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ORIGINAL ARTICLE

Local systems with tame, and a unipotent, local monodromy

R.P. (Bob) Langlands'a sevgi ve saygılarımızla

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Abstract

We compute the cardinality of a set of Galois-invariant isomorphism classes of irreducible rank two $\overline{\mathbb{Q}}_\ell$-smooth sheaves on $X_1 - S_1$, where X_1 is a smooth projective absolutely irreducible curve of genus g over a finite field $\mathbb{F}_q$ and S_1 is a reduced divisor, with pre-specified tamely ramified monodromy data at S, *including precisely one point of principal unipotent monodromy, twisted by a tame character*. Equivalently, we compute the number of the corresponding automorphic representations. The approach is based on using an explicit form of the trace formula for GL(2), extending the work "Counting local systems with tame ramification" to include a Steinberg (= special) component, twisted by a tame character, by employing a pseudo-coefficient thereof.

Keywords Local systems · ℓ-adic representations · Automorphic representations · Tame and Steinberg representations · Pseudo-coefficients · Explicit trace formula · Curves over finite fields · Counting

Mathematics Subject Classification 11S37 · 11R39 · 11F70 · 11F72 · 22E35 · 22E55 · 11G20 · 14H30

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1 Introduction

Let X_1 be a smooth projective absolutely irreducible curve over a finite field $\mathbb{F}_q$ of cardinality q. Let $F_1 = k(X_1)$ be the field of rational functions on X_1 over $\mathbb{F}_q$, and $\mathbb{A}$ its ring of adèles. Let $|X_1|$ be the set of closed points of X_1. Denote by S_1 a nonzero reduced divisor on X_1, namely a finite subset of $|X_1|$, a disjoint union of three subsets $S_1^{(1,1)}$, $S_1^{(2)}$, $S_1^{D^+}$, $|S_1^{D^+}| = 1$. Denote by $\mathbb{F} = \cup_{m>0}\mathbb{F}_{q^m}$ a separable closure of $\mathbb{F}_q$, by $(X_m, S_m) = (X_1, S_1) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ the base change to the extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ of degree m in $\mathbb{F}$, and by $(X, S) = (X_1, S_1) \otimes_{\mathbb{F}_q} \mathbb{F}$ to the geometric case. Define analogously S^*, S_m^*, $*= (1,1)$, (2), D^+. The function field $F_m = k(X_m)$ of $X_m = X_1 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ over $\mathbb{F}_{q^m}$ is $F_m = F_1 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$. Fix a rational prime $\ell, \ell \nmid q$.

The geometric Frobenius Fr topologically generates $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. It acts on $\mathbb{F}$, hence on the scheme $X - S = (X_1 - S_1) \otimes_{\mathbb{F}_q} \mathbb{F}$, and by transport of structures on the set of isomorphism classes of $\overline{\mathbb{Q}}_\ell$-smooth sheaves on $X - S$. The latter coincides with the pullback Frob^* by the Frobenius endomorphism $\mathrm{Frob} : X - S \to X - S$ (see [3, Lemma 1.3]). For $s \in S$ let $X_{(s)}$ be the completion $\mathrm{Spec}\widehat{\mathcal{O}}_{(s)}$ of X at s, where $\widehat{\mathcal{O}}_{(s)}$ is the completion of the local ring $\mathcal{O}_{(s)}$ of X at s. Put $X_{(s)}^* = X_{(s)} - \{s\}$ for the punctured space. Assume given for each $s \in S$ a $\overline{\mathbb{Q}}_\ell$-smooth sheaf $\mathcal{R}(s)$ of rank $r \geq 1$ on $X_{(s)}^*$, up to isomorphism. The morphism $\mathrm{Frob} : X \to X$ defines a morphism again denoted $\mathrm{Frob} : X_{(s)}^* \to X_{(\mathrm{Frob}(s))}^*$. Assume the family $\mathcal{R} = (\mathcal{R}(s); s \in S)$ is Frob-*compatible*: for each $s \in S$ the sheaf $\mathcal{R}(s)$ is isomorphic to the inverse image of $\mathcal{R}(\mathrm{Frob}(s))$ by Frob, namely $\mathcal{R}(s) \simeq \mathrm{Frob}^*(\mathcal{R}(\mathrm{Frob}(s)))$.

Let $E_r(X, S; \mathcal{R})$ be the set of isomorphism classes of irreducible rank r $\overline{\mathbb{Q}}_\ell$-smooth sheaves $\mathcal{F}$ on $X - S$ satisfying: the inverse image $\mathcal{F}|X_{(s)}^*$ of $\mathcal{F}$ on $X_{(s)}^*$ is isomorphic to $\mathcal{R}(s)$. Since $\mathcal{R}$ is Frob-compatible, the functor Frob^* defines a permutation, denoted $\mathbb{V}$ (see [2, section 2.2]), of the set $E_r(X, S; \mathcal{R})$. The *counting problem* is to compute the cardinality $|E_r(X, S; \mathcal{R})^{\mathbb{V}^m}|$ of the points in $E_r(X, S; \mathcal{R})$ fixed by $\mathbb{V}^m$, $m \geq 1$.

In this work we fix regular tame rank two Frob-compatible ($\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-stable) monodromy $\mathcal{R} = \mathcal{R}^{(1,1)} \cup \mathcal{R}^{(2)} \cup \mathcal{R}^{D^+}$, $\mathcal{R}^i = (\mathcal{R}(s); s \in S^i)$ $(i = (1,1), (2), D^+)$ in general position (defined in [2, section 2], see also [9], and below), satisfying the compatibility assumption. Here the $\overline{\mathbb{Q}}_\ell$-smooth sheaf $\mathcal{R}(s)$ on $X_{(s)}^*$ is semisimple of principal series type at the $s \in S^{(1,1)}$, semisimple of discrete series type at the $s \in S^{(2)}$, and with principal unipotent monodromy twisted by an unramified character at the $s \in S^{D^+}$.

The case where S_1 is empty in rank $r = 2$ is due to Drinfeld [5]. For a detailed proof see [7, section 3]. This is based on an explicit computation of the trace formula for $G = \mathrm{GL}(2)$ in [6]. Cases with local principal unipotent monodromy $S_1 = S_1^{D^+}$ for general rank $r \geq 2$ when $|S_1| \geq 2$, were studied with Deligne in [3] using masses of categories, in [7] also when $|S_1| = 1$ in rank two, and again for general rank and $|S_1| \geq 2$ using the trace formula in [8].

The notion of general position was introduced in Deligne [2], where a geometric approach – that Deligne attributes to Arinkin – to the counting problem is

presented, in the context of general rank $r \geq 2$, and principal series local monodromy. It is reminiscent of the general position notion of Deligne-Lusztig [4].

In all these works where the trace formula is applied, we use a simple form of the trace formula in higher rank, or the trace formula for the rank one group GL(2). We do not employ the general rank Arthur trace formula, adapted by L. Lafforgue [12] to the positive characteristic case, due to its complexity.

The present work deals in rank two with the case where in addition to tame semisimple ramification at the sets $S_1^{(1,1)}$, $S_1^{(2)}$, there is principal unipotent monodromy twisted by a tame character at the set $S_1^{D^+}$, taken here to consists of a single place. In [9] we dealt with the case where $S_1^{D^+}$ is empty, using the explicit trace formula for GL(2) of [6]. We use the same technology here, except that we make use of a pseudo-coefficient of the Steinberg (= special) representation at the place of $S_1^{D^+}$, as in [7], where $S_1^{(1,1)}$, $S_1^{(2)}$ are empty (and $|S_1^{D^+}| = 1$). The advantage of the present work is in its brevity, and thus we hope clarity.

We also deal with the case where $|S_1^{D^+}| \geq 2$ in [10] on using a simple form of the trace formula (which we find to be less computational, hence more satisfactory: we could use the technique of the present work with the pseudo-coefficient of Steinberg at each place of $S_1^{D^+}$ also when $|S_1^{D^+}| > 1$). It can be shown that [10] generalizes to any rank $r \geq 2$.

We develop here then the approach based on automorphic forms and the use of the trace formula to deal with tame ramification, in rank two. The translation between $\overline{\mathbb{Q}}_\ell$-smooth sheaves and automorphic representations is via the well-known Galois-automorphic dictionary of L. Lafforgue [12]. The pseudo-coefficients for (families of) tamely ramified principal series and discrete series (cuspidal, monomial) local components of [9], and of Steinberg components as in [7], are used to compute the number of automorphic representations of $GL(2, \mathbb{A})$ unramified outside S_1, with tame ramification specified by the regular tame Frob -compatible data $\mathcal{R}$ in general position and satisfying compatibility, at the places of S_1 (= $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-orbits in S), via the local Langlands correspondence. This is the technical heart of the work, as in [3, 5, 7–11].

As noted in [9], the labels "principal series" and "discrete series" for the rank two $\overline{\mathbb{Q}}_\ell$-smooth sheaves $\mathcal{R}(s)$ are transferred from the representation theoretic to the geometric realms; the superscripts (1, 1) and (2) allude to partitions. In rank r partitions $\lambda \vdash r$ determined by Deligne-Lusztig theory [4] would occur.

In contrast to [3, 5, 7, 8], here and in [9] we do not deal with the question of geometrization of the number $|E_2(X, S; \mathcal{R})^{\mathbb{V}^m}|$, namely showing it has the form of the number of $\mathbb{F}_{q^m}$-points on some variety over $\mathbb{F}_q$, thus given by a linear combination with integral coefficients of Weil-numbers. What we do is give an explicit computation of $|E_r(X, S; \mathcal{R})^{\mathbb{V}}|$. We use the result of the present work to establish this geometrization in [10], for $|S_1^{D^+}| \geq 1$, not only when $|S_1^{D^+}| \geq 2$ as the computational result of [10] would permit. See the paragraph following Theorem 2.1 below for elaboration.

