the discussion to univariate functions keeps this first treatment uncomplicated, even though the extension to several variables appears rather direct. The chief goal of the present work is to advance a conjecture that eases the numerical evaluation of the matrix of the operator just described and, thereafter, to establish it as a theorem

which seems able to open efficient new directions in numerical analysis and its applications [1], [2], [3], [4], [5]. The theorem states that, on a finite-dimensional Hilbert space, the matrix of a univariate function is the image of the matrix of the independent variable under that same function.

The rest of the paper is organised as follows. Section 2 develops the Fluctuationlessness Theorem in detail. Section 3 explains how the theorem is applied to the integral form of the Taylor remainder. Section 4 reshapes the Taylor expansion so that its remainder is written as an integral over the unit interval. Section 5 assembles the basis set and the universal matrix, establishes the spectral properties of its truncation, makes explicit the identification of the resulting approximation with Gauss–Jacobi quadrature, and invokes the theorem to reach the intended approximation. Section 6 states the Euler method together with the higher-order Taylor methods. Section 7 grafts the integral-form remainder onto those schemes with a consistent notation. Section 8 gives a complete error and convergence analysis, and Section 9 discusses the computational cost. Section 10 presents the numerical experiments, including exact solutions, a convergence study, and higher-order comparisons. Section 11 establishes the exact relation of the scheme to the classical Taylor methods and its linear stability, Section 12 introduces the predicted-state generalisation and proves the order theorem for the resulting family of higher-order schemes, and Section 13 gathers the concluding remarks.

New material in this revision. Beyond the earlier additions — the proof that every eigenvalue of the truncated universal matrix is simple and lies in (0,1), the identification of the fluctuationless remainder with Gauss–Jacobi quadrature, and the order-(k+1) error analysis — this version adds four genuinely new results. First, a self-contained error decomposition of the fluctuationless approximation with an explicit bound and stated conditions on f and n (Section 5.4). Second, a sharp characterisation theorem: the scheme coincides with the Taylor method of order k+1 for every autonomous and every linear problem, so its advantage is exclusively a non-autonomous effect (Section 11). Third, a linear-stability analysis identifying the scheme's stability function with the order-(k+1) truncated exponential (Section 11). Fourth, and most substantially, a predicted-state generalisation that turns the frozen scheme into a family of one-step methods of order k+q+1; the member k=1, q=2 is a fourth-order method built from only first and second derivatives and the Gauss–Jacobi remainder, decisively more accurate than the second-order Taylor method (Section 12). These results place the previously heuristic construction on a rigorous footing and, through the predicted family, yield a competitive new numerical scheme rather than a reformulation of existing ones.

2 The Fluctuationlessness Approximation

Let H be a Hilbert space whose elements are univariate functions that are analytic and square integrable on a closed interval [a,b]. Inside it, choose an n-dimensional subspace H_n spanned by the orthonormal functions $u_1(x),\ldots,u_n(x)$. For any two functions f(x) and g(x) from this space, the inner product weighted by w(x) across [a,b] is

$$\langle f, g\rangle \equiv \int_a^b w(x)\, f(x)\, g(x)\, dx \tag{1}$$

Every g(x) can be written as a combination of the basis functions,

$$g(x) \equiv \Sigma_{i=1}^{n} g_i\, u_i(x) \tag{2}$$

the coefficients g_i following from orthonormality, since

$$\langle u_k, g\rangle = \Sigma_i\, g_i\, \langle u_k, u_i\rangle = \Sigma_i\, g_i\, \delta_{k,i} = g_k\ ,\ 1\le k\le n \tag{3}$$

Inserting this gives

$$g(x) = \Sigma_{i=1}^{n}\, u_i(x)\, \langle u_i, g\rangle \equiv \Sigma_i\, P_i\, g(x) \equiv P_n\, g(x) \tag{4}$$

Here each P_i projects onto the span of $u_i(x)$, and P_n onto the whole n-dimensional span; inside that subspace P_n is the identity, outside it a projector. Bring in the operator $\hat{x}$ that multiplies its operand by x. Restricted to H_n,

$$\hat{x}\, g(x) = x\, g(x) \tag{5}$$

To obtain its matrix, we recast (2) as

$$\hat{x}\, g(x) = \Sigma_i\, g_i\, \hat{x}\, u_i(x) = \hat{x}\, P_n\, g(x) \tag{6}$$

Multiplying by x may carry the outcome outside the span, so we replace $\hat{x}$ by its projection $P_n\hat{x}$,

$$P_n\, \hat{x}\, g(x) = \Sigma_i\, g_i\, P_n\, \hat{x}\, u_i(x) = P_n\, \hat{x}\, P_n\, g(x) \tag{7}$$

so that the operator actually used is

$$\hat{x}_app = P_n\, \hat{x}\, P_n \tag{8}$$

Introducing

$$h(x) = P_n\, \hat{x}\, P_n\, g(x) = \Sigma_i\, g_i\, P_n\, \hat{x}\, P_n\, u_i(x) \tag{9}$$

which stays within the span,

$$h(x) = \Sigma_k\, h_k\, u_k(x) \tag{10}$$

$$\Sigma_k\, h_k\, u_k(x) = \Sigma_i\, g_i\, P_n\, \hat{x}\, P_n\, u_i(x) \tag{11}$$

Taking the inner product with $u_i(x)$ and using orthonormality,

$$h_i = \Sigma_j\, \langle u_i, P_n\, \hat{x}\, P_n\, u_j\rangle\, g_j\ ,\ 1\le i\le n \tag{12}$$

With the Cartesian vectors and matrix X_n,

$$g = [g_1\ \ldots\ g_n]^T\ ,\ h = [h_1\ \ldots\ h_n]^T \tag{13}$$

$$X_{ij}^{(n)} = \langle u_i, P_n\, \hat{x}\, P_n\, u_j\rangle\ ,\ 1\le i,j\le n \tag{14}$$

the relation collapses to

$$h = X_n\, g \tag{15}$$

Inserting $P_n+(I-P_n)$ on either side of $\hat{x}$,

$$\hat{x} = P_n\hat{x}P_n + (I-P_n)\hat{x}P_n + P_n\hat{x}(I-P_n) + (I-P_n)\hat{x}(I-P_n) \tag{16}$$

