GENERAL RECURRENCES IN MODULAR GROUPS

Ramazan Dikici
Atatürk Üniversitesi
K. K. Eğitim Fakültesi
Matematik Eğitimi Bölümü
ERZURUM

&

Engin Özkan
Atatürk Üniversitesi
Fen-Edebiyat Fakültesi
Matematik Bölümü
ERZURUM

Abstract

We have proved two original theorems concerning the periodicity and Wall number of the 2-step general Fibonacci sequences constructed by two generating elements of modular group.

Key Words: Nilpotent Group, General Recurrences Relations, 2-Step Fibonacci Sequences, Wall Number, Fundamental Period.

1.INTRODUCTION

The study of Fibonacci sequences in groups began with the earlier work of Wall [1] where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid eighties, Wilcox extended the problem to abelian groups [2]. Aydın and Smith proved in [3] that the lengths of ordinary 2-step Fibonacci sequences are equal to the length of 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency clas 4 and exponent a prime number p. The theory has been generalized in [4] to the ordinary 3-step Fibonacci sequences in finite nilpotent groups of nilpotency class 2 and exponent p.

From now on, f_n is reserved to designate the Fibonacci sequences constructed by the two generating elements of the modular group and p is a prime. 2-step Fibonacci sequence in group is defined by $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ which is constructed by two generating elements of the group. Since we work on modulo p, this sequence would ultimately be periodic and we call the shortest period of this sequence fundamental period and denote by *k(G).* Moreover, *k(p)* denotes the fundamental period of the ordinary 2-step Fibonacci sequence 0, 1, 1, 2, 3, ... taken modula a distinguished prime *p.* We sometimes call it Wall's number after a pioneer in this area.

The aim of this paper is to prove similar theorems to that of [4]. In addition to this, a theorem related to the periodicity of this sequences is given. Moreover, the group is chosen to be modular one. Clearly, the modular group is not abelian. Of course, nilpotent groups

are close to abelian in some sense, and one expects linear methods to be effective in this context.

2.THE MAIN RESULTS

Let G be a group and let $x, y \in G$. If every element of G can be written as a word

$$x^{\alpha_1} y^{\alpha_2} x^{\alpha_3} \dots x^{\alpha_{n-1}} y^{\alpha_n}$$

where $\alpha_i \in Z$, $1 \le i \le n$, then we say that x and y generate G and G is a 2-generator group. Although cyclic groups are 2-generator groups according to this definition we are only interested here in 2-generator groups which can not be generated by a single element.

A modular group is given by the following definition

DEFINITION 2.1: The set of all Möbius transformations of the form

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where a, b, c, d are integers with $ad-bc=1$, is called the modular group under the composition of transformations [5].

This group can be represented by 2x2 integer matrices as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $detA=1$ provided we identify each matrix with its negative, since A and $-A$ represent the same transformation.

THEOREM 2.2: The modular group Γ is generated by two matrices

$$T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

That is every A in Γ can be expressed in the form $A = T^{n_1} S T^{n_2} \dots S T^{n_k}$ where the n_i are integers.This representation is not unique [5].

We now express the first theorem concerning the Fibonacci sequences in modular group.

THEOREM 2.3: Let Γ be a modular group generated by $S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $k(\Gamma) \neq k(p)$.

PROOF: The group Γ is a finitely generated infinite group. Take the generators $S, T \in \Gamma$. We shall construct the 2-step Fibonacci sequence starting with this generating pair, where the product is to be understood as matrix multiplication. It can also be considered the composition of functions when the matrices are thought as functions. Thus, we have the sequence

$$S, T, ST, TST, ST^2ST, \ldots \qquad (1)$$

We first of all, demonstrate some data about Wall number of ordinary -step Fibonacci sequence for various primes *p.* This is,

p	***k(p)***
2	3
3	8
5	20
7	16
11	10
13	28
17	36
19	18
23	48

Figure 1

There is a direct connection between p and k(p) in the case of cyclic groups. k(p) stands for k(G). Wall distinguishes this connection in [1] and append a list of p and k(p) where 5<p<2000. We have used a computer code to calculate the next term of (1). We have more data but for the hospitality of the journal a small part of it is given. This is,

p	***k(p)*+1 th element of (1)**	**expected value**
2	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
5	$\begin{pmatrix} -3 & 0 \\ 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
7	$\begin{pmatrix} -5 & -2 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
11	$\begin{pmatrix} -1 & -3 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
13	$\begin{pmatrix} 5 & 0 \\ -5 & -5 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
17	$\begin{pmatrix} 3 & 13 \\ -8 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
19	$\begin{pmatrix} 15 & 1 \\ 3 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
23	$\begin{pmatrix} -21 & -13 \\ -12 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Figure 2

