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POLYNOMIALS OF BALANCING, BALANCING-LUCAS SEQUENCES AND k-BALANCING, k-BALANCING-LUCAS SEQUENCES

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Abstract. In this work, we introduce Balancing, Balancing-Lucas polynomials, and $k-$Balancing, $k-$Balancing-Lucas polynomials. We exhibit some important theorems: Cassini, Catalan, Vajda, d'Ocagne, and Honsberger identities for Balancing, Balancing-Lucas polynomials, and $k-$Balancing, $k-$Balancing-Lucas polynomials. We give the concepts of recurrence relation, characteristic equation, roots of the characteristic equation, Binet formula, and generating function for polynomials, which are important for number sequences.
Keywords: Balancing polynomials, Balancing-Lucas polynomials, $k-$Balancing polynomials, $k-$Balancing-Lucas polynomials, Binet formula.

1. Introduction

Number sequences continue to attract the attention of researchers for a long time. Number sequences, especially Fibonacci sequences, find application in many

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departments of mathematics as well as in other branches of science. Many researchers have studied Fibonacci numbers and new number sequences created by their generalizations [4, 6, 11, 12, 17, 18, 25, 28].

There is much research about the polynomials of the number sequences [14, 16, 19, 20, 27].

The notion of balancing numbers $\{B_n\}_{n=0}^{\infty}$ has been introduced by Behera and Panda [1]. These numbers have been studied extensively in the last twenty years.

The concept of balancing numbers came into existence after an article [1] by Behera and Panda, who defined a balancing number n as a solution of the Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$$

calling r as the balancer corresponding to n. First, a few balancing numbers are $1, 6, 35, 204$ and 1189.

The sequence of balancing numbers has been studied extensively, and many generalizations of these numbers have been made [2, 3, 5, 7–10, 13, 15, 21–24, 26].

In [13], Liptai et al. generalized the concept of balancing numbers.

In this paper, we examined some identities of the balancing polynomials and the k-balancing polynomials based on the studies on balancing numbers.

Let us now give some basic definitions that we use for our study.

Definition 1.1. The Balancing numbers sequence is denoted by $\{B_n\}_{n=0}^{\infty}$. We can define the recurrence relation of Balancing numbers by

$$B_n = 6B_{n-1} - B_{n-2},\ n \geqslant 2$$

with the initial terms, $B_0 = 0, \quad B_1 = 1$ [1].

Theorem 1.1. *Binet formula for Balancing numbers as follows,*

$$B_n = \frac{{t_1}^n - {t_2}^n}{t_1 - t_2}.$$

Note that,

1. $t_1 = 3 + \sqrt{8}$ *and* $t_2 = 3 - \sqrt{8}$,
2. $t_1 t_2 = 1$,
3. $t_1 + t_2 = 6$ *[1].*

Definition 1.2. Balancing polynomials $B_n(x)$ are defined by

$$B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x),\ n \geqslant 2$$

with $B_0(x) = 0$ and $B_1(x) = 1$ [7]. Obviously, for $x = 1$ we obtain $B_n(1) = B_n$ namely Balancing numbers.

Definition 1.3. The Lucas- Balancing numbers C_n are defined by

$$C_n = 6C_{n-1} - C_{n-2}\ , \quad n \geqslant 2$$

with $C_0 = 1$ and $C_1 = 3$ [7].

Definition 1.4. The Lucas- Balancing polynomials $C_n(x)$ are defined by

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x)\,,\ n \geqslant 2$$

with $C_0(x) = 1$ and $C_1(x) = 3x$ [7]. Obviously, for $x = 1$ we obtain $C_n(1) = C_n$, namely Lucas-Balancing numbers.

Definition 1.5. Binet's formula for Lucas-Balancing polynomials is as follows,

$$C_n(x) = \frac{t^n(x) + t^{-n}(x)}{2}$$

where $t(x) = 3x + \sqrt{9x^2-1},\ t^{-1}(x) = 3x - \sqrt{9x^2-1}$ [7].

Definition 1.6. The Binet formula for Balancing polynomials is as follows,

$$B_n(x) = \frac{t^n(x) - t^{-n}(x)}{t(x) - t^{-1}(x)} \quad [7]\,.$$

Definition 1.7. For any positive number k, the $k-$Balancing numbers, denoted by $\{B_{k,n}\}_{n=0}^{\infty}$. We can define the recurrence relation of $k-$Balancing numbers by

$$B_{k,n} = 6kB_{k,n-1} - B_{k,n-2}\ \ , n \geqslant 2$$

with the initial terms, $B_{k,0} = 0,\ B_{k,1} = 1$ [25].