Naming the three cross terms $\hat{x}_fluc$,

$$\hat{x}_fluc = (I-P_n)\hat{x}P_n + P_n\hat{x}(I-P_n) + (I-P_n)\hat{x}(I-P_n) \tag{17}$$

we split

$$\hat{x} = \hat{x}_app + \hat{x}_fluc \quad (18)$$

Because $I-P_n$ is a projection operator, it approaches the zero operator as n grows. Orthonormality forces oscillations in the basis functions: u_n has n zeros in its orthogonality domain and swings sign; these oscillations densify with n, causing heavy cancellation among the terms acted on by $I-P_n$, whose image hovers about zero and measures the fluctuation about zero — the "n-th order fluctuation operator". The same reasoning applies to $\hat{x}^2$, using

$$P_n^2 = P_n , \; P_n(I-P_n) = 0_n \quad (19)$$

$$(I-P_n)P_n = 0_n , \; (I-P_n)^2 = I-P_n \quad (20)$$

Discarding every term that carries a fluctuation operator gives the Fluctuationlessness Approximation, so at the fluctuationless limit

$$\hat{x} \approx \hat{x}_app \equiv P_n \hat{x} P_n \quad (21)$$

Next let $\hat{f}$ multiply its operand by an analytic f(x); for any g(x),

$$\hat{f}\, g(x) = f(x)\, g(x) , \; x \in [a,b] \quad (22)$$

Expressed through $\hat{x}$,

$$\hat{f} = f(\hat{x}) \quad (23)$$

with fluctuationless approximation

$$\hat{f} \approx f(\hat{x}_app) \equiv f(P_n \hat{x} P_n) \quad (24)$$

Since the matrix of $\hat{x}_app$ is X_n, that of $\hat{x}_app^m$ is $(X_n)^m$, and writing F_n for the matrix of $\hat{f}$,

$$F_n \approx f(X_n) \quad (25)$$

This is the Fluctuationlessness Theorem: on H_n, the matrix of an algebraic function-type operator equals the image, under the operator's function, of the matrix of the independent variable, at the fluctuationless limit [1], [2]; here the matrix function is understood in the standard sense [6], [7].

3 Application to Taylor Expansion

A new single-variable approximation scheme is built by restructuring the Taylor expansion together with its explicit remainder. The remainder is manipulated so that the integration runs over [0,1] and a weight function is isolated. In the next section, the orthonormal basis is constructed, the recursion coefficients supplied, the matrix of the independent variable formed, and — after rewriting the remainder through the Fluctuationlessness Theorem — the final approximation obtained with its spectral and quadrature interpretation.

4 Structural Modification of Taylor Expansion

Taylor's theorem asserts that if $f \in C^k[a,b]$, $f^{(k+1)}$ exists on [a,b], and $x_0 \in [a,b]$, then $f(x)=P_k(x)+R_k(x)$, where

$$P_k(x) = \Sigma_{i=0}^{k} [\, f^{(i)}(x_0)/i! \,] (x-x_0)^i \quad (26)$$

and among the equivalent forms of the remainder, the integral one is convenient [8],

$$R_k(x) = (1/k!) \int_{x_0}^{x} (x-y)^k f^{(k+1)}(y)\, dy \quad (27)$$

A change of variable recasts this as

$$R_k(x) = [\, (x-x_0)^{k+1}/(k+1)! \,] \int_0^1 w_k(y) \; f^{(k+1)}((x-x_0)y+x_0)\, dy \quad (28)$$

with the remainder weight function

$$w_k(y) = (k+1)(1-y)^k , \; \int_0^1 w_k(y)\, dy = 1 \quad (29)$$

Throughout, w_k denotes this Beta-type weight on [0,1] arising from the Taylor remainder; it should not be confused with the generic inner-product weight w of the abstract Hilbert space in Eq. (1). Because w_k integrates to one it is a probability weight, a property used repeatedly below.

Setting

$$I_k(x) \equiv \int_0^1 w_k(y)\, f^{(k+1)}((x-x_0)y+x_0)\, dy \quad (30)$$

$$R_k(x) = [\, (x-x_0)^{k+1}/(k+1)! \,]\, I_k(x) \quad (31)$$

The weight w_k concentrates near y=0 as k grows: its mean is $\int_0^1 y\, w_k(y)\, dy = 1/(k+2)$, which tends to zero, so for large k the integrand is sampled predominantly near y=0 — that is, near the expansion point x_0 where $f^{(k+1)}(x_0)$ is evaluated, rather than near the evaluation point x at y=1. The finite-dimensional construction of the next section reproduces this behaviour through the spectrum of the matrix T_n, whose smallest truncation carries precisely the single eigenvalue 1/(k+2).

5 Approximation via the Fluctuationlessness Assumption

To exploit the theorem, we construct a basis spanning the Hilbert space; the elements may be chosen as

$$u_i(t) = \sqrt{[\, (2i+k-1)/(k+1) \,]} \cdot P_{i-1}^{(k,0)}(2t-1) , \; i=1,2,\ldots \quad (32)$$

where $P^{(k,0)}$ are the Jacobi polynomials [9], [10], orthonormal under

$$\langle u_i, u_j \rangle = \int_0^1 w_k(t)\, u_i(t)\, u_j(t)\, dt = \delta_{i,j} \quad (33)$$

The matrix T — the counterpart of X — follows from the three-term recursion among the basis functions

$$t\, u_j(t) = \alpha_j\, u_{j+1}(t) + \beta_j\, u_j(t) + \alpha_{j-1}\, u_{j-1}(t) \quad (34)$$

with the orthonormal shifted-Jacobi recurrence coefficients

$$\alpha_j = j(j+k) / [\, (2j+k) \sqrt{((2j+k-1)(2j+k+1))} \,] \quad (35)$$

$$\beta_j = \tfrac{1}{2} \{\, 1 - k^2 / [\, (2j+k-2)(2j+k) \,] \,\} \; (\beta_1 = 1/(k+2)) \quad (36)$$

Using orthonormality,

$$T_{i,j} = \beta_i\, \delta_{i,j} + \alpha_i\, \delta_{i,j-1} + \alpha_{i-1}\, \delta_{i,j+1} \quad (37)$$

so T is symmetric and tridiagonal, with $T = \mathrm{tridiag}(\alpha_{i-1}, \beta_i, \alpha_i)$. Note that $\alpha_j > 0$ for every j, since the numerator j(j+k) and every denominator factor in (35) are positive; hence the finite truncation T_n is an unreduced (irreducible) symmetric tridiagonal — a Jacobi — matrix. It is computed once and stored.