General rank cases with wild ramification are studied in [11]. The present work replaces our preliminary notes "Automorphic forms with tame components".

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2 Counting local systems

We start with stating our arithmetic result. Our tale concerns then $X_1/\mathbb{F}_q$ and $F_1 = k(X_1)$ as above, and a set $S_1 = S_1^{D^+} \sqcup S_1^{(1,1)} \sqcup S_1^{(2)}$, where $S_1^{D^+} = \{u\}$ is a singleton. Put $S_1' = S_1^{D^+} \sqcup S_1^{(2)}$ and $S_1'' = S_1^{(1,1)} \sqcup S_1^{(2)}$. The notation (1, 1) and (2) indicates partitions of 2. As explained in [9], an extension of [9] and the work here to general rank r will use Deligne-Lusztig [4] and be stated in terms of partitions $\lambda \vdash r$.

Let $N_1^* = |S_1^*|$ be the cardinality of S_1^*. We consider here the case: $N_1^{D^+} = 1, N_1'' \geq 1$.

For $s_1 \in |X_1|$, the set of closed points in X_1, denote by $k(s_1) = k_{s_1}$ the residue field of the completion F_{1s_1} of the function field $F_1 = k(X_1)$ at s_1. It is $\mathcal{O}_{s_1}/\mathfrak{m}_{s_1}$, where $\mathcal{O}_{s_1}$ is the ring of integers of F_{1s_1} and $\mathfrak{m}_{s_1}$ is its maximal ideal. This $k(s_1)$ is $\mathbb{F}_{q_{s_1}}$, $q_{s_1} = q^{d(s_1)}$, $d(s_1)$ is the degree $\deg(s_1)$ of s_1, namely the number of points in the Frob-orbit $s_1 \otimes_{\mathbb{F}_1} \mathbb{F}$ in S.

We consider tame ramification data $\mathcal{R}(s)$ on $X_{(s)}^*$, $s \in S$, of the following types.

The *principal series* type consists of pairs $(\varepsilon_{1s_1}, \varepsilon_{2s_1})$, for $s_1 \in S_1^{(1,1)}$; here $\varepsilon_{is_1} : k_{s_1}^\times = \mathcal{O}_{s_1}^\times/(1 + \varpi_{s_1}\mathcal{O}_{s_1}) \to \overline{\mathbb{Q}}_\ell^\times$ are characters. Put $q_{s_1} = |k_{s_1}| = q^{d_{s_1}}$ and $\omega_{s_1} = \varepsilon_{1s_1}\varepsilon_{2s_1}$. The *discrete series* type consists of $(\eta_{s_1}, \overline{\eta}_{s_1} = \eta_{s_1}^{q_{s_1}})$, for $s_1 \in S_1^{(2)}$, where $\eta_{s_1} : k_{2s_1}^\times = \mathcal{O}_{E_{s_1}}^\times/(1 + \varpi_{s_1}\mathcal{O}_{s_1}) \to \overline{\mathbb{Q}}_\ell^\times$ is a character. Here $k_{2s_1} \simeq \mathbb{F}_{q_{s_1}^2}$ is a quadratic field extension of $k_{s_1} = \mathbb{F}_{q_{s_1}}$, the residue field of the ring of integers of a quadratic unramified field extension E_{s_1} of the local field F_{s_1} of F at s_1.

We also fix a tame character $\varepsilon_{s_1} : k(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$ at $s_1 \in S_1^{D^+}$. The monodromy $\mathcal{R}(s)$ at $s \in S^{D^+}$ is taken to be principal unipotent twisted by the character ε_s.

In addition to being *regular*: $\varepsilon_{1s_1} \neq \varepsilon_{2s_1}$, $\eta_{s_1} \neq \overline{\eta}_{s_1}$, they are assumed to satisfy *compatibility*:

$$\varepsilon_{s_1}^2 \cdot \prod_{s_1 \in S_1^{(1,1)}} \omega_{s_1} \cdot \prod_{s_1 \in S_1^{(2)}} \eta_{s_1} = 1$$

on $\mathbb{F}_q^\times$ ($\varepsilon_{s_1}^2$ is the central character ω_ν of $\mathrm{St}_{s_1} \otimes \varepsilon_{s_1} \circ \det$), and be in *general position*:

$$\prod_{s \in S^{D^+}} \varepsilon_s \cdot \prod_{s \in S^{(1,1)}} \prod_{0 \leq j < d_{s_1}} \varepsilon_s^{(j)} \cdot \prod_{s \in S^{(2)}} \prod_{0 \leq j < d_{s_1}} \eta_s^{(j)} \neq 1$$

on $\widehat{\mathbb{Z}}(1)$, see below, where $\varepsilon_s^{(j)} \in \{\varepsilon_{1s}^j, \varepsilon_{2s}^j\}$ and $\eta_s^{(j)} \in \{\eta_s^j, \overline{\eta}_s^j\}$. The last notion is a slight variant to include $s \in S^{D^+}$ of that introduced by Deligne [2, section 2] to force the local systems we count to be irreducible, not only indecomposable as is

implied by taking the local data to have unipotent monodromy at least at one place: $|S^{D^+}| \geq 1$. It is used in the computation of the elliptic contribution to the geometric side of the trace via the example at the end of section 2 of [10] or [9] (the former extends the latter to the case where there is unipotent monodromy).

We compute the cardinality

$$T = T(S_1; \{(\varepsilon_{1s_1}, \varepsilon_{2s_1}); s_1 \in S_1^{(1,1)}\}; \{(\eta_{s_1}, \overline{\eta}_{s_1}); s_1 \in S_1^{(2)}\}; \{\varepsilon_{s_1}; s_1 \in S_1^{D^+}\})$$

of the set of equivalence classes of $\overline{\mathbb{Q}}_\ell$-smooth sheaves $\mathcal{R}$, fixed by the Frobenius, thus by $\mathbb{V}$, such that the pullback of $\mathcal{R}$ to $X^*_{(s)}$ is $\mathcal{R}(s)$ up to isomorphism.

The zeta function of X_1 over $\mathbb{F}_q$ is

$$\zeta(X_1, t) = \prod_i \det(1 - t \cdot \mathrm{Frob}; H^i(X))^{(-1)^{i+1}} = \frac{\mathbf{h}_1(t)}{(1-t)(1-qt)}.$$

Here

$$\mathbf{h}_1(t) = \sum_{0 \leq i \leq 2g} a_i t^i = \det(1 - t \cdot \mathrm{Frob}; H^1(X)) = \prod_\alpha (1 - \alpha t),$$

$a_i \in \mathbb{Z}$, is a polynomial of degree $2g$ over $\mathbb{Z}$ satisfying the functional equation $\mathbf{h}_1(1/qt) = q^{-g}t^{-2g}\mathbf{h}_1(t)$, or $a_{2g-i} = q^{g-i}a_i$, where $g = \mathrm{genus}(X_1)$. The Picard number of X_1 over $\mathbb{F}_q$ is $h_1 = \mathbf{h}_1(1)$. Write h_1' for $\mathbf{h}_1'(1)$. Note that $a_0 = \mathbf{h}_1(0) = \det(1; H^1(X))$ is 1. The complex absolute values of the $2g$ algebraic numbers α are $q^{1/2}$. These α can be labeled so that $\alpha_{g+i} = \overline{\alpha}_i = q\alpha_i^{-1}$. Assume $(\ell, q) = 1$

Theorem 2.1 *Let X_1 be a geometrically irreducible projective smooth curve over $\mathbb{F}_q$ of genus g, S_1 a reduced divisor on X_1, $X = X_1 \otimes_{\mathbb{F}_q} \mathbb{F}$, $S = S_1 \otimes_{\mathbb{F}_q} \mathbb{F}$. Suppose S_1 is the disjoint sum of $S_1^{D^+}$, $S_1^{(1,1)}$ and $S_1^{(2)}$. Assume $S_1'' \neq \emptyset$, $|S_1^{D^+}| = 1$. For $s_1 \in S_1$ choose $s \in S$ above s_1.*

For each $s_1 \in S_1^{(1,1)}$ choose a pair of characters $\varepsilon_{1,s_1}, \varepsilon_{2,s_1} : k(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$. Here $k(s_1)$ is the residue field of X_1 at s_1. Its cardinality is $q_{s_1} = q^{d_{s_1}}$ where $d_{s_1} = d(s_1)$ is $\deg(s_1)$.

For each $s_1 \in S_1^{(2)}$ choose characters $\eta_{s_1}, \overline{\eta}_{s_1} : k_2(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$, where $\overline{\eta}_s = \eta_s^{q^{\deg(s_1)}}$, and $k_2(s_1)$ is a quadratic extension of $k(s_1)$, so $x \mapsto x^{q^{\deg(s_1)}}$ generates $\mathrm{Gal}(k_2(s_1)/k(s_1))$.

Assume the data $\{\varepsilon_{1,s_1}, \varepsilon_{2,s_1}; s_1 \in S_1^{(1,1)}\}$ and $\{\eta_{s_1}, \overline{\eta}_{s_1}; s_1 \in S_1^{(2)}\}$ are **regular** *: $\varepsilon_{1,s_1} \neq \varepsilon_{2,s_1}$ for all $s_1 \in S_1^{(1,1)}$ and $\eta_{s_1} \neq \overline{\eta}_{s_1}$ for all $s_1 \in S_1^{(2)}$.*

View $s \in S$ above $s_1 \in S_1^{(1,1)}$ (thus $s \in s_1 \otimes_{\mathbb{F}_q} \mathbb{F}$) as an embedding $s : k(s_1)^\times \hookrightarrow \mathbb{F}^\times$, to define the projection $\mathrm{pr}_s : \widehat{\mathbb{Z}}(1) \to \widehat{\mathbb{Z}}(1)/\langle x^{q_{s_1}} - x\rangle \simeq k(s_1)^\times$, and the character $\varepsilon_{is} = \varepsilon_{is_1} \circ \mathrm{pr}_s : \widehat{\mathbb{Z}}(1) \to k(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$. See [2, section 2] or [9, section 2].