Definition 1.8. For any positive number k Binet's formula for $k-$Balancing numbers are as follows,

$$B_{k,n} = \frac{{t_1}^n - {t_2}^n}{t_1 - t_2}$$

where $t_1 = 3k + \sqrt{9k^2-1}\qquad ,\quad t_2 = 3k - \sqrt{9k^2-1}$ [25].

Definition 1.9. Let k be any positive integer. The $k-$Balancing-Lucas numbers, denoted by $\{B_{k,n}\}_{n=0}^{\infty}$.We can define the reccurence relation of $k-$Balancing numbers by

$$C_{k,n} = 6kC_{k,n-1} - C_{k,n-2}\ \ , n \geqslant 2$$

with the initial terms, $C_{k,0} = 1, \quad C_{k,1} = 3k$ [24].

Definition 1.10. Let k be any positive integer.The Binet formula for $k-$Balancing-Lucas numbers are as follows,

$$C_{k,n} = \frac{{t_1}^n + {t_2}^{-n}}{2}$$

where $t_1 = 3k + \sqrt{9k^2-1},\ t_2 = 3k - \sqrt{9k^2-1}$ [24].

2. Some Identities of Balancing and Balancing-Lucas Polynomials

In this section, we have calculated the identities that are important for the Balancing and Balancing-Lucas polynomials for the Cassini, Catalan, d'Ocagne, etc. number sequences.

Theorem 2.1. *(Cassini Identity) For $n \geqslant 1$, we have*

$$C_{n-1}(x) C_{n+1}(x) - C_n^2(x) = \frac{C_2(x) - 1}{2}.$$

Proof. By using the Binet formula of Balancing-Lucas polynomials, we have $C_{n-1}(x) C_{n+1}(x) - C_n^2(x)$

$$\begin{aligned}
&= \left(\frac{t^{n-1}(x) - t^{-n+1}(x)}{2}\right)\left(\frac{t^{n+1}(x) - t^{-n-1}(x)}{2}\right) - \left(\frac{t^n(x) - t^{-n}(x)}{2}\right)^2 \\
&= \frac{1}{4}\left(t^{-2}(x) + t^2(x) - 1 - 1\right) \\
&= \frac{t^2(x) + t^{-2}(x)}{4} - \frac{1}{2} \\
&= \frac{C_2(x) - 1}{2}.
\end{aligned}$$

□

Theorem 2.2. *(Catalan Identity) For $n \geqslant p$,*

$$C_{n+p}(x) C_{n-p}(x) - C_n^2(x) = \frac{C_{2p}(x) - 1}{2}\ .$$

Proof. By using the Binet formula of Balancing-Lucas polynomials, we have

$$\begin{aligned}
C_{n+p}(x) C_{n-p}(x) - C_n^2(x) &= \left(\frac{t^{n+p}(x) + t^{-n-p}(x)}{2}\right)\left(\frac{t^{n-p}(x) + {t_1}^{-n+p}(x)}{2}\right) \\
&- \left(\frac{t^n(x) + t^{-n}(x)}{2}\right)^2 \\
&= \frac{1}{4}\left(t^{-2p}(x) + t^{2p}(x) - 1 - 1\right) \\
&= \frac{t^{2p}(x) + t^{-2p}(x)}{4} - \frac{1}{2} \\
&= \frac{C_{2p}(x) - 1}{2}.
\end{aligned}$$

If we take $p = 1$ in the Catalan identity, we obtain the Cassini identity. □

Theorem 2.3. *(d'Ocagne Identity) Let n be any non-negative integer and $m \geqslant n$ then we have*

$$C_m(x) C_{n+1}(x) - C_n(x) C_{m+1}(x) = B_{n-m}(x) \left(9x^2 - 1\right).$$

Proof. For the proof, we use the Binet formula of Balancing-Lucas polynomials, we have

$$\begin{aligned}
&C_m(x) C_{n+1}(x) - C_n(x) C_{m+1}(x) \\
&= \left(\frac{t^m(x) + t^{-m}(x)}{2}\right)\left(\frac{t^{n+1}(x) + t^{-n-1}(x)}{2}\right) \\
&\quad - \left(\frac{t^n(x) + t^{-n}(x)}{2}\right)\left(\frac{t^{m+1}(x) + t^{-m-1}(x)}{2}\right) \\
&= \frac{t^{m-n-1}(x) + t^{-m+n+1}(x) - t^{n-m-1}(x) - t^{-n+m+1}(x)}{4} \\
&= \frac{t^{m-n}(x)\left(t^{-1}(x) - t(x)\right) + t^{n-m}(x)\left(t(x) - t^{-1}(x)\right)}{\left(t(x) - t^{-1}(x)\right)^2} \\
&= \frac{\left(t(x) - t^{-1}(x)\right)\left(t^{n-m}(x) - t^{m-n}(x)\right)}{4} \\
&= B_{m-n}(x)\left(9x^2 - 1\right).
\end{aligned}$$