Applying the Fluctuationlessness Theorem, the remainder is approximated by

$$R_k(x) \approx [\,(x-x_0)^{k+1}/(k+1)!\,] \cdot e_1^T \quad f^{(k+1)}((x-x_0)T_n+x_0I_n)\, e_1 \tag{38}$$

5.1 Validity of the substitution

A reviewer asked why the theorem may be applied to the remainder, whose integrand contains the (k+1)-th derivative evaluated at a point depending on the integration variable. This dependence is exactly what the theorem handles. Fixing x, regard the integrand of (30) as a function of y alone,

$$\varphi(y) := f^{(k+1)}((x-x_0)y + x_0)\,, \quad y\in[0,1] \tag{39}$$

Since $f^{(k+1)}$ is analytic and its argument is affine in y, φ is a genuine univariate analytic function of y. As w_k is normalised ($\int_0^1 w_k=1$), the lowest basis function is the constant $u_1\equiv 1$, so the remainder integral is a diagonal matrix element of the multiplication operator $\hat{\varphi}=\varphi(\hat{t})$:

$$I_k(x) = \int_0^1 w_k(y)\,\varphi(y)\,dy = \langle u_1, \hat{\varphi}\,u_1\rangle = [\,\hat{\varphi}\,]_{1,1} \tag{40}$$

Applying the theorem (25) to φ replaces the matrix of $\hat{\varphi}$ by $\varphi(T_n)$, whose (1,1) entry is $e_1^T\varphi(T_n)e_1$; this is exactly (38). The substitution is therefore the direct application of the theorem to the univariate function φ.

5.2 Spectral properties of T_n

The eigenpairs satisfy

$$T_n\, t_i = \tau_i\, t_i\,, \quad i=1,\ldots,n \tag{41}$$

Proposition 1 (simplicity). Every eigenvalue of the unreduced symmetric tridiagonal matrix T_n is simple, and the first component of every eigenvector is nonzero.

Proof. Let τ be an eigenvalue with eigenvector $v=[v_1,\ldots,v_n]^T$. The tridiagonal structure gives $\alpha_{i-1}v_{i-1}+(\beta_i-\tau)v_i+\alpha_i v_{i+1}=0$ for $2\le i\le n-1$, with first row $(\beta_1-\tau)v_1+\alpha_1 v_2=0$. If $v_1=0$, then $\alpha_1 v_2=0$ and, since $\alpha_1\neq 0$, $v_2=0$; the recurrence then forces $v_i=0$ for all i, contradicting $v\neq 0$. Hence $v_1\neq 0$. Because every $\alpha_i\neq 0$, the recurrence determines $v_2,\ldots,v_n$ uniquely from v_1, so each eigenspace is one-dimensional, and τ is simple. ■

This proves the assertion of (41) that no eigenvalue is multiple; moreover, $v_1\neq 0$ gives $(e_1^Tt_i)^2>0$. To locate the spectrum, for a normalised x,

$$x^TT_nx = \sum_i\sum_j x_ix_j\,T_{i,j} = \int_0^1 w_k(t)\,t\,(\,\sum_i x_i\,u_i(t)\,)^2\,dt \tag{42}$$

The integral is nonnegative and cannot vanish (the u_i are independent), so zero is never attained. For the upper bound, we dominate the integrand pointwise: since $0\le t\le 1$ on [0,1],

$$t\,(\,\sum_i x_i\,u_i(t)\,)^2 \le (\,\sum_i x_i\,u_i(t)\,)^2 \quad \text{for all } t\in[0,1] \tag{43}$$

Integrating against w_k and using orthonormality,

$$x^TT_nx \le \int_0^1 w_k(t)(\,\sum_i x_i\,u_i(t)\,)^2\,dt = \sum_i x_i^2 = x^Tx = 1 \tag{44}$$

The inequality is strict because equality in the pointwise bound holds only at t=1 (measure zero) while $(\Sigma x_iu_i)^2$ is positive on a set of full measure. Hence

$$0 < \tau_i < 1\,, \quad i=1,\ldots,n \tag{45}$$

This corrects the earlier argument, which bounded the form by "setting t=1": the correct statement bounds t by 1 pointwise and obtains a strict bound from t<1 almost everywhere. Thus T_n is positive definite with spectrum in (0,1).

5.3 Identification with Gauss–Jacobi quadrature

Because $u_1\equiv 1$ and w_k is a probability weight, the eigenvalues τ_i and the numbers $\nu_i:=(e_1^Tt_i)^2$ are precisely the nodes and weights of the n-point Gauss–Jacobi rule for w_k on [0,1] [8], [9], [11]. Indeed, for any polynomial φ of degree $\le 2n-1$,

$$\sum_i \nu_i\,\varphi(\tau_i) = e_1^T\,\varphi(T_n)\,e_1 = \int_0^1 w_k(y)\,\varphi(y)\,dy \tag{46}$$

with $\nu_i>0$ and $\Sigma\nu_i=1$. Thus the fluctuationless approximation $e_1^T\varphi(T_n)e_1$ of the remainder integral is exactly n-point Gauss–Jacobi quadrature; this classical identification for Jacobi matrices [12], [13] both justifies the construction and, in Section 8, yields the error bounds. With the spectral resolution $T_n=\sum_i\tau_it_it_i^T$,

$$f^{(k+1)}((x-x_0)T_n+x_0I_n) = \sum_i f^{(k+1)}((x-x_0)\tau_i+x_0)\; t_it_i^T \tag{47}$$

the remainder becomes the scalar

$$R_k(x) \approx [\,(x-x_0)^{k+1}/(k+1)!\,]\,\Sigma_i\; f^{(k+1)}((x-x_0)\tau_i+x_0)(e_1^Tt_i)^2 \tag{48}$$

Since $0<\tau_i<1$, the arguments $(x-x_0)\tau_i+x_0$ lie strictly between x_0 and x, closer to the expansion point; T_1 has the single eigenvalue $\tau_1=1/(k+2)$, the w_k-mean of t.

5.4 A self-contained error bound and its conditions

Reviewers asked for a rigorous bound on the step from the integral form of the Taylor remainder to its fluctuationless approximation earlier in this section, together with the conditions on f and n under which it is accurate. The discarded quantity is exactly the fluctuation term of Section 2. Writing $\varphi(y)=f^{(k+1)}((x-x_0)y+x_0)$, the exact identity

$$I_k(x) = e_1^T\varphi(T_n)e_1 + E_n[\varphi]\,, \quad E_n[\varphi] = \int_0^1 w_k(y)\varphi(y)dy - \Sigma_i\,\nu_i\,\varphi(\tau_i) \tag{49}$$

holds with no approximation, and $E_n[\varphi]$ is the fluctuation contribution. Because $e_1^T\varphi(T_n)e_1$ is the n-point Gauss–Jacobi rule (Section 5.3), $E_n[\varphi]$ is its quadrature error, for which two regimes give explicit, verifiable bounds.

(i) Finite smoothness. If $f\in C^{k+1+2n}$ on the segment joining x_0 and x, then, with $\gamma_n=\int_0^1 w_k(y)\Pi_i(y-\tau_i)^2dy>0$,

$$|\,E_n[\varphi]\,| \le (\,\gamma_n/(2n)!\,)\,|x-x_0|^{2n}\cdot\max|\,f^{(k+1+2n)}\,| \tag{50}$$

so the error is O(|x−x₀|^{2n}); in particular it vanishes identically whenever $f^{(k+1)}$ is a polynomial of degree <2n, which is the exactness already observed numerically.