Also we fix a tame character $\varepsilon_{s_1} : k(s_1)^\times \hookrightarrow \mathbb{F}^\times$, and put $\varepsilon_s = \varepsilon_{s_1} \circ \mathrm{pr}_s : \widehat{\mathbb{Z}}(1) \to k(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$, $s \in S^{D^+}$ above s_1.

For $s \in S$ above $s_1 \in S_1^{(2)}$ denote by s also an embedding $k_2(s_1)^\times \hookrightarrow \mathbb{F}^\times$ whose restriction to $k(s_1)^\times$ is s, so that η_{s_1} defines $\eta_s = \eta_{s_1} \circ \mathrm{pr}_s : \widehat{\mathbb{Z}}(1) \to k_2(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$.

Then ε_{is}, ε_s and η_s are characters of $\widehat{\mathbb{Z}}(1)$, ε_{is_1}, ε_{s_1} and η_{s_1} are characters of $k(s_1)^\times$ and $k_2(s_1)^\times$.

For each $s \in S^{(1,1)} = S_1^{(1,1)} \otimes_{\mathbb{F}_q} \mathbb{F}$, resp. $s \in S^{(2)} = S_1^{(2)} \otimes_{\mathbb{F}_q} \mathbb{F}$, let $\mathcal{R}(s)$ be the rank two $\overline{\mathbb{Q}}_\ell$-smooth sheaf corresponding to the representation ρ_s of the inertia group I_s that factorizes via $\widehat{\mathbb{Z}}(1) = I_s/P_s$ and is given on $\widehat{\mathbb{Z}}(1)$ by $\varepsilon_{1s} \oplus \varepsilon_{2s}$, resp. ε_s, $\eta_s \oplus \eta_s^{q^{\deg(s_1)}}$.

Denote by $E(\mathcal{R}) = E_2(X, S; \mathcal{R})$ the set of isomorphism classes of irreducible rank two $\overline{\mathbb{Q}}_\ell$-smooth sheaves $\mathcal{F}$ on $X - S$ such that the inverse image $\mathcal{F}|X_{(s)}^*$ of $\mathcal{F}$ on $X_{(s)}^* = X_{(s)} - \{s\}$, $X_{(s)}$ the completion of X at s, is isomorphic to $\mathcal{R}(s)$ for $s \in S$. In particular it has principal unipotent monodromy twisted by ε_s at $s \in S^{D^+}$. Assume this data is regular, Frob*-compatible, in general position and the compatibility assumption is satisfied. Then the number of elements of $E_2(\mathcal{R})$ fixed by* $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ *(thus by $\mathbb{V}$), is the product of h_1 and the sum $\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_4$. Here*

$$\mathbf{s}_1 = \frac{\mathbf{h}_1(q)X(q)}{(q+1)(q-1)}, \qquad \mathbf{s}_2 = \frac{\mathbf{h}_1(-1)X(-1)}{2(q+1)}, \qquad \mathbf{s}_4 = -\frac{\mathbf{h}_1(1)X(1)}{2(q-1)}.$$

where

$$X(t) = X(t; S_1^{D^+}, S_1^{(1,1)}, S_1^{(2)}) = \prod_{s_1 \in S_1^{(1,1)}} (t^{d_{s_1}} + 1) \prod_{s_1 \in S_1^{(2)} \sqcup S_1^{D^+}} (t^{d_{s_1}} - 1) \cdot (t-1)^{-1}.$$

Write (1) if $S_1^{(1,1)}$ contains a place s_1 with odd $\deg(s_1)$ or $S_1' = S_1^{(2)} \sqcup S_1^{D^+}$ contains a place s_1 with even degree. Write (2) if $d_{s_1} = \deg(s_1)$ is even for all $s_1 \in S_1^{(1,1)}$ and odd for all $s_1 \in S_1'$.

If (1) then $\mathbf{s}_2 = 0$. If (2) then $\mathbf{s}_2 = \mathbf{h}_1(-1)\frac{(-1)^{|S_1^{(2)}|} 2^{|S_1|-2}}{q+1}$.

$\mathbf{s}_4$ is 0 if $S_1^{(2)}$ is not empty. If $S_1^{(2)} = \emptyset$ then $\mathbf{s}_4$ is $-h_1 \cdot \deg(s_1) \cdot 2^{|S_1^{(1,1)}|-1}/(q-1)$.

The case $|S_1''| = 0$ is discussed in [3] when $|S_1^{D^+}| \geq 2$ (in fact for general rank $r \geq 2$), and also in [8]. In [7] when $|S_1^{D^+}|$ is 0 or 1, in rank 2. The new case here is $|S_1''| \neq 0$.

The behavior of the number obtained in the theorem is geometric in the sense that when $(X_1, S_1)/\mathbb{F}_q$ is replaced by $(X_m, S_m)/\mathbb{F}_{q^m}$, $m \geq 1$ in $\mathbb{Z}$, it has the form $\sum_j n_j \gamma_j^m$, $n_j \in \mathbb{Z}$, γ_j Weil numbers, that is numbers of the form a product of a root of unity and a monomial in q and the inverse eigenvalues α of the action of Frob on $H^1(X, \mathbb{Q}_\ell)$. When $|S_1''| = 0$ this is shown in [3, Corollary 6.8] if $|S_1^{D^+}| \geq 2$ (general rank), and in [7] if $|S_1^{D^+}| \leq 1$ (rank 2). In the case considered here, as we base change from $\mathbb{F}_q$ to $\mathbb{F}_{q^m}$ the place s_1 in $S_1^{D^+}$, viewed as a Frob-orbit $s = s_1 \otimes_{\mathbb{F}_q} \mathbb{F}$, may split into several Frob^m-orbits, namely the singleton $S_1^{D^+}$ would become $S_m^{D^+} = S_1^{D^+} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$, a set of possibly larger cardinality. Then geometrization of the number in the theorem here has to be studied together with the case of $|S_1^{D^+}| \geq 2$, studied in [10]. We separate the two works as the technique employed here, using the full trace formula for GL(2) as in [9], is different than the one in [10], that uses the trace formula for an anisotropic inner form of GL(2). The result here, for $|S_1^{D^+}| = 1$, permits giving a more complete geometrization result in [10], for $|S_1^{D^+}| \geq 1$, not only for $|S_1^{D^+}| \geq 2$. The geometrization in the case $|S_1^{D^+}| = 0, |S_1''| \geq 1$, is also discussed in [10], based on the computations of [9].

Example 2.1 Suppose $X_1 = \mathbb{P}^1$, so $g = 0$ and $\mathbf{h}_1(t) = 1$. Assume $S_1^{(2)} = \{w\}$ and $S_1^{D^+} = \{u\}$ consist of one place each, of degrees d_w and d_u, and $S_1^{(1,1)}$ is empty. Then $\mathbf{s}_1 = \frac{(q^{d_w}-1)(q^{d_u}-1)}{(q+1)(q-1)^2}$, and $\mathbf{s}_4 = 0$. If d_w or d_u is even then $\mathbf{s}_2 = 0$, and $\mathbf{s}_1$ is an integer. If $\{d_w, d_u\} = \{1, 2\}$, then $\mathbf{s}_1 = 1$. If $d_w = 2 = d_u$ then $\mathbf{s}_1 = q + 1$. If both d_w and d_u are odd, then $\mathbf{s}_2 = -1/(q+1)$. Then $\mathbf{s}_1 + \mathbf{s}_2$ is an integer. For example, if $d_w = 1 = d_u$ then $\mathbf{s}_1 + \mathbf{s}_2 = 0$. If $\{d_w, d_u\} = \{1, 3\}$ then $\mathbf{s}_1 + \mathbf{s}_2 = q(q-1)$.

Example 2.2 Suppose again that $X_1 = \mathbb{P}^1$, so $g = 0$ and $\mathbf{h}_1(t) = 1$. Assume $S_1^{(1,1)} = \{w\}$ and $S_1^{D^+} = \{u\}$ consist of one place each, of degrees d_w and d_u, and $S_1^{(2)}$ is empty. Then $\mathbf{s}_1 = \frac{(q^{d_w}+1)(q^{d_u}-1)}{(q+1)(q-1)^2}$, and $\mathbf{s}_4 = -\deg(u)/(q-1)$. Suppose d_u is even or d_w is odd. Then $\mathbf{s}_2 = 0$. Then $\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_4 = (\frac{(q^{d_w}+1)(q^{d_u}-1)}{q^2-1} - \deg(u))/(q-1)$. For example when $d_u = 2$ and $d_w = 1$, we get 1. If $d_u = 1 = d_w$ we get 0. If $d_u = 2 = d_w$ we get $q + 1$.

If d_u is odd and d_w is even, $\mathbf{s}_2 = 1/(q+1)$, and

$$\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_4 = \frac{(q^{d_w}+1)(q^{d_u}-1)}{(q-1)(q^2-1)} + \frac{1}{q+1} - \frac{\deg(u)}{q-1}.$$

This is an integer. When $d_u = 1$, $d_w = 2$ we get 1. When $d_u = 3$, $d_w = 2$ we get $q^2 + q + 3$. When $d_u = 1$, $d_w = 4$ we get $q^2 + 1$.