□

Theorem 2.4. *(Vajda Identity) If i and j are any two natural numbers then we have*

$$i)\ B_{n+i}(x) B_{n+j}(x) - B_n(x) B_{n+i+j}(x) = B_j(x) B_i(x),$$

$$ii)\ C_{n+i}(x) C_{n+j}(x) - C_n(x) C_{n+i+j}(x) = \frac{C_{i-j}(x) - C_{i+j}(x)}{2}.$$

Proof. For the proof, we use the Binet formula of Balancing polynomials, we have $i)\ B_{n+i}(x) B_{n+j}(x) - B_n(x) B_{n+i+j}(x)$

$$\begin{aligned}
&\left(\frac{t^{n+i}(x) - t^{-n-i}(x)}{t(x) - t^{-1}(x)}\right)\left(\frac{t^{n+j}(x) - t^{-n-j}(x)}{t(x) - t^{-1}(x)}\right) \\
&\quad - \left(\frac{t^n(x) - t^{-n}(x)}{t(x) - t^{-1}(x)}\right)\left(\frac{t^{n+i+j}(x) - t^{-n-i-j}(x)}{t(x) - t^{-1}(x)}\right) \\
&= \frac{-t^{i-j}(x) - t^{-i+j}(x) + t^{-i-j}(x) + t^{i+j}(x)}{\left(t(x) - t^{-1}(x)\right)^2} \\
&= \frac{t^i(x)\left(t^j(x) - t^{-j}(x)\right) - t^{-i}(x)\left(t^j(x) - t^{-j}(x)\right)}{\left(t(x) - t^{-1}(x)\right)^2} \\
&= \frac{\left(t^j(x) - t^{-j}(x)\right)\left(t^i(x) - t^{-i}(x)\right)}{\left(t(x) - t^{-1}(x)\right)\left(t(x) - t^{-1}(x)\right)} \\
&= B_j(x) B_i(x).
\end{aligned}$$

The other part is shown similarly to i). $\square$

Theorem 2.5. *(Honsberger Identity) We have,*

$$\begin{aligned} i)\ B_{k-1}(x)B_n(x)+B_k(x)B_{n+1}(x) &= \frac{2\left(t(x)+t^{-1}(x)\right)C_{n+k}(x)-C_{n-k+1}(x)}{\left(t(x)-t^{-1}(x)\right)^2}, \\ ii)\ C_{k-1}(x)C_n(x)+C_k(x)C_{n+1}(x) &= C_1(x)C_{n+k}(x)+C_{k-n-1}(x). \end{aligned}$$

Proof. $i)$ $B_{k-1}(x)B_n(x)+B_k(x)B_{n+1}(x)$

$$\begin{aligned} &= \left(\frac{t^{k-1}(x)-t^{-k+1}(x)}{t(x)-t^{-1}(x)}\right)\left(\frac{t^n(x)-t^{-n}(x)}{t(x)-t^{-1}(x)}\right) \\ &\quad+\left(\frac{t^k(x)-t^{-k}(x)}{t(x)-t^{-1}(x)}\right)\left(\frac{t^{n+1}(x)-t^{-n-1}(x)}{t(x)-t^{-1}(x)}\right) \\ &= \frac{t^{n+k-1}(x)-t^{-n+k-1}(x)-t^{n-k+1}(x)+t^{-n-k+1}(x)}{\left(t(x)-t^{-1}(x)\right)^2} \\ &\quad+\frac{t^{n+k+1}(x)-t^{-n+k-1}(x)-t^{n-k+1}(x)+t^{-n-k-1}(x)}{\left(t(x)-t^{-1}(x)\right)^2} \\ &= \frac{t^{n+k}(x)\left(t(x)+t^{-1}(x)\right)+t^{-n-k}(x)\left(t(x)+t^{-1}(x)\right)}{\left(t(x)-t^{-1}(x)\right)^2} \\ &\quad-\frac{2\left(t^{n-k+1}(x)+t^{-n+k-1}(x)\right)}{\left(t(x)-t^{-1}(x)\right)^2} \\ &= \frac{\left(t(x)+t^{-1}(x)\right)\left(t^{n+k}(x)+t^{-n-k}(x)\right)-2\left(t^{n-k+1}(x)+t^{-n+k-1}(x)\right)}{\left(t(x)-t^{-1}(x)\right)^2} \\ &= \frac{2\left(t(x)+t^{-1}(x)\right)C_{n+k}(x)-C_{n-k+1}(x)}{\left(t(x)-t^{-1}(x)\right)^2}. \end{aligned}$$