As it can be seen from the both of Figure 1 and Figure 2, the proof of Theorem 2.3 is obvious. This result has a great importance in respect to group theory. At the same time, it tells us that the fundamental period of the sequence made by two generating elements of modular group is different from that of cyclic groups.

THEOREM 2.4: Let G be a non-trivial finite p-group of exponent p and nilpotency class 2. Then $k(G)=k(p)$ [6].

A modular group Γ is given by the presentation

$$< a,b: a^4 = b^p = (ab)^3 = (ab^4 ab^{(p+1)\backslash 2})^2 = 1 > .$$

As matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ maps to a and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$maps to b under the homomorphism from to
$SL(2, p)$ to $PSL(2, p)$.

The 2-step general Fibonacci sequence constructed by any two generating pair is defined by

$$S, T, S^m T^l, T^m (S^m T^l)^l, (S^m T^l)^m (T^m (S^m T^l)^l)^l, \ldots \qquad (2)$$

where $l, m \in Z$. We now prove the second original theorem of this document.

THEOREM 2.5: Let Γ be a modular group generated by $T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The series constructed by this generating pair in a sense of (2) is periodic and ultimately returns to its starting values.

PROOF: Take the generators S, $T \in \Gamma$. We shall construct the 2-step general Fibonacci sequence starting with this generating pair. Thus, we have the sequence

$$S, T, S^m T^l, T^m (S^m T^l)^l, (S^m T^l)^m (T^m (S^m T^l)^l)^l, \ldots$$

where $l, m \in Z$. Our claim is now that this sequence returns to its starting pair after finitely many steps. The group Γ is also generated by S and R_1 where $R=\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. The generators S and R_1 of the group Γ satisfy the relations $S^4 = {R_1}^3 = I$ and $R_1 S^2 = S^2 R_1$ and these are indeed defining relations for Γ. Moreover, these generators satisfy

$$S^{\varepsilon_1} {R_1}^{\alpha_1} S^{\varepsilon_2} {R_1}^{\alpha_2} \ldots S^{\varepsilon_n} {R_1}^{\alpha_n} {S_1}^{\varepsilon_{n+1}} = I$$

where, $\alpha_i, \varepsilon_i \in Z$. Although the proof of above relation is not intricate it is omitted here and we do not wish to presume too much on the hospitality of this journal. We have, by using above defining relations

$$S^{\varepsilon_1} R_1^{\alpha_1} S R_1^{\alpha_2} \ldots S\ R_1^{\alpha_n} S_1^{\varepsilon_{n+1}} = I\,.$$

Since R_1=-R , T=SR and $-R = S^{-1}T$ we have

$$S = R_1^{-\alpha_1} S^{-\varepsilon_2} S_1^{-\varepsilon_{n+1}} R_1^{-\alpha_n} S \ldots R_1^{-\alpha_2} \tag{3}$$

and

$$T = S R_1^{-\alpha_1+1} S^{-\varepsilon_1} S_1^{-\varepsilon_{n+1}} R_1^{-\alpha_n} S \ldots R_1^{-\alpha_2} S \tag{4}$$

From (3) and (4), we conclude that the sequence (2) has a consequtive pair *S, T* after finitely many steps. So we are done. However, we conjecture that there is a connection between the period of the sequence (2) and the Wall number *k(p).* We also conjecture that there exists similar connections to that of [1] between *p* and *k(G).*

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GRUPLARDA 3-BASAMAK GENEL FİBONACCİ DİZİLERİ

ÖZET

Modüler grubun iki geren elemanı ile oluşturulan 2-basamak genel Fibonacci dizilerinin Wall sayıları ve periyodikliği ile ilgili olarak iki orijinal teorem ispat edilmiştir.