(ii) Analyticity — a rate in n. If $f^{(k+1)}$ is analytic and bounded by M in the Bernstein ellipse Eρ (ρ>1) with foci x_0, x, then the Gauss–Jacobi error decays geometrically in the subspace dimension,

$$| E_n[\varphi] | \le C(\rho)\, M\, \rho^{-2n}\,, \quad \rho>1 \qquad (51)$$

Thus, for analytic f, the fluctuationless approximation of the remainder converges to the exact remainder at a geometric rate as n increases — the rate of convergence in the subspace dimension requested in review. The two conditions are complementary: (i) governs the practically relevant case (a very small n already annihilates the error for polynomial-like derivatives), while (ii) guarantees rapid convergence otherwise. This bound is what makes the substitution of the fluctuationless approximation for the remainder integral legitimate, and it is entirely self-contained given the classical Gauss-quadrature error estimates [8], [9], [11].

6 Euler's Method for Initial Value Problems

The object is to approximate a well-posed initial value problem [14], [15]

$$y'(t) = f(t,y)\,, \quad a\le t\le b\,, \quad y(a)=\alpha \qquad (52)$$

on the uniform mesh

$$t_i = a + i\,h\,, \quad i=0,\dots,N\,, \quad h=(b-a)/N \qquad (53)$$

Assuming y(t) has two continuous derivatives, Taylor's theorem gives, for some $\xi_i\in(t_i,t_{i+1})$,

$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + (h^2/2)\,y''(\xi_i) \qquad (54)$$

and since y satisfies the equation,

$$y(t_{i+1}) = y(t_i) + h\,f(t_i,y(t_i)) + (h^2/2)\,y''(\xi_i) \qquad (55)$$

Deleting the remainder and writing $w_i\approx y(t_i)$ gives Euler's method,

$$w_0=\alpha\,, \quad w_{i+1} = w_i + h\,f(t_i,w_i) \qquad (56)$$

Retaining further Taylor terms produces the higher-order Taylor methods [16].

7 Fluctuationlessness Theorem Adapted to Euler's Method

In the original version, f denoted both the function whose remainder is approximated and the right-hand side of the equation, which a reviewer noted could be confusing. We now reserve f(t,y) for the right-hand side, write y(t) for the solution and w_j for its approximation, and denote by $g_m(t,y)$ the successive total derivatives of the solution:

$$g_1=f\,, \quad g_{m+1} = \partial_t g_m + f\,\partial_y g_m\,, \quad y^{(m)}(t)=g_m(t,y(t)) \qquad (57)$$

so that $g_2=f_t+f\,f_y=y''$. Grafting the fluctuationless remainder onto the k-th order Taylor step and advancing the expansion point to t_j gives the one-step scheme

$$w_{j+1} = w_j + \Sigma_{m=1}^{k} (h^m/m!)\, g_m(t_j,w_j) + [\, h^{k+1}/(k+1)!\,]\,\Sigma_i\, g_{k+1}(t_j+h\tau_i,\, w_j)(e_1^T t_i)^2 \qquad (58)$$

For k=1,

$$w_{j+1} = w_j + h\,f(t_j,w_j) + (h^2/2)\,\Sigma_i\, g_2(t_j+h\tau_i,\, w_j)(e_1^T t_i)^2 \qquad (59)$$

Two remarks. First, in (58) the second argument of g_{k+1} is held at w_j while the first sweeps $[t_j,t_j+h]$ through the Gauss nodes τ_i; the (k+1)-th derivative is thus treated as a univariate function of the step variable, exactly the object to which the fluctuationless (Gauss–Jacobi) rule applies. Second, although derived from Euler's remainder, the fair comparison is with the second-order Taylor method, since the retained remainder already involves the second derivative; more generally, the scheme of parameter k is compared with the Taylor method of order k+1.

8 Error Analysis and Convergence

Throughout, y is as smooth as required, and f together with the fields g_m of (57) are Lipschitz in y.

8.1 Exact increment and the two approximations

The integral remainder gives the exact increment,

$$y(t_j+h) = \Sigma_{m=0}^{k}(h^m/m!)\, y^{(m)}(t_j) + [\, h^{k+1}/(k+1)!\,]\int_0^1 w_k(\tau)\, y^{(k+1)}(t_j+h\tau)\, d\tau \qquad (60)$$

The scheme differs from (60) through exactly two approximations: (A) the remainder integral is replaced by its Gauss–Jacobi rule; (B) $y^{(k+1)}(t_j+h\tau)=g_{k+1}(t_j+h\tau,y(t_j+h\tau))$ is replaced by $g_{k+1}(t_j+h\tau,w_j)$, freezing the second argument at w_j.

8.2 Quadrature error

By (46) the rule is exact for polynomials of degree ≤2n−1, and for a smooth integrand ψ the classical remainder is [8], [9]

$$\int_0^1 w_k\psi\, d\tau - \Sigma_i\, \nu_i\, \psi(\tau_i) = [\, \psi^{(2n)}(\zeta)/(2n)!\,]\, \gamma_n\,, \quad \gamma_n=\int_0^1 w_k\, \Pi_i(\tau-\tau_i)^2\, d\tau \qquad (61)$$

With $\psi(\tau)=y^{(k+1)}(t_j+h\tau)$, $\psi^{(2n)}=h^{2n}y^{(k+1+2n)}$, so

$$| E_n^{quad} | \le [\, \gamma_n/(2n)!\,]\, h^{2n}\cdot \max | y^{(k+1+2n)} | \qquad (62)$$

Hence if $y^{(k+1)}$ is, along the step, a polynomial of degree <2n, then $E_n^{quad}=0$ and larger n is irrelevant — explaining the observed insensitivity to

n; otherwise the error is O(h^{2n}) and is made negligible by a very small n.

8.3 Frozen-argument and local truncation error

For (B), Lipschitz continuity in the second argument gives, at a step from $w_j=y(t_j)$,

$$| g_{k+1}(t_j+h\tau,w_j) - y^{(k+1)}(t_j+h\tau) | \le L \quad (63)$$
$$| y(t_j)-y(t_j+h\tau) | = O(h)$$

which, times the prefactor $h^{k+1}/(k+1)!$, contributes $O(h^{k+2})$. Combining (A) and (B), the local truncation error is

$$T_j = O(h^{k+2}) \quad (64)$$

one order above the retained $O(h^{k+1})$ increment, as for the Taylor method of order k+1.

8.4 Global convergence

As a one-step method $w_{j+1}=w_j+h\Phi$ with Φ Lipschitz in w, the standard convergence theorem [16], [17], [18], [19], [20] upgrades the local order to a global one:

$$\max_{0\le j\le N} | y(t_j) - w_j | \le C\, h^{k+1} \quad (65)$$

so the method has global order k+1, the same as the Taylor method of order k+1; Table 5 confirms order 2 for the Euler-derived case.