The other part is shown similar to i). $\square$

3. Some Identities of the $k-$Balancing and $k-$Balancing-Lucas Polynomials

In this section, we defined $k-$Balancing and $k-$Balancing-Lucas polynomials as generalizations of Balancing and Balancing-Lucas polynomials. We gave the recurrence relations, Binet formulas, and generating functions of these polynomials. Then, we calculated some important identities.

Definition 3.1. Let k be any positive integer.The $k-$Balancing polynomials are defined by

$$B_{k,n}(x)=6kxB_{k,n-1}(x)-B_{k,n-2}(x) \quad ,n\geqslant 2$$

with $B_{k,0}(x)=0, \quad B_{k,1}(x)=1$

Table 3.1: The first few terms of $k-$Balancing polynomials

n	0	1	2	3	4	...
$B_{k,n}(x)$	0	1	$6kx$	$36k^2x^2-1$	$216k^3x^3-12kx$	...

Table 3.2: The first few terms of $k-$Balancing-Lucas polynomials

n	0	1	2	3	4	...
$C_{k,n}(x)$	0	3k	$6kx$	$18k^2x-1$	$108k^3x^2-12kx$	...

The characteristic equation of $k-$Balancing polynomials is

$$t^2-6kxt+1=0.$$

The roots of the characteristic equation are as follows

$$t_1(x)=3k+\sqrt{9k^2x^2-1} \quad , \quad t_2(x)=3k-\sqrt{9k^2x^2-1}.$$

Note that $t_1(x)+t_2(x)=6k$ and $t_1(x)t_2(x)=1$.

Similarly, let k be any positive integer. $k-$Balancing-Lucas polynomials are defined by,

$$C_{k,n}(x)=6kxC_{k,n-1}(x)-C_{k,n-2}(x) \quad ,n\geqslant 2$$

with $C_{k,0}(x)=1, \quad C_{k,1}(x)=3k$.

Theorem 3.1. *The Binet formula of $k-$Balancing polynomials and $k-$Balancing-Lucas polynomials as follows,*

$$\begin{aligned} i)\ B_{k,n}(x) &= \frac{{t_1}^n(x)-{t_2}^n(x)}{t_1(x)-t_2(x)}, \\ ii)\ C_{k,n}(x) &= c_1(x){t_1}^n(x)+c_2(x){t_2}^n(x) \end{aligned}$$

where $c_1(x)=\frac{3k-t_2(x)}{t_1(x)-t_2(x)}$,

$$c_2(x)=\frac{t_1(x)-3k}{t_1(x)-t_2(x)}.$$

Proof. i) Binet formula of $k-$Balancing polynomials has the form:

$$B_{k,n}(x)=A{t_1}^n(x)+B{t_2}^n(x)$$

For $n=0$, $A+B=0$. For $n=1$, $At_1+Bt_2=1$. If we solve these two equations, we obtain

$$B=\frac{1}{t_2(x)-t_1(x)},$$

$$A = \frac{-1}{t_2(x) - t_1(x)}.$$

Thus, the Binet formula of $k-$Balancing polynomials is

$$B_{k,n}(x) = \frac{{t_1}^n(x) - {t_2}^n(x)}{t_1(x) - t_2(x)}.$$

The other part is shown similar to i).

We can also mention that

$$B_{k,-n}(x) = \frac{{t_1}^{-n}(x) - {t_2}^{-n}(x)}{t_1(x) - t_2(x)}.$$

□

Theorem 3.2. *The generating functions for $k-$Balancing and $k-$Balancing-Lucas polynomials are given by*

$$\begin{aligned} i)\ G(x,z) &= \sum_{n=0}^{\infty} B_{k,n}(x) z^n = \frac{B_{k,1}(x) z}{z^2 - 6kxz + 1}, \\ ii)\ g(x) &= \sum_{n=0}^{\infty} C_{k,n}(x) z^n = \frac{1 + 3kx}{1 - (1-6k)x^2}. \end{aligned}$$