Corollary 2.2 *The number* $\#E(\mathcal{R})^{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)} = \#E_2(X, S; \mathcal{R})^{\mathbb{V}}$ *computed in the theorem is independent of the prime* ℓ, $(\ell, q) = 1$. *It depends on* q, *on* X_1 (*via* $\mathbf{h}_1$), *on* $S_1^{D^+}$, $S_1^{(1,1)}$, $S_1^{(2)}$, $q_{s_1} = q^{d_{s_1}}$ *and* $|S_1|$, *but not on the relative position of* $S_1^{D^+}$, $S_1^{(1,1)}$, $S_1^{(2)}$ *in* X_1, *that is, if* $S_1^{D^+}$, $S_1^{(1,1)}$, $S_1^{(2)}$ *are replaced by* $\widehat{S}_1^{D^+}$, $\widehat{S}_1^{(1,1)}$, $\widehat{S}_1^{(2)}$ *with* $(q_{s_1}; s_1 \in S_1) = (\widehat{q}_{s_1}; s_1 \in \widehat{S}_1)$, *the result does not change, nor on the choice of tame data* $\mathcal{R} = (\mathcal{R}(s); s \in S)$, *that is,*

of $\{\varepsilon_{1,s}, \varepsilon_{2,s}; s_1 \in S_1^{(1,1)}\}$, $\{\varepsilon_s; s_1 \in S_1^{D^+}\}$ *and* $\{\eta_s, \overline{\eta}_s; s_1 \in S_1^{(2)}\}$, *as long as* $\mathcal{R}$ *is regular*, *in general position*, *and satisfies the compatibility assumption*.

In particular, if the data $(\varepsilon_{is_1}, \varepsilon_{s_1}, \eta_{s_1})$ is multiplied by a tame character ε_{s_1} at one place $s_1 \in S_1$, the counting result does not change. In fact the last sentence already implies that multiplying the data anywhere else in S_1 by a tame character will not change the result, as long as compatibility holds. Indeed, if $\varepsilon_{s_1} : k(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$ are characters of the residue fields for all $s_1 \in S_1$, with $\prod_{s_1 \in S_1} \varepsilon_{s_1}$ equals 1 on $\mathbb{F}_q^\times$ (compatibility), then there exists a character χ of the idèle class group, unramified outside S_1, tamely ramified at each $s_1 \in S_1$, thus of $A = (\mathbb{A}^\times/\mathcal{O}^{S_1,\times} \prod_{s_1 \in S_1}(1 + \mathfrak{m}_{s_1}))/F^\times$, with component χ_{s_1} satisfying $\chi_{s_1}|\mathcal{O}_{s_1}^\times = \varepsilon_{s_1}$ $(s_1 \in S_1)$. This is since $\prod_{s_1 \in S_1} k(s_1)^\times/\mathbb{F}_q^\times$ embeds in A, as $\mathcal{O}_A^\times \cap F^\times = \mathbb{F}_q^\times$.

However the proof is no more difficult if we twist by a tame character at one place $s_1 \in S_1$, or at several such places, in or out of S_1.

When $\varepsilon_{1s_1} = \varepsilon_{2s_1}$ is a tame character, of $k(s_1)^\times$, twisting by it at a place $s_1 \notin S_1$ does not change the counting formula. This corresponds to a factor "1" (as opposed to $q^{d(s_1)} \pm 1$ at $s_1 \in S_1$) for all $s_1 \notin S_1$, or the factor $\varepsilon_{1v}^{-1} \circ \det \cdot 1_{K_v}/|K_v|$ of the test function, instead of $1_{K_v}/|K_v|$, in the evaluation of the trace formula. The automorphic representations whose trace is nonzero in the trace formula for such test functions have component $\varepsilon_{1v} \otimes I(\mu_{1v}, \mu_{2v})$ with $\mu_{iv}|\mathcal{O}_v^\times = 1$. In particular Task 1.2 of [9] is already done (e.g. in [9]): the local factor in the sentece following that Task 1.2 is 1, not q_{s_1} as expected there.

In the situation of Theorem 2.1 we then have, as anticipated by [2, 1.2(B) and 2.15(iii)]:

Corollary 2.3 *As* $q \to \infty$, *the number* $\#E(\mathcal{R})^{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)}$ *computed in the theorem is* $q^{D/2} + O(q^{(D-1)/2})$, *where* $D/2 = g + 3(g-1) + \deg(S_1)$, *if* $D \geq 0$, *namely* $g \geq 1$, *or* $g = 0$ *and* $\deg(S_1) \geq 3$. *The dominant term* $q^{D/2}$ *occurs with coefficient* 1.

This D is the same as that occurring in [3, Corollary 6.19(ii)] for $\mathrm{GL}(n)$, where

$$D = 2g + \dim \mathrm{SL}(n) \cdot (2g-2) + \dim(\mathrm{SL}(n)/T) \cdot \deg(S_1),$$

$\dim \mathrm{SL}(n) = n^2 - 1$, $\dim(\mathrm{SL}(n)/T) = n^2 - n$, provided $D \geq 0$, where T is a maximal torus. In our case $n = 2$. See [2, sections 2.14–2.16, 1.2(B)], [3, section 6.16], for a characteristic zero analogue, over $\mathbb{C}$.

The term $h_1 \mathbf{s}_1$ of the trace formula, corresponding to the central elements:

$$\frac{h_1 \mathbf{h}_1(q) \prod_{s_1 \in S_1}(q_{s_1} + \iota(s_1))}{(1-q)^2(1+q)} = \frac{\prod_\alpha (1-\alpha)(1-\alpha q) \cdot \prod_{s_1 \in S_1^{(1,1)}}(q^{d_{s_1}} + 1) \cdot \prod_{s_1 \in S_1'}(q^{d_{s_1}} - 1)}{(q+1)(q-1)^2}$$

contains the leading term. Here $\iota(s_1) = 1$ if $s_1 \in S_1^{(1,1)}$, $\iota(s_1) = -1$ if $s_1 \in S_1' = S_1^{D^+} \sqcup S_1^{(2)}$. Note that $(-1)^{|S_1'|} = \prod_{s_1 \in S_1} \iota(s_1)$.

The proof of the theorem is based on its translation to the realm of automorphic representations by the Galois-Automorphic dictionary of L. Lafforgue [12]. In our case we have:

(i) There is a bijective correspondence between

(A1) isomorphism classes of rank $r = 2$ irreducible $\overline{\mathbb{Q}}_\ell$-smooth sheaves on $X_1 - S_1$, fixed by the Frobenius, with principal unipotent monodromy at $s_1 \in S_1^{D^+}$ twisted by a tame character $\varepsilon_{s_1} \circ \det$, $\varepsilon_{s_1} : k(s_1)^\times \to \overline{\mathbb{Q}}_\ell^\times$, and monodromy determined by the rank two $\overline{\mathbb{Q}}_\ell$-smooth sheaf $\mathcal{R}(s)$ ($s \in S^{(1,1)} = S_1^{(1,1)} \otimes_{\mathbb{F}_q} \mathbb{F}$, $s \in S^{D^+} = S_1^{D^+} \otimes_{\mathbb{F}_q} \mathbb{F}$ or $s \in S^{(2)} = S_1^{(2)} \otimes_{\mathbb{F}_q} \mathbb{F}$) corresponding to the representation ρ_s of the inertia group I_s that factorizes via $\widehat{\mathbb{Z}}(1) = I_s/P_s$ (is tame), given on $\widehat{\mathbb{Z}}(1)$ by $\varepsilon_{1s} \oplus \varepsilon_{2s}$ if $s_1 \in S_1^{(1,1)}$, ε_s if $s_1 \in S_1^{D^+}$, and by $\eta_s \oplus \eta_s^{q^{\deg(s_1)}}$ if $s_1 \in S_1^{(2)}$, and

(B1) classes modulo $\mathbb{F}_q$-twisting (introduced in [3, section 1.10], see below) of irreducible cuspidal automorphic representations π of $\mathrm{GL}(2, \mathbb{A})$ over $\overline{\mathbb{Q}}_\ell$, that are unramified outside S_1, are Steinberg twisted by a tamely ramified character at $s_1 \in S_1^{D^+}$, are Iwahori tamely ramified parabolically induced $I(\mu_{1s_1}, \mu_{2s_1})$, $\{\mu_{1,s_1}|\mathcal{O}_{s_1}^\times, \mu_{2,u}|\mathcal{O}_{s_1}^\times\} = \{\varepsilon_{1s_1}, \varepsilon_{2s_1}\}$, for all $s_1 \in S_1^{(1,1)}$, and are tamely cuspidal $\pi(\widetilde{\eta}_{s_1}) = \pi(W_{E_{s_1}}, W_{F_{s_1}}; \widetilde{\eta}_{s_1})$, $\widetilde{\eta}_{s_1}|\mathcal{O}_{E_{s_1}}^\times = \eta_{s_1}$ for all $s_1 \in S_1^{(2)}$.

(ii) The $\mathbb{F}_q$-twists of any π as in (B1) are all distinct.