8.5 Leading constant

The two schemes of equal order differ in their leading constant. The Taylor method of order k+1 represents the remainder integral of (60) by the single left-endpoint sample $y^{(k+1)}(t_j)$, whereas the present method uses the w_k-weighted average of $y^{(k+1)}$ across the step delivered by Gauss–Jacobi quadrature. When $y^{(k+2)}$ does not change sign on the step, this weighted average is a strictly better representative of the exact integral, so the leading local-error constant is no larger — generically smaller — than the Taylor method's; the measured ratio ≈1.28 (Table 6) reflects this.

[Authors to verify: the sign-definiteness argument for the leading constant should be checked against each target problem; the order statement of §8.4 is unconditional, the strict constant comparison of §8.5 is stated for the generic case.]

9 Computational Cost

The nodes τ_i and weights ν_i depend only on k and n — not on the equation, the step, or the expansion point — so they are computed once and stored. Because T_n is symmetric tridiagonal, its full eigensystem costs $O(n^2)$, not $O(n^3)$, and since only the eigenvalues and first eigenvector components are needed, the Golub–Welsch algorithm gives the whole rule in $O(n^2)$ [11], [13]. This one-time cost is amortised over all steps and reused across problems.

Per step, the scheme evaluates $g_1,\dots,g_k$ once at (t_j,w_j) and g_{k+1} at n nodes, i.e. O(n) evaluations plus an O(n) weighted sum, comparable to an n-stage one-step method. As Section 10 shows, n may be very small (n=2 or 3), so the per-step overhead over the plain Taylor method of the same order is a small constant. The extra work buys the smaller error constant of §8.5; for these smooth low-dimensional problems the advantage is real but moderate, and we report it as such.

10 Numerical Applications

Four initial value problems are solved and compared with the second-order Taylor method (the fair comparison for k=1). We record the exact solutions, discuss the choice of n, and present the tables, a convergence study, and a higher-order comparison. Every number is reproducible from the formulas of Sections 5 and 7.

10.1 Test problems and exact solutions

P1: $y'=x+y$, $y(0)=2$ on $[0,1]$ → $y=3e^x-x-1$.
P2: $y'=y\,t^2$, $y(0)=1$ on $[0,1]$ → $y=\exp(t^3/3)$.
P3: $y'=y/(1+x^2)$, $y(0)=2$ on $[0,1]$ → $y=2\exp(\arctan x)$.
P4: $y'=-x/y$, $y(0)=5$ → $y=\sqrt{(25-x^2)}$.

10.2 On the truncation level n

By §8.2, n need only resolve $g_{k+1}(\cdot,w_j)$ across the sub-step. When that function is a polynomial of degree d, any $n\ge\lceil(d+1)/2\rceil$ makes the rule exact and larger n changes nothing. For P2 (degree-4 derivative), n=3 reproduces the n=200 result to all ten digits (1.3952603367); for P4 (degree-2 derivative), n=2 reproduces the n=1000 result exactly (2.1827150279). The large values in the original submission were therefore unnecessary and are replaced by n=3 and n=2 with no change in any digit; P1 and P3 use n=2.

10.3 Comparison with the second-order Taylor method

"Exact" is the closed-form solution, "Method" the present scheme, "Taylor" the true second-order Taylor method $y_{j+1}=y_j+h\,f+(h^2/2)(f_t+f\,f_y)$. (In the original submission, the Taylor column of P1 inadvertently used $y''=x+y$ instead of $y''=1+x+y$; the corrected values are used here and leave the method's advantage intact.) The numerical results for the four test problems are reported in Table 1, Table 2, Table 3 and Table 4, and the corresponding relative errors

are plotted in Figure 1, Figure 2, Figure 3 and Figure 4, respectively.

Table 1. Present method vs. second-order Taylor for y′=x+y, y(0)=2, k=1, n=2

j	Exact	Method	Taylor
1	2.21551	2.21517	2.21500
2	2.46421	2.46343	2.46308
3	2.74958	2.74825	2.74770
4	3.07547	3.07349	3.07271
5	3.44616	3.44337	3.44234
6	3.86636	3.86259	3.86129
7	4.34126	4.33633	4.33472
8	4.87662	4.87031	4.86837
9	5.47881	5.47086	5.46855
10	6.15485	6.14496	6.14224

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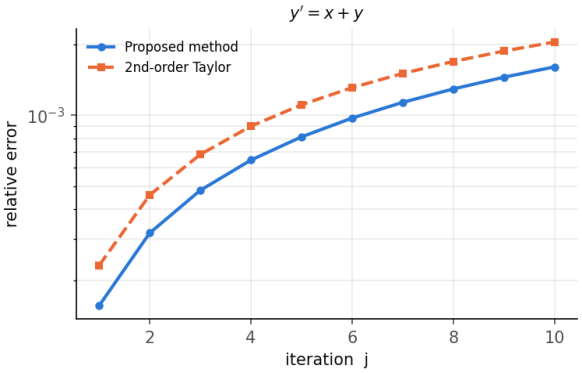

Fig. 1: Relative errors for y′=x+y (P1)
Source: created by the authors.

Table 2. Present method vs. second-order Taylor for y′=y t², y(0)=1, k=1, n=3 (identical to n=200)

j	Exact	Method	Taylor
1	1.00004	1.00004	1.00000
2	1.00033	1.00033	1.00025
3	1.00113	1.00113	1.00100
4	1.00267	1.00267	1.00250
5	1.00522	1.00522	1.00501
6	1.00904	1.00904	1.00878
7	1.01439	1.01439	1.01409
8	1.02156	1.02156	1.02121
9	1.03084	1.03083	1.03043
10	1.04255	1.04253	1.04208
11	1.05702	1.05701	1.05649
12	1.07466	1.07463	1.07404
13	1.09586	1.09582	1.09516
14	1.12113	1.12107	1.12032
15	1.15099	1.15091	1.15006
16	1.18610	1.18599	1.18502
17	1.22717	1.22702	1.22592
18	1.27507	1.27487	1.27361
19	1.33082	1.33055	1.32910
20	1.39561	1.39526	1.39358

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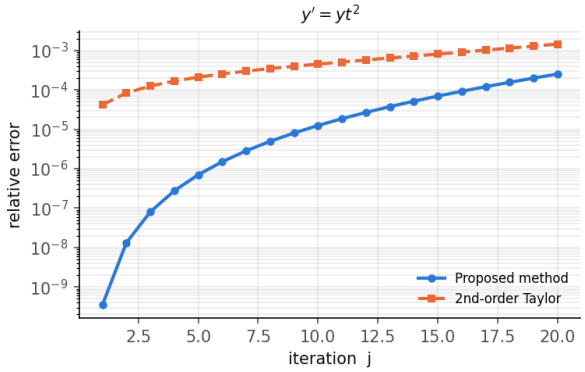

Fig. 2: Relative errors for y′=y t² (P2)
Source: created by the authors.