Proof.

$$G(x,z) = B_{k,0}(x) + B_{k,1}(x) z + B_{k,2}(x) z^2 + B_{k,3}(x) z^3 + \dots \tag{3.1}$$

Multiplying both side of 3.1 by z^2 and $-6kxz$ respectively, we get

$$-6kxzG(x,z) = -6kxzB_{k,0}(x) - 6kxz^2 B_{k,1}(x) - 6kxz^3 B_{k,2}(x) + \dots$$

$$z^2 G(x,z) = B_{k,0}(x) z^2 + B_{k,1}(x) z^3 + B_{k,2} z^5 + \dots$$

If we do the necessary mathematical operations from here, we get the following equation.

$$G(x,z)\left(z^2 - 6kxz + 1\right) = B_{k,0}(x) + B_{k,1}(x) z - 6kxzB_{k,0}(x)$$

We obtain,

$$G(x,z) = \frac{B_{k,1}(x) z}{z^2 - 6kxz + 1}.$$

The other part is shown similar to i). □

There are some important properties for sequences of numbers. They have some applications, for example, the Cassini identity, which is a special case of the Catalan identity, is used in the proof of Curry's Paradox. That's why we examine these important identities in this article.

Theorem 3.3. *(Cassini Identity)*

$$\begin{aligned} i)\ B_{k,n-1}(x)\,B_{k,n+1}(x) - {B^2}_{k,n}(x) &= -1,\ n \geqslant 1, \\ ii)\ C_{k,n+1}(x)\,C_{k,n-1}(x) - C^2_{k,n}(x) &= (3k - t_2(x))\,(t_1(x) - 3k). \end{aligned}$$

Proof. $i)$ For the proof, we use the Binet formula of $k-$Balancing polynomials we have

$B_{k,n-1}(x).B_{k,n+1}(x) - {B^2}_{k,n}(x)$

$$\begin{aligned} &= \left(\frac{{t_1}^{n-1}(x) - {t_2}^{n-1}(x)}{t_1(x) - t_2(x)}\right)\left(\frac{{t_1}^{n+1}(x) - {t_2}^{n+1}(x)}{t_1(x) - t_2(x)}\right) - \left(\frac{{t_1}^{n}(x) - {t_2}^{n}(x)}{t_1(x) - t_2(x)}\right)^2 \\ &= \frac{{t_1}^{2n}(x) - {t_1}^{n-1}(x)\,{t_2}^{n+1}(x) - {t_2}^{n-1}(x)\,{t_1}^{n+1}(x) + {t_2}^{2n}(x)}{(t_1(x) - t_2(x))^2} \\ &\quad - \frac{{t_1}^{2n}(x) + 2{t_1}^{n}(x)\,{t_2}^{n}(x) + {t_2}^{2n}(x)}{(t_1(x) - t_2(x))^2} \\ &= \frac{{t_1}^{n}(x)\,{t_2}^{n}(x)\left(2 - \frac{t_1(x)}{t_2(x)} - \frac{t_2(x)}{t_1(x)}\right)}{(t_1(x) - t_2(x))^2} \\ &= -(t_1(x)\,t_2(x))^{n-1} = -1. \end{aligned}$$

The other part is shown similar to $i)$. □

Theorem 3.4. *(Catalan Identity) For $n \geqslant p$,*

$$\begin{aligned} i)\ B_{k,n+p}(x)\,B_{k,n-p}(x) - {B^2}_{k,n}(x) &= -{B_{k,p}}^2(x), \\ ii)\ C_{k,n+p}(x)\,C_{k,n-p}(x) - C^2_{k,n}(x) &= c_1(x)\,c_2(x)\,({t_1}^{p}(x) - {t_2}^{p}(x))^2. \end{aligned}$$

Proof. $i)$

$B_{k,n+p}(x)\,B_{k,n-p}(x) - {B^2}_{k,n}(x)$

$$\begin{aligned} &= \left(\frac{{t_1}^{n+p}(x) - {t_2}^{n+p}(x)}{t_1(x) - t_2(x)}\right)\left(\frac{{t_1}^{n-p}(x) - {t_2}^{n-p}(x)}{t_1(x) - t_2(x)}\right) - \left(\frac{{t_1}^{n}(x) - {t_2}^{n}(x)}{t_1(x) - t_2(x)}\right)^2 \\ &= \frac{-{t_1}^{n+p}(x)\,{t_2}^{n-p}(x) - {t_2}^{n+p}(x)\,{t_1}^{n-p}(x) + 2