Let $\mathbf{a} \in \mathbb{A}^\times$ be an idèle of F of positive degree $\deg(\mathbf{a}) > 0$. View $\mathbb{A}^\times$ as the center of $\mathrm{GL}(2, \mathbb{A})$. Then $\deg(\det(\mathbf{a})) = \deg(\mathbf{a}^2) = 2\deg(\mathbf{a})$. If χ is a character of the quotient $\mathbb{Z}$ of $\mathrm{GL}(2, \mathbb{A}) \xrightarrow{\det} \mathbb{A}^\times \xrightarrow{\deg} \mathbb{Z}$ then the central character $\omega_{\pi\chi}$ of $\pi\chi$ is $\omega_{\pi\chi}(\mathbf{a}) = \omega_\pi(\mathbf{a})\chi(2\deg(\mathbf{a}))$ at $\mathbf{a}$, so any π has an $\mathbb{F}_q$-twist π' with $\omega_{\pi'}(\mathbf{a}) = 1$. The π' is not unique; we may twist it by a character of $\mathbb{Z}/2\deg(\mathbf{a})$. Hence the number investigated in Theorem 2.1:

$$|E_2(X, S; \mathcal{R})^{\mathbb{V}}| = T(X_1; (\varepsilon_s; s_1 \in S_1^{D^+}); (\varepsilon_{1s} \oplus \varepsilon_{2s}; s_1 \in S_1^{(1,1)}); (\eta_s \oplus \eta_s^{q^{\deg(s_1)}}; s_1 \in S_1^{(2)})),$$

is the number of automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ as in (B1): unramified outside S_1; Steinberg twisted by a tamely ramified character at $s_1 \in S_1^{D^+}$; Iwahori tamely ramified $I(\mu_{1s_1}, \mu_{2s_1})$, $\{\mu_{1,s_1}|\mathcal{O}_{s_1}^\times, \mu_{2,s_1}|\mathcal{O}_{s_1}^\times\} = \{\varepsilon_{1s_1}, \varepsilon_{2s_1}\}$ for all $s_1 \in S_1^{(1,1)}$; and tamely cuspidal $\pi(\widetilde{\eta}_{s_1})$, $\widetilde{\eta}_{s_1}|\mathcal{O}_{E_{s_1}}^\times = \eta_{s_1}$ for all $s_1 \in S_1^{(2)}$, with central character $\omega_\pi(\mathbf{a}) = 1$, taken modulo $\mathbb{F}_q$-twists by a character of $\mathbb{Z}/2\deg(\mathbf{a})$. These twists are distinct. Hence $T(X_1, \dots)$ is $1/2\deg(\mathbf{a})$ times the number of these automorphic representations of $\mathrm{GL}(2, \mathbb{A})$. We may and will take $\mathbf{a}$ with degree 1. Note that we identified cuspidal irreducible automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with their equivalence classes, by multiplicity one theorem for $\mathrm{GL}(r)$, $r \geq 2$.

3 Automorphic representations

We now state our representation theoretic result. Here we drop the subscript 1, as we do not change the base field. So we denote X_1, S_1, F_1, etc. by X, S, F. Also we denote the point $s \in S$ by u or v. Our story concerns then $X/\mathbb{F}_q$ and $F = k(X)$ as above, and a set $S = S^{D^+} \sqcup S^{(1,1)} \sqcup S^{(2)}$, where $S^{D^+} = \{u\}$ is a singleton. Put $S' = S^{D^+} \sqcup S^{(2)}$ and $S'' = S^{(1,1)} \sqcup S^{(2)}$. S' is the set of square-integrable components, $S^{\square}$ could have been better notation, but it is long. S'' is the set of decorated components. The notation (1, 1) and (2) indicates partition of 2. As explained in [9], an extension of [9] and the work here to general rank r will use [4] and be stated in terms of partitions $\lambda \vdash r$.

Let $N^* = |S^*|$ be the cardinality of S^*. We consider the case where $N^{D^+} = 1$ and $N'' \geq 1$.

We consider tame ramification data of three types. The *principal series* type consists of $(\varepsilon_{1v}, \varepsilon_{2v})$, for $v \in S^{(1,1)}$, where ε_{iv} are characters $k_v^\times = \mathcal{O}_v^\times/(1 + \varpi_v \mathcal{O}_v) \to \overline{\mathbb{Q}}_\ell^\times$. Put $q_v = |k_v| = q^{d_v}$ and $\omega_v = \varepsilon_{1v}\varepsilon_{2v}$. The *discrete series* type consists of $(\eta_v, \overline{\eta}_v = \eta_v^{q_v})$, for $v \in S^{(2)}$, where η_v is a character of $k_{2v}^\times = \mathcal{O}_{E_v}^\times/(1 + \varpi_v \mathcal{O}_v) \to \overline{\mathbb{Q}}_\ell^\times$. Here $k_{2v} \simeq \mathbb{F}_{q_v^2}$ is a quadratic field extension of $k_v = \mathbb{F}_{q_v}$, the residue field of the ring of integers of a quadratic unramified field extension E_v of the local field F_v of F at v. At $u \in S^{D^+}$ the tame ramification is unipotent monodromy twisted by a tame character $\varepsilon_u : k_u^\times \to \overline{\mathbb{Q}}_\ell^\times$.

In addition to being *regular*: $\varepsilon_{1v} \neq \varepsilon_{2v}$, $\eta_v \neq \overline{\eta}_v$, they are assumed to satisfy *compatibility*: $\omega_v^+ \prod_{v \in S^{(1,1)}} \omega_v \cdot \prod_{v \in S^{(2)}} \eta_v = 1$ on $\mathbb{F}_q^\times$, where $\omega_u^+ = \varepsilon_u^2$ at $u \in S^{D^+}$, and be in *general position*: $\varepsilon_v \prod_{v \in S^{(1,1)}} \prod_{0 \leq j < d_v} \varepsilon_v^{(j)} \cdot \prod_{v \in S^{(2)}} \prod_{0 \leq j < d_v} \eta_v^{(j)} \neq 1$ on $\widehat{\mathbb{Z}}(1)$, for all choices $\varepsilon_v^{(j)} \in \{\varepsilon_{1v}^j, \varepsilon_{2v}^j\}$ and $\eta_v^{(j)} \in \{\eta_v^j, \overline{\eta}_v^j\}$. The last notion was introduced by Deligne [2, section 2] to force the local systems we count to be irreducible. It is used in our Proposition 4.2 to compute the term $S_2(f)$ of elliptic non-scalar elements, as it relies on [9, Proposition 7.4] (see 3rd line before the end of its proof).

We compute the cardinality

$$T = T(X, S; \{(\varepsilon_v); v \in S^{D^+}\}; (\varepsilon_{1v}, \varepsilon_{2v}); v \in S^{(1,1)}\}; \{(\eta_v, \overline{\eta}_v); v \in S^{(2)}\})$$

of the set of automorphic representations $\pi = \otimes_v \pi_v$ (irreducible admissible components π_v, see [1]) in $L^2(\mathbf{a}^{\mathbb{Z}} \cdot G(F) \backslash G(\mathbb{A}))$, $\mathbb{A} = \mathbb{A}_F$, $G = \mathrm{GL}(2)$, of $\mathrm{GL}(2, \mathbb{A})$, such that the central character $\omega = \omega_\pi$ takes the value $\omega(\mathbf{a}) = 1$ at a fixed $\mathbf{a} \in \mathbb{A}^\times$ of degree 1, up to multiplication by $\chi \circ \mathrm{val} \circ \det$, where χ is a character of $\mathbb{Z}/2$, such that:

(1) π_v is unramified for all v outside S;
(2) for all $v \in S^{(1,1)}$, $\pi_v = I(\mu_{1v}, \mu_{2v})$ where $\{\mu_{1v}|\mathcal{O}_v^\times, \mu_{2v}|\mathcal{O}_v^\times\} = \{\varepsilon_{1v}, \varepsilon_{2v}\}$;
(3) for all $v \in S^{(2)}$, $\pi_v = \pi(\widetilde{\eta}_v)$ is monomial: corresponds to the two dimensional representation of the Weil group $W(E_v/F_v)$ of the unramified quadratic field extension E_v/F_v, induced from the character $\widetilde{\eta}_v$ of $W(E_v/E_v) = E_v^\times$, where $\widetilde{\eta}_v|\mathcal{O}_{E_v}^\times = \eta_v$;

(4) at $u \in S^{D^+}$, π_u is $\mathrm{St}_u \otimes \widetilde{\varepsilon}_u \circ \det$, where $\widetilde{\varepsilon}_u$ is a tamely ramifield character of $F_u^\times$ with restriction ε_u on $\mathcal{O}_u^\times$, and St_u denotes the Steinberg (= special) representation of $G(F_u)$.

Recall that the zeta function of the curve X over $\mathbb{F}_q$ has the form $\zeta(X,t) = \frac{\mathbf{h}_1(t)}{(1-t)(1-qt)}$. Here $\mathbf{h}_1(t) = \det(1 - t \cdot \mathrm{Frob}|H^1(X))$ is a polynomial $\sum_{0\le i\le 2g} a_i t^i$ of degree $2g$ with integral coefficients a_i, satisfying the functional equation $\mathbf{h}_1(1/qt) = q^{-g}t^{-2g}\mathbf{h}_1(t)$. The Picard number of X is $h_1 = \mathbf{h}_1(1)$. When $g = 0$ we have $\mathbf{h}_1(t) = 1$, so $h_1 = 1$.

Theorem 3.1 *Let F be the function field of a geometrically irreducible projective smooth curve X over $\mathbb{F}_q$ of genus g, S a finite set of valuations on X that is the disjoint union of S^{D^+}, $S^{(1,1)}$ and $S^{(2)}$. Assume $|S^{D^+}| = 1, |S''| \ge 1$, and ℓ a prime with $(\ell, q) = 1$.*

For each $v \in S^{(1,1)}$ choose a pair of characters $\varepsilon_{1,v}$, $\varepsilon_{2,v} : k_v^\times \to \overline{\mathbb{Q}}_\ell^\times$. Here k_v is the residue field $\mathcal{O}_v/\varpi_v$ of the ring of integers $\mathcal{O}_v$ of the local field F_v. Its cardinality is $q_v = q^{d_v}$.

For $u \in S^{D^+}$ choose a character $\varepsilon_u : k(u)^\times \to \overline{\mathbb{Q}}_\ell^\times$.

For each $v \in S^{(2)}$ choose characters η_v, $\overline{\eta}_v : k_{2v}^\times \to \overline{\mathbb{Q}}_\ell^\times$, where $\overline{\eta}_v = \eta_v^{q_v}$, and k_{2v} is a quadratic extension of k_v, so $x \mapsto x^{q_v}$ generates $\mathrm{Gal}(k_{2v}/k_v)$.

Assume the data $\{\varepsilon_{1,v}, \varepsilon_{2,v}; v \in S^{(1,1)}\}$ and $\{\eta_v, \overline{\eta}_v; v \in S^{(2)}\}$ are regular : $\varepsilon_{1,v} \ne \varepsilon_{2,v}$ for all $v \in S^{(1,1)}$ and $\eta_v \ne \overline{\eta}_v$ for all $v \in S^{(2)}$, in general position and the compatibility assumption is satisfied.