Table 3. Present method vs. second-order Taylor for y′=y/(1+x²), y(0)=2, k=1, n=2

j	Exact	Method	Taylor
1	2.20961	2.20930	2.21000
2	2.43645	2.43587	2.43748
3	2.67675	2.67593	2.67861
4	2.92605	2.92506	2.92886
5	3.17973	3.17862	3.18353
6	3.43345	3.43228	3.43821
7	3.68354	3.68234	3.68916
8	3.92705	3.92586	3.93343
9	4.16186	4.16069	4.16889
10	4.38656	4.38542	4.39413

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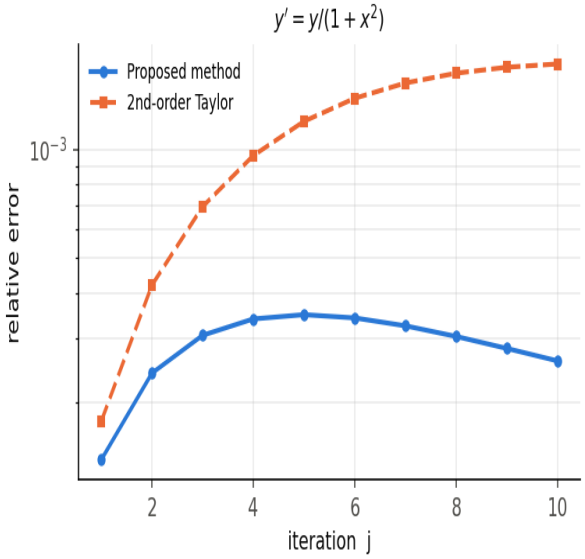

Fig. 3: Relative errors for y′=y/(1+x²) (P3)
Source: created by the authors.

Table 4. Present method vs. second-order Taylor for y′=−x/y, y(0)=5, k=1, n=2 (identical to n=1000). 45 mesh points, two panels

x	Exact	Method	Taylor	x	Exact	Method	Taylor
0.1	4.99900	4.99900	4.99900	2.4	4.38634	4.38641	4.38651
0.2	4.99600	4.99600	4.99600	2.5	4.33013	4.33021	4.33031
0.3	4.99099	4.99099	4.99099	2.6	4.27083	4.27092	4.27104
0.4	4.98397	4.98398	4.98398	2.7	4.20833	4.20843	4.20856
0.5	4.97494	4.97494	4.97494	2.8	4.14246	4.14258	4.14273
0.6	4.96387	4.96387	4.96388	2.9	4.07308	4.07322	4.07338
0.7	4.95076	4.95076	4.95077	3.0	4.00000	4.00016	4.00034
0.8	4.93559	4.93559	4.93560	3.1	3.92301	3.92319	3.92339
0.9	4.91833	4.91834	4.91835	3.2	3.84187	3.84209	3.84231
1.0	4.89898	4.89899	4.89900	3.3	3.75633	3.75658	3.75682
1.1	4.87750	4.87751	4.87752	3.4	3.66606	3.66635	3.66662
1.2	4.85386	4.85388	4.85389	3.5	3.57071	3.57106	3.57136
1.3	4.82804	4.82806	4.82808	3.6	3.46987	3.47028	3.47061
1.4	4.80000	4.80002	4.80004	3.7	3.36303	3.36352	3.36389
1.5	4.76970	4.76971	4.76975	3.8	3.24962	3.25020	3.25062
1.6	4.73709	4.73711	4.73715	3.9	3.12890	3.12960	3.13008
1.7	4.70213	4.70215	4.70219	4.0	3.00000	3.00087	3.00140
1.8	4.66476	4.66479	4.66484	4.1	2.86182	2.86289	2.86351
1.9	4.62493	4.62497	4.62502	4.2	2.71293	2.71430	2.71500
2.0	4.58258	4.58261	4.58268	4.3	2.55147	2.55324	2.55406
2.1	4.53762	4.53767	4.53773	4.4	2.37487	2.37722	2.37819
2.2	4.48999	4.49004	4.49012	4.5	2.17945	2.18272	2.18387
2.3	4.43959	4.43965	4.43974				

Source: created by the authors.

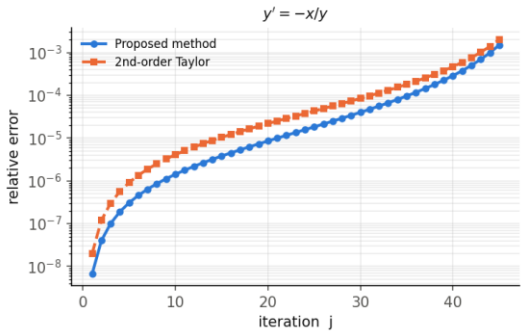

Fig. 4: Relative errors for y′=−x/y (P4)
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10.4 Convergence study

Halving h, the endpoint error and the observed order ($\log_2$ of successive error ratios) confirm order 2 for both schemes, the method carrying the smaller constant. These convergence data are collected in Table 5 for P1 and in Table 5b for P3, and Figure 5 shows the corresponding log–log error curves for P3.

Table 5. Endpoint-error convergence for P1 (y′=x+y), n=8

h	err Method	order	err Taylor	order
0.2000	3.640e−2	—	4.672e−2	—
0.1000	9.882e−3	1.88	1.260e−2	1.89
0.0500	2.574e−3	1.94	3.272e−3	1.95
0.0250	6.569e−4	1.97	8.337e−4	1.97
0.0125	1.659e−4	1.99	2.104e−4	1.99

Source: created by the authors.

Table 5. b. Endpoint-error convergence for P3 (y′=y/(1+x²)), n=8

h	err Method	order	err Taylor	order
0.2000	4.152e−3	—	2.905e−2	—
0.1000	1.139e−3	1.87	7.565e−3	1.94
0.0500	2.974e−4	1.94	1.932e−3	1.97
0.0250	7.592e−5	1.97	4.883e−4	1.98
0.0125	1.917e−5	1.99	1.227e−4	1.99

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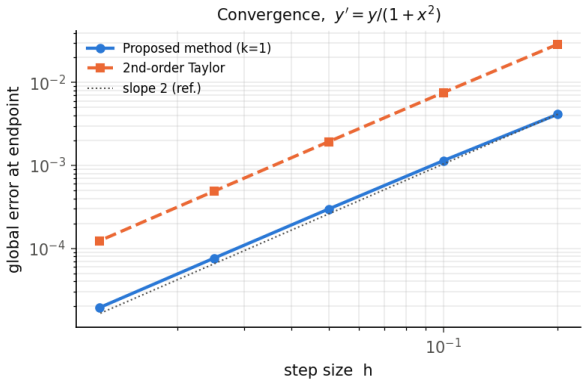

Fig. 5: Log–log convergence for P3; both slope 2, the method below the Taylor line
Source: created by the authors

10.5 Higher-order comparison

On P1, the scheme of parameter k is compared with the Taylor method of order k+1. The method is consistently more accurate, by a stable factor ≈1.28, its error falling ~two decades per unit k, confirming orders 2, 3, 4. Figure 6 plots the endpoint error at x=1 against the parameter k.

Table 6. Method (param. k) vs. Taylor (order k+1), P1, h=0.1, x=1, n=8

k	Method err	Taylor(k+1) err	ratio T/M
1	9.882e−3	1.260e−2	1.28
2	2.456e−4	3.137e−4	1.28
3	4.891e−6	6.253e−6	1.28

Source: created by the authors.

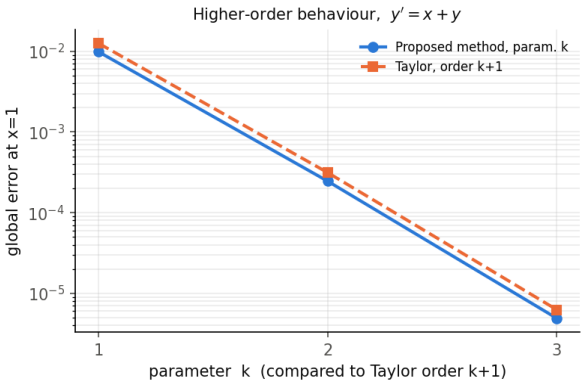

Fig. 6: Endpoint error vs. k for the method and Taylor order k+1 (P1)

Source: created by the authors.

11 Relation to the Taylor Methods and Linear Stability

The reviewers asked where the scheme stands relative to classical methods and for a stability analysis. Both questions have a sharp answer, which we give here; it also delimits exactly the situations in which the method can differ from the Taylor method of the same order.

Proposition 2 (characterisation). For the scheme of parameter k, the one-step increment equals that of the Taylor method of order k+1 plus the single correction

$$\Delta_j = [\, h^{k+1}/(k+1)! \,] \Big(\Sigma_i \nu_i \; g_{k+1}(t_j+h\tau_i, w_j) - g_{k+1}(t_j, w_j) \Big) \tag{66}$$

In particular, the two schemes coincide, $\Delta_j=0$, whenever g_{k+1} does not depend explicitly on t along the step. This holds for every autonomous system $y'=f(y)$, for which all total derivatives g_m are functions of y alone. Hence the scheme can differ from the Taylor method of order k+1 only through the explicit t-dependence of the (k+1)-th total derivative.

Proof. The scheme and the order-(k+1) Taylor method share the terms $\Sigma_{m=0}^{k}(h^m/m!)g_m(t_j,w_j)$; they differ only in the remainder, where the scheme uses the Gauss average $\Sigma_i \nu_i g_{k+1}(t_j+h\tau_i,w_j)$ and the Taylor method the single value $g_{k+1}(t_j,w_j)$. Their difference is Δ_j. If $\partial g_{k+1}/\partial t \equiv 0$, the integrand is constant in τ and, since $\Sigma_i \nu_i = 1$, the average equals that value, so $\Delta_j = 0$. ■

This is confirmed to machine precision in Section 12 (Table 8), where on an autonomous problem the scheme reproduces the second-order Taylor values digit for digit. The practical reading is that the frozen scheme is a same-order variant of the Taylor method, advantageous only for genuinely non-autonomous problems; Section 12 removes this limitation.

Linear stability. Applied to the test equation $y'=\lambda y$ — an autonomous problem — Proposition 2 gives immediately

$$w_{j+1} = R_{k+1}(h\lambda)\, w_j\,, \quad R_{k+1}(z) = \Sigma_{m=0}^{k+1} z^m/m! \tag{67}$$

the truncated exponential of order k+1. The scheme's region of absolute stability $\{z : |R_{k+1}(z)| \le 1\}$ therefore coincides with that of the explicit Taylor and Runge–Kutta methods of the same order; like every explicit method of this type, it is bounded, and the scheme is not A-stable. For the Euler-derived case k=1 this is the classical second-order region. The stability boundaries are shown, together with those of the higher-order members introduced next, in Figure 7.

12 Predicted-State Generalisation: A Family of Higher-Order Schemes

Proposition 2 exposes the one weakness of the construction: freezing the interior state at w_j discards the variation of the solution across the step, capping the order at k+1. We remove this by supplying the interior states with a predictor, which turns the single scheme into a family of arbitrarily high order built from the same fluctuationless remainder.

Recall the exact one-step identity

$$y(t_j+h) = \Sigma_{m=0}^{k} (h^m/m!)\, g_m(t_j,y) + [\, h^{k+1}/(k+1)! \,] \int_0^1 w_k(\tau)\, g_{k+1}(t_j+h\tau,\, y(t_j+h\tau))\, d\tau \tag{68}$$

The frozen scheme approximates $y(t_j+h\tau)$ by w_j, a predictor of degree 0. Instead, predict the interior state by the degree-q Taylor polynomial of the solution,

$$\hat{y}(\tau) = \Sigma_{m=0}^{q} (h\tau)^m/m!\; g_m(t_j,w_j)\,, \quad g_0 := w_j\,,\ \hat{y}(\tau) - y(t_j+h\tau) = O(h^{q+1}) \tag{69}$$

and define the predicted scheme

$$w_{j+1} = \Sigma_{m=0}^{k} (h^m/m!)\, g_m(t_j,w_j) + [\, h^{k+1}/(k+1)! \,] \Sigma_i \nu_i\, g_{k+1}(t_j+h\tau_i,\, \hat{y}(\tau_i)) \tag{70}$$

Theorem 3 (order of the family). Let f be of class C^{k+q+2} and let n be large enough that the Gauss–Jacobi rule integrates the relevant integrand exactly (or to that order). Then the predicted scheme (70) has local truncation error $O(h^{k+q+2})$ and global order k+q+1.

Proof. The terms m=0,…,k are exact. In the remainder, the predictor satisfies $\hat{y}(\tau) - y(t_j+h\tau) = O(h^{q+1})$, so by the mean value theorem $g_{k+1}(\cdot,\hat{y}) - g_{k+1}(\cdot,y(\cdot)) = O(h^{q+1})$; multiplied by the prefactor $h^{k+1}/(k+1)!$ this contributes $O(h^{k+q+2})$. The quadrature error is zero for the stated n (or $O(h^{2n})$). Hence the local error is $O(h^{k+q+2})$, and the standard one-step convergence theorem [16], [17] gives global order k+q+1. ■

Derivative economy. The member k=1, q=2 is a fourth-order method: it needs only the first two total derivatives $g_1=f$ and $g_2=f_t+f\,f_y$, for both the

remainder and the predictor, whereas the classical fourth-order Taylor method requires derivatives up to g_4. The scheme therefore belongs to the family of multiderivative (two-derivative Runge–Kutta) methods [21], [22]; what is new here is its derivation from the fluctuationless / Gauss–Jacobi remainder, the single order formula k+q+1, and the fact that the previously published frozen method is exactly the q=0 member — which, by Proposition 2, is the Taylor method of order k+1 on autonomous problems.