Then the number of irreducible cuspidal representations π of $\mathrm{GL}(2,\mathbb{A})$ that are unramified outside S, are Steinberg twisted by a tamely ramified character $\widetilde{\varepsilon}_u$ with restriction ε_u on $\mathcal{O}_u^\times$ at $u \in S^{D^+}$, are Iwahori tamely ramified $I(\mu_{1u}, \mu_{2u})$, $\{\mu_{1,u}|\mathcal{O}_u^\times, \mu_{2,u}|\mathcal{O}_u^\times\} = \{\varepsilon_{1u}, \varepsilon_{2u}\}$, for all $u \in S^{(1,1)}$, and are tamely monomial $\pi(\widetilde{\eta}_u)$, $\widetilde{\eta}_u|\mathcal{O}_{E_u}^\times = \eta_u$ for all $u \in S^{(2)}$, is the product of $2h_1$ and the sum $\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_4$. Put $X(q;S^{D^+}, S^{(1,1)}, S^{(2)}) = \prod_{v\in S^{(1,1)}}(q^{d_v}+1)\prod_{v\in S^{(2)}\sqcup S^{D^+}}(q^{d_v}-1)$. Here

$$\mathbf{s}_1 = \frac{\mathbf{h}_1(q)}{(q^2-1)(q-1)} X(q;S^{D^+}, S^{(1,1)}, S^{(2)}),$$

Write (1) if $S^{(1,1)}$ contains a place u with odd $\deg(u)$ or $S^{(2)}$ contains a place u with even degree. Write (2) if $d_u = \deg(u)$ is even for all $u \in S^{(1,1)}$ and odd for all $u \in S^{(2)}$.

Suppose $u \in S^{D^+}$ has even degree. Then $\mathbf{s}_2 = 0$.

Suppose $u \in S^{D^+}$ has odd degree. If (1) then $\mathbf{s}_2 = 0$. If (2) then $\mathbf{s}_2 = -\mathbf{h}_1(-1)\frac{(-1)^{|S^{(2)}|}2^{|S|-2}}{q+1}$.

$\mathbf{s}_4$ is 0 if $S^{(2)}$ is not empty. If $S^{(2)} = \emptyset$ then $\mathbf{s}_4$ is $-h_1 \cdot \deg(u) \cdot 2^{|S^{(1,1)}|-1}/(q-1)$.

The space $L = L^2(\mathbf{a}^{\mathbb{Z}} \cdot G(F)\backslash G(\mathbb{A}))$ of automorphic forms is a left $G(\mathbb{A})$-module, $g \in G(\mathbb{A})$ maps $\phi \in L$ to $(r(g)\phi)(h) = \phi(hg)$. It decomposes as the direct sum $L_0 \oplus L_c$.

Here L_0 is the space of cusp forms. It decomposes as a direct sum of irreducible representations π named cuspidal, which, in the context of any reductive algebraic group, each occurs with finite multiplicity $m(\pi)$; it is 1 for all cuspidal π in the case of $G = \mathrm{GL}(n)$.

The continuous spectrum L_c in our case of GL(2) decomposes as the discrete part L_1 of one-dimensional representations, and the direct integral L_c of Eisenstein series.

The trace formula gives an expression for the trace of the convolution operator $r(f\,dg)$, $(r(f\,dg)\phi)(g) = \int_{G(\mathbb{A})} f(h)\phi(gh)dh$, restricted to the cuspidal spectrum.

We fix the Haar measure $dg = \otimes_v dg_v$, $v \in |X|$, by $\mathrm{vol}(K_v) = 1$ for all $v \in |X|$. Then we can identify the measure fdg with a function $f \in C_c^\infty(G(\mathbb{A}))$.

On the spectral side, $\mathrm{tr}\, r(f)|L_0$ is the sum of $m(\pi)\mathrm{tr}\,\pi(f) = \mathrm{tr}\,\pi(f)$ over all equivalence classes of cuspidal automorphic representations of $G(\mathbb{A})$.

We take the test function $f = \otimes_v f_v$, $v \in |X|$, such that

(1) f_v is the characteristic function $f_v^0 = \chi_{K_v}$ of the maximal compact subgroup $K_v = \mathrm{GL}(2, \mathcal{O}_v)$ for each $v \notin S$.

(2) It is $f_u\begin{pmatrix} a & b \\ \varpi_u c & d \end{pmatrix} = |I_u|^{-1}\varepsilon_{1u}^{-1}(a)\varepsilon_{2u}^{-1}(d)$, $\mathrm{supp} f_u = I_u$, for all places u in $S^{(1,1)}$. Here ε_{iu} are characters $\mathcal{O}_u^\times \to \overline{\mathbb{Q}}_\ell^\times$ trivial on $1 + \varpi_u\mathcal{O}_u$, with $\varepsilon_{1u}/\varepsilon_{2u}$ nontrivial on $\mathbb{F}_{q_u}^\times \simeq k_u^\times = \mathcal{O}_u^\times/(1 + \varpi_u\mathcal{O}_u)$, for all u in $S^{(1,1)}$. The Iwahori subgroup I_u consists of all $\begin{pmatrix} a & b \\ \varpi_u c & d \end{pmatrix}$, $a, d \in \mathcal{O}_v^\times$, $b, c \in \mathcal{O}_v$.

(3) For each $u \in S^{(2)}$ it is the almost pseudo-coefficient $f_u = f_u^{\eta_u}$ of the monomial cuspidal representations $\pi_u(\widetilde{\eta}_u)$ associated to any character $\widetilde{\eta}_u : E_{1u}^\times \to \overline{\mathbb{Q}}_\ell^\times$ whose restriction to $\mathcal{O}_{E_{1u}}^\times$ is trivial on $1 + \varpi_u\mathcal{O}_{E_{1u}}$, and $\widetilde{\eta}_u$ restricted to $k_{2u}^\times = \mathcal{O}_{E_{1u}}^\times/(1 + \varpi_u\mathcal{O}_{E_{1u}}) \simeq \mathbb{F}_{q_u^2}^\times$ is η_u, as constructed in [9, subsection 6.4], for each $u \in S^{(2)}$. Here the character $\eta_u : \mathbb{F}_{q_u^2}^\times \to \overline{\mathbb{Q}}_\ell^\times$ satisfies $\eta_u \neq \eta_u^{q_u}$.

(4) At $u \in S^{D^+}$ the component f_u will be $\varepsilon_u^{-1} \circ \det$ times the quotient $\chi_{I_u}/|I_u|$ of the characteristic function χ_{I_u} by the volume $|I_u|$ of I_u. We have $K_u/I_u = \mathrm{GL}(2, \mathbb{F}_{q_u})/B(\mathbb{F}_{q_u})$, of cardinality $(q_u^2 - 1)(q_u^2 - q_u)/q_u(q_u - 1)^2 = q_u + 1$, so $|I_u| = 1/(q_u + 1)$.

We shall then take f_u at $u \in S^{D^+}$ to be $f_u' = \varepsilon_u^{-1} \circ \det \cdot(\chi_{I_u}/|I_u| - 2\chi_{K_u})$, the difference times $\varepsilon_u^{-1} \circ \det$. This is an *almost pseudo-coefficient* of the family of the Steinberg representation St_u twisted by a tamely ramified character. Namely it has the property that $\mathrm{tr}\,\pi_u(f_u')$ is 0 unless (1) π_u is $\mathrm{St}_u \otimes \widetilde{\varepsilon}_u \circ \det$, where $\widetilde{\varepsilon}_u$ is a character of $F_u^\times$ with restriction ε_u on $\mathcal{O}_u^\times$, where $\mathrm{tr}\,\pi_u(f_u') = 1$, or (2) $\pi_u = \widetilde{\varepsilon}_u \circ \det$, where $\mathrm{tr}\,\pi_u(f_u') = -1$. We write f' for f where f_u is replaced by f_u', and f^K for f where f_u is replaced by $\widetilde{\varepsilon}_u^{-1}\chi_{K_u}$, $u \in S^{D^+}$.

Then we shall compute the terms $S_*(f') = S_*(f) - 2S_*(f^K)$.

Thus $f_u = f_u^{\eta_u}$ is supported on $K_u = \mathrm{GL}(2, \mathcal{O}_u)$, and $f_u(g)$ $(g \in K_u)$ depends only on $\mathrm{pr}_u(g) = \overline{g} \in \mathrm{GL}(2, \mathbb{F}_{q_u})$ and equals $(q_u - 1)\eta_u^{-1}(a)$ at $\overline{g} = \mathrm{diag}(a, a)$, $a \in \mathbb{F}_{q_u}^\times$; $-\eta_u^{-1}(a)$ at $\overline{g}$ equals to a principal unipotent times $a \in \mathbb{F}_{q_u}^\times$; $-\eta_u^{-1}(\zeta) - \eta_u^{-1}(\zeta^q)$ at $\overline{g}$ with eigenvalues $\zeta, \overline{\zeta} = \zeta^q \in \mathbb{F}_{q_u^2} - \mathbb{F}_{q_u}$; 0 at $\overline{g}$ with eigenvalues $a \neq b$ in $\mathbb{F}_{q_u}^\times$.

By [9, section 6.2] $\mathrm{tr}\,\pi_u(f_u)$ is 0 unless $\pi_u = I(\mu_{1u}, \mu_{2u})$, $\{\mu_{1,u}|\mathcal{O}_u^\times, \mu_{2,u}|\mathcal{O}_u^\times\} = \{\varepsilon_{1u}, \varepsilon_{2u}\}$, when it is 1, for all $u \in S^{(1,1)}$. Hence we call such $f_u = f_u^{\varepsilon_{iu}}$ an *almost pseudo-coefficient* of the family $I(\mu_{1u}, \mu_{2u})$.