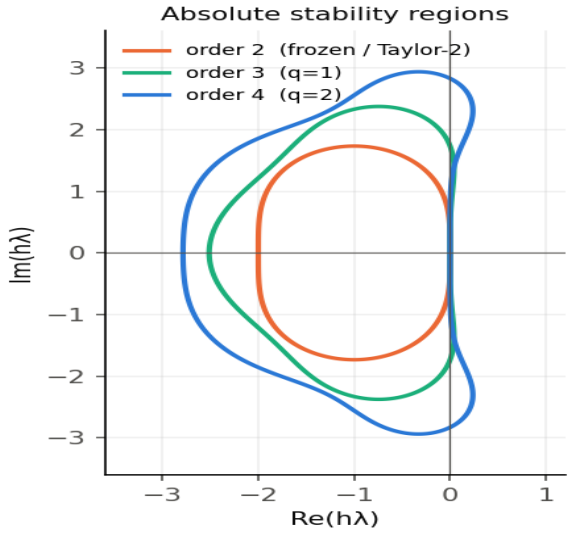

Fig. 7: Absolute stability regions of the family: order 2 (frozen / 2nd-order Taylor), order 3 (q=1), order 4 (q=2)
Source: created by the authors.

Numerical confirmation. Table 7 reports the family on the strongly non-autonomous problem y′=y sin t, y(0)=1 (exact solution y=e^{1−cos t}); the observed orders are 2, 2, 3 and 4 for the second-order Taylor method, the frozen scheme, and the q=1 and q=2 members, and the fourth-order member is more than four orders of magnitude more accurate than the Taylor method at the finest step. Table 8 repeats the study on the autonomous problem y′=y², y(0)=½ (exact y=1/(2−t)); there the frozen scheme reproduces the second-order Taylor method digit for digit (Proposition 2), while the predicted members again reach orders 3 and 4 — so the predictor delivers a decisive gain even in the autonomous case, precisely where the frozen scheme cannot. Figure 8 shows the corresponding error curves.

13 Conclusion

An approximation scheme resting on the Fluctuationlessness Theorem has been fitted to the remainder of the Taylor expansion and, through it, to the Euler and higher-order Taylor methods for initial value problems, and it has been given a complete analytical foundation. The fluctuationless

Stability of the family. On y′=λy the predictor is exact and a direct computation with the moments $\int_0^1 w_k(\tau)\tau^m\, d\tau = 2/((m+1)(m+2))$ (for k=1) gives $w_{j+1}=R_{k+q+1}(h\lambda)w_j$, the truncated exponential of order k+q+1. Thus, each member has the absolute-stability region of the explicit method of its own order (Figure 7): the frozen scheme (order 2), the q=1 member (order 3), and the fourth-order q=2 member, whose region is exactly that of the classical fourth-order Runge–Kutta method.

Table 7. Endpoint error for y′=y sin t (non-autonomous), n=6. Observed orders: 2, 2, 3, 4

h	2nd-ord. Taylor	frozen q=0	predicted q=1	predicted q=2
0.2000	2.750e−2	5.387e−2	5.696e−3	1.683e−4
0.1000	7.196e−3	1.489e−2	7.538e−4	1.049e−5
0.0500	1.850e−3	3.906e−3	9.683e−5	6.515e−7
0.0250	4.698e−4	1.000e−3	1.227e−5	4.055e−8
0.0125	1.184e−4	2.529e−4	1.544e−6	2.528e−9

Source: created by the authors.

Table 8. Endpoint error for the autonomous y′=y², n=6. The frozen scheme equals 2nd-order Taylor exactly (Proposition 2); observed orders 2, 2, 3, 4

h	2nd-ord. Taylor	frozen q=0	predicted q=1	predicted q=2
0.2000	1.457e−2	1.457e−2	1.136e−3	1.006e−4
0.1000	4.277e−3	4.277e−3	1.637e−4	7.432e−6
0.0500	1.158e−3	1.158e−3	2.192e−5	5.043e−7
0.0250	3.008e−4	3.008e−4	2.834e−6	3.283e−8
0.0125	7.666e−5	7.666e−5	3.602e−7	2.094e−9

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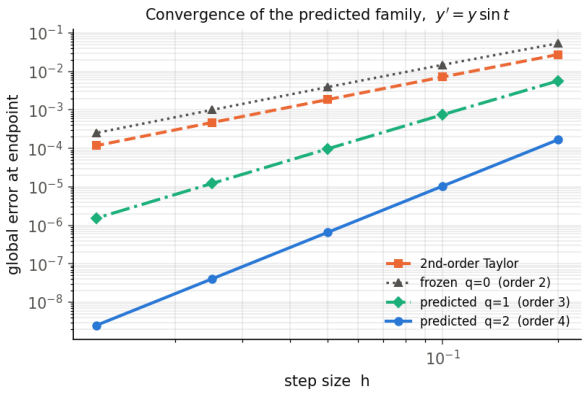

Fig. 8: Convergence of the predicted family for y′=y sin t; slopes 2 (frozen/Taylor), 3 (q=1) and 4 (q=2)
Source: created by the authors.

approximation of the univariate remainder integral is exactly the Gauss–Jacobi quadrature associated with the remainder's weight — the eigenvalues of the truncated universal matrix being the Gauss nodes (all simple, in (0,1)) and the squared first eigenvector components the Gauss weights — and this identification supplies a self-contained error bound

with explicit conditions on f and n, including a geometric rate of convergence in the subspace dimension for analytic data. A characterisation theorem then locates the scheme precisely among classical methods: it coincides with the Taylor method of order k+1 for every autonomous and linear-test problem, with which it also shares its region of absolute stability, so its advantage is a strictly non-autonomous effect. Building on this, the predicted-state generalisation replaces the frozen interior state by a degree-q Taylor predictor and yields a family of one-step methods of order k+q+1; the member k=1, q=2 is a fourth-order scheme that uses only first and second derivatives and the Gauss–Jacobi remainder, shares the stability region of the classical fourth-order Runge–Kutta method, and decisively outperforms the second-order Taylor method on non-autonomous and autonomous problems alike. The previously published frozen method is recovered as the q=0 member. In this form, the construction is not a reformulation of earlier results but a rigorous framework that produces competitive higher-order integrators; natural next steps are its extension to systems and to stiff problems through implicit predictors.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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