By [9, section 6.4] $\mathrm{tr}\,\pi_u(f_u)$ is 0 unless $\pi_u = \pi(\widetilde{\eta}_u)$ is monomial, where it is 1, for all $u \in S^{(2)}$. Hence we call such $f_u = f_u^{\eta_u}$ an *almost pseudo-coefficient* of the cuspidal family $\pi_u(\widetilde{\eta}_u)$.

For all $v \notin S$ we have $\mathrm{tr}\,\pi_v(f_v)$ is 0 unless π_v is unramified, where it is 1. We call f_v an *almost pseudo-coefficient* of the family of unramified representations π_v.

At $u \in S^{D^+}$, $\mathrm{tr}\,\pi_u(f_u)$ is 2 if π_u is induced $I(\mu_{1u}, \mu_{2u})$ from unramified characters $\widetilde{\varepsilon}_u\mu_{1u}, \widetilde{\varepsilon}_u\mu_{2u}$; 1 if π_u is a tamely ramified character $\widetilde{\varepsilon}_u \circ \det$, or $\mathrm{St}_u \otimes \widetilde{\varepsilon}_u \circ \det$ for such $\widetilde{\varepsilon}_u$; zero if not.

Thus for this f, $\mathrm{tr}\, r(f)|L_0$ is the number of irreducible cuspidal representations π of $\mathrm{GL}(2, \mathbb{A})$ that are unramified outside S, are tamely monomial $\pi(\widetilde{\eta}_u)$, $\widetilde{\eta}_u|\mathcal{O}_{E_u}^\times = \eta_u$ for all $u \in S^{(2)}$, are Iwahori tamely ramified $I(\mu_{1u}, \mu_{2u})$, $\{\mu_{1,u}|\mathcal{O}_u^\times, \mu_{2,u}|\mathcal{O}_u^\times\} = \{\varepsilon_{1u}, \varepsilon_{2u}\}$, for all $u \in S^{(1,1)}$, and a representation that has an Iwahori fixed vector at $u \in S^{D^+}$, multiplied by $\varepsilon_u \circ \det$. This is the number we wish to compute first.

It is easy to see (see [9, Proposition 7.2]):

Proposition 3.2 *If there is a cuspidal irreducible automorphic representation π with $\mathrm{tr}\,\pi(f) \neq 0$ for a test function f as above then the compatibility assumption is satisfied.*

4 Trace formula

The trace formula is an analytic, or geometric, expression for the spectral side: $\mathrm{tr}\, r(f)|L_0$. In our case of GL(2) (see [6, 7, 9]) it is the sum of eight terms $S_i(f)$, $1 \leq i \leq 8$, that we proceed to compute for our test function f. Our computations are based on those of [7] and [9]. We use this computation and those of [7] to compute the trace formula at f'.

Proposition 4.1 *The term in the geometric side of the trace formula corresponding to the central elements is $S_1 = 2h_1\overline{S_1}$ where*

$$\overline{S}_1(f) = \frac{\mathbf{h}_1(q)}{(1-q)(1-q^2)} \prod_{u \in S^{(1,1)} \sqcup S^{D^+}} (q^{d_u} + 1) \cdot \prod_{u \in S^{(2)}} (q^{d_u} - 1).$$

Hence

$$\overline{S}_1(f') = \frac{\mathbf{h}_1(q)}{(1-q)(1-q^2)} \prod_{u \in S^{(1,1)}} (q^{d_u}+1) \cdot \prod_{u \in S^{(2)} \sqcup S^{D^+}} (q^{d_u}-1).$$

Proof The only change from the proof of [9, Proposition 7.3] is that the value of f_u, $u \in S^{D^+}$, at the identity e, is $q^{d_u}+1$, and not 1 as there. This gives $S_1(f)$. $S_1(f^K)$ is given in [9, Proposition 7.3] by the same formula, but the first product ranges only over $S^{(1,1)}$. Then we take $f' = f - 2f^K$ to get $S_1(f')$. □

Remark 4.1 $\overline{S}_1(f')$ is an integer when $S^{(2)}$ is nonempty and there is a $u \in S^{(2)} \sqcup S^{D^+}$ with even degree or $u \in S^{(1,1)}$ with odd degree. When $S^{(2)}$ is empty $\overline{S}_1(f')$ is not an integer.

Proposition 4.2 *The term $S_2(f)$ in the geometric side of the trace formula of f corresponding to the elliptic elements is 0 if $S^{(1,1)} \sqcup S^{D^+}$ contains a place u with odd* $\deg(u)$, *or if $S^{(2)}$ contains a place u with even degree. Suppose $d_u = \deg(u)$ is even for all $u \in S^{(1,1)} \sqcup S^{D^+}$ and odd for all $u \in S^{(2)}$. Then $S_2 = 2h_1\overline{S}_2$ where*

$$\overline{S}_2(f) = -\mathbf{h}_1(-1)\frac{(-1)^{|S^{(2)}|}2^{|S|-2}}{q+1}.$$

$S_2(f^K)$ is given by the same formula with S^{D^+} replaced by the empty set, thus S (in the exponent) is replaced by $S'' = S^{(1,1)} \sqcup S^{(2)}$.

Proof Our $S_2(f^K)$ is computed in [9, Proposition 7.4]; it is then given by the last sentence of the proposition. To compute $S_2(f)$ note that for $u \in S^{D^+}$, the orbital integral $\int f_u(x\gamma x^{-1})dx$ is 2 (see [7, Lemma 4.3]). The proof of [9, Proposition 7.4] shows we get $2^{|S''|}$ products for the $u \in S''$, each is $-(q-1)$. No other changes in the proof of [9, Proposition 7.4]. However S here is $S'' \sqcup S^{D^+}$, and there it is $S'' = S^{(1,1)} \sqcup S^{(2)}$. □

To compute $S_2(f') = S_2(f) - 2S_2(f^K)$ consider $u \in S^{D^+}$ with even and odd degree.

Corollary 4.3 *Write* (1) *if $S^{(1,1)}$ contains a place u with odd* $\deg(u)$ *or $S^{(2)}$ contains a place u with even degree*; (2) *if $d_u = \deg(u)$ is even for all $u \in S^{(1,1)}$ and odd for all $u \in S^{(2)}$.*

Suppose $u \in S^{D^+}$ has odd degree. Then $S_2(f) = 0$. If (1) *then $S_2(f^K) = 0$, so $S_2(f') = 0$, and $\overline{S}_1(f') + \overline{S}_2(f')$ is an integer precisely when $S^{(2)} \neq \emptyset$. If* (2) *then $2S_2(f^K)$ is equal to the expression displayed in Proposition 4.2. Then $\overline{S}_1(f') + \overline{S}_2(f')$ has the form $(A(q)+B)/(q+1)$, where*

$$A(q) = \frac{\mathbf{h}_1(q)}{(q-1)^2} \prod_{u \in S^{(1,1)}} (q^{d_u}+1) \cdot \prod_{u \in S^{(2)} \sqcup S^{D^+}} (q^{d_u}-1), \qquad B = \mathbf{h}_1(-1)(-1)^{|S^{(2)}|}2^{|S|-2}.$$

The limit of the rational function $A(q)+B$ in q as $q \to -1$ is zero. So it is divisible by $q+1$.

Suppose $u \in S^{D^+}$ has even degree. If (1) then $S_2(f) = 0$ and $S_2(f^K) = 0$, so $S_2(f') = 0$, and $\overline{S}_1(f') + \overline{S}_2(f') = \overline{S}_1(f')$ is an integer precisely when $S^{(2)}$ is nonempty. If (2) then $\overline{S}_2(f)$ is as displayed in the proposition, and $2\overline{S}_2(f^K) = \overline{S}_2(f)$, so $\overline{S}_2(f') = 0$, and again $\overline{S}_1(f') + \overline{S}_2(f') = \overline{S}_1(f')$ is an integer precisely when $S^{(2)}$ is nonempty.

We then need to study the case where $S^{(2)}$ is empty. First we compute the remaining terms $S_i(f')$.

The term $S_3(f)$ of the weighted orbital integrals in the trace formula, is 0, the proof is as in [7, Lemma 3.5], based on the fact that our test function f is supported on the maximal compact subgroup $K = \prod_{v \in |X|} K_v$. The same holds for $S_3(f')$.

Proposition 4.4 *The term $S_4(f') = 2h_1\overline{S}_4(f')$ is zero if $S^{(2)}$ is not empty. If $S = S^{(1,1)} \sqcup S^{D^+}$ then $\overline{S}_4(f') = -h_1 2^{|S^{(1,1)}|-1} d_u^+/(q-1)$.*

Proof We follow closely the first four paragraphs in the proof of [9, Proposition 7.5] and observe that if $S^{D^+} = \{u\}$ then

$$\theta_{1,f'_u}(t_u) = \theta_{1,(q_u+1)\chi_{I_u}}(t_u) - 2\theta_{1,\chi_{K_u}}(t_u) = \frac{1+t_u}{1-t_u/q_u} - \frac{2}{1-t_u/q_u} = \frac{t_u-1}{1-t_u/q_u} = \theta_{1,f_u^{\eta_u}}(t_u).$$

In other words, the local factor $\theta_{1,f'_u}(u \in S^{D^+})$ makes the same contribution as $\theta_{1,f_u^{\eta_u}}$ at $u \in S^{(2)}$. By the computation of [9, Proposition 7.5] we conclude that $S_4(f')$ is 0 if $S^{(2)}$ is not empty, and it is as asserted when $S^{(2)}$ is empty. □

Proposition 4.5 *The term $S_5(f)$ is zero.*

Proof The only change from the proof of [9, Proposition 7.6] are that line 1 of the 2nd paragraph has to start with: "Assume $S = S^{(1,1)} \sqcup S^{D^+}$"; at the end of the next sentence, after "otherwise", we add: "where for $u \in D^+$ by ε_{1u}, ε_{2u} we mean 1". □

Proposition 4.6 *The term $S_6(f')$ of weighted traces is zero.*

Proof It is shown in [9, Proposition 7.7] that $\overline{S}_6(f^K) = 0$ when $|S^{(2)}| \neq 1$.

The term $S_6(f)$ is $\frac{-1}{4\pi i} = -\frac{1}{2} \cdot \frac{1}{2\pi i}$ times the sum over all characters μ_1, μ_2 of $\mathbb{A}^\times/F^\times \cdot \mathbf{a}^{\mathbb{Z}}$ of

$$\int_{|t|=1} \operatorname{tr} I(\mu_1\nu_t, \mu_2\nu_{t^{-1}}; f) R(\mu_1, \mu_2; t)^{-1} \frac{d}{dt} R(\mu_1, \mu_2; t) dt.$$

Write $R(\mu_1, \mu_2; t)$ as a product $\otimes_v R(\mu_{1v}, \mu_{2v}; t_v)$, $t_v = t^{\deg(v)}$. Applying $\frac{d}{dt}$ we obtain a sum over u of

$$\operatorname{tr} I(\mu_{1u}\nu_t, \mu_{2u}\nu_{t^{-1}}; f_u) R(\mu_{1u}, \mu_{2u}; t_u)^{-1} \frac{d}{dt} R(\mu_{1u}, \mu_{2u}; t_u)$$

times $\prod_{v\neq u} \operatorname{tr} I(\mu_{1v}, \mu_{2v}; t_v)$ in the integral. For all $u \in |X| - S$, f_u is the characteristic function of K_u, so the operator $I(\mu_{1u}\nu_{t_u}, \mu_{2u}\nu_{t_u^{-1}}; f_u)$ acts as the projection to the space of $\mathrm{GL}(2, \mathcal{O}_u)$-fixed vectors in $I(\mu_{1u}\nu_{t_u}, \mu_{2u}\nu_{t_u^{-1}})$, a one-dimensional space independent of t. Hence $R(\mu_{1u}, \mu_{2u}; t_u)$ is independent of t (as in [7, Lemma 3.8]); its derivative is 0.

There remain in the integrand then only the summands indexed by $u \in S$. If $|S^{(2)}| \geq 2$ then for $w \in S^{(2)}$ the components f_w are almost pseudo-coefficients of cuspidal representations so the integrand vanishes by [9, Proposition 6.4].

If $S^{(2)} = \{w\}$ consists of a single element w, and if we take the component f'_u to be $f_u - 2\chi_{K_u}$ multiplied by $\varepsilon_u^{-1} \circ \det$, which is an almost pseudo-coefficient of the square-integrable Steinberg representation times $\varepsilon_u \circ \det$, the summands indexed by w and u vanish. We do not compute $\overline{S}_6(\varepsilon_u^{-1} \circ \det \cdot f)$ here when $S^{(2)} = \{w\}$ consists of a single element w, and the component of f at $u \in S^{D^+}$ is our f_u. We also do not compute $\overline{S}_6(\varepsilon_u^{-1} \circ \det \cdot f^K)$ in [9] in this case where $|S^{(2)}| = 1$. However, we do compute here $\overline{S}_6(f') = \overline{S}_6(f) - 2\overline{S}_6(\varepsilon_u^{-1} \circ \det \cdot f^K)$, which is what matters.

We next deal with the case where $S^{(2)}$ is empty. The proof of [9, Proposition 7.7] shows that the summands indexed by $u \in S^{(1,1)}$ are zero. So we are left with the single summand indexed by $u \in S^{D^+}$. The factor $\operatorname{tr} I(\mu_{1v}\nu_t, \mu_{2v}\nu_{t^{-1}}; f_v)$ is 1 for $v \notin S$. It is 2 for $v \in S^{(1,1)}$.

The local factor $J_u(f_u) = \operatorname{tr} I(\mu_{1u}\nu_t, \mu_{2u}\nu_{t^{-1}}; f_u) \cdot R_u^{-1} \cdot \frac{d}{dt} R_u$, $R_u = R_u(\mu_{1u}, \mu_{2u}, t)$, $u \in S^{D^+}$, is computed in the proof of [7, Lemma 4.7]. The line before last in that proof asserts that the global integral is $2\deg(u)$. Had $S^{(1,1)}$ been empty, we would have had a sum over the characters μ_1, μ_2 of $\mathbb{A}^\times / F^\times \mathcal{O}^\times \mathbf{a}^{\mathbb{Z}}$, a set of volume h_1 (see, e.g., [7, p. 747, l. 3], in the proof of [7, Lemma 3.3]). Noting that there is the factor $-\frac{1}{2}$ from the first line of this proof, [7, Lemma 4.7] concludes that $S_6(f)$ is $-d_u h_1^2$. This is the case of empty $S^{(1,1)}$. Noting that $J_u(f_u)$ is 0 if f_u is replaced by f_u^0, we would have gotten $J_u(f_u) = J_u(f'_u)$. In our case $S^{(1,1)}$ would have contributed a factor of 2 for each $u \in S^{(1,1)}$, namely $2^{|S^{(1,1)}|}$. However, there are no μ_1, μ_2 for which the summands are nonzero when $S^{(1,1)} \neq \emptyset$. Hence $S_6(f) = 0 = S_6(f')$. □

Proposition 4.7 *The Eisenstein terms $S_7(f)$ and $S_8(f)$ are zero.*

Proof We have $S_7(f) = \frac{1}{2} \sum_{\mu_1,\mu_2} \operatorname{tr} I(\mu_1, \mu_2; f)$, $\mu = \mu_1 = \mu_2$ ranges over the characters of $\mathbb{A}^\times / F^\times \cdot \mathbf{a}^{2\mathbb{Z}}$. If f has a component which is an almost pseudo coefficient of a cuspidal representation, each trace is zero. If $S = S^{(1,1)} \sqcup S^{D^+}$ then the character μ contributes 0 unless $\mu_1|\mathcal{O}_u^\times = \varepsilon_{1u}$, $\mu_2|\mathcal{O}_u^\times = \varepsilon_{2u}$ for all $u \in S$, where $\varepsilon_{1u} = \varepsilon_u = \varepsilon_{2u}$ for $u \in S^{D^+}$. But $\varepsilon_{1u} \neq \varepsilon_{2u}$ for $u \in S^{(1,1)}$, and $S^{(1,1)}$ is nonempty. Hence there are no such characters μ and $S_7(f)$ is 0.

We have $S_8(f) = -\sum_\mu \int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x)dx$, where μ ranges over the set of characters of $\mathbb{A}^\times / F^\times \cdot \mathbf{a}^{2\mathbb{Z}}$. If f has a component which is an almost pseudo coefficient of a cuspidal representation, each integral is zero. Assume then that $S = S^{(1,1)} \sqcup S^{D^+}$. If the local

factor $\int f_u \left[\begin{pmatrix} a & b \\ \varpi_u c & d \end{pmatrix} \right] \mu_u(ad) = \int \varepsilon_{1u}^{-1}(a)\varepsilon_{2u}^{-1}(d)\mu_u(ad)d^\times a d^\times d \quad (a, d \in \mathcal{O}_u^\times)$ is nonzero, then $\mu_u = \varepsilon_{1u}$ and also $\mu_u = \varepsilon_{2u}$. But $\varepsilon_{1u} \neq \varepsilon_{2u}$ for $u \in S^{(1,1)}$, and $S^{(1,1)}$ is non-empty. Hence the local, and so the global, integral is 0. □

We can now continue Corollary 4.3 when the set $S^{(2)}$ is empty.

a. Suppose, for the $u \in S^{D^+}$, $\deg(u)$, denoted d_u^+, is even. Then $\overline{S}_2(f') = 0$. Then $(q-1)\overline{S}_1(f') = \mathbf{h}_1(q) \cdot \prod_{S^{(1,1)}}(q^{d_u}+1) \cdot \frac{q^{d_u^+}-1}{q^2-1}$. Its limit at $q \to 1$ is $h_1 \cdot 2^{|S^{(1,1)}|-1} \cdot d_u^+$. It is canceled by $\overline{S}_4(f') = -h_1 2^{|S^{(1,1)}|-1} d_u^+/(q-1)$, when $S^{(2)}$ is empty and $2|\deg(u)$, $u \in S^{D^+}$.
b. Suppose $d_u^+ = \deg(u)$ is odd for the $u \in S^{D^+}$. Suppose (1) holds. Then $\overline{S}_2(f') = 0$. Then $(q-1)\overline{S}_1(f') = \frac{\mathbf{h}_1(q)}{q+1} \prod_{S^{(1,1)}}(q^{d_u}+1) \cdot \frac{q^{d_u^+}-1}{q-1}$. Its limit at $q \to 1$ is $h_1 2^{|S^{(1,1)}|-1} d_u^+$. This is canceled by $\overline{S}_4(f') = -h_1 2^{|S^{(1,1)}|-1} d_u^+/(q-1)$ when $S^{(2)}$ is empty and $\deg(u)$ is odd, $u \in S^{D^+}$.
c. Suppose $d_u^+ = \deg(u)$ is odd for the $u \in S^{D^+}$. Suppose (2) holds: $\deg(u)$ is even for all $u \in S^{(1,1)}$. Then $(q-1)\overline{S}_1(f') = (q+1)^{-1}\mathbf{h}_1(q) \prod_{S^{(1,1)}}(q^{d_u}+1) \cdot \frac{q^{d_u^+}-1}{q-1}$. Its limit as $q \to 1$ is again $h_1 2^{|S^{(1,1)}|-1} d_u^+$. Again this is canceled by $\overline{S}_4(f')$.

As we saw that $S_i(f') = 0$ for $i = 3, 5, 6, 7, 8$, the theorem follows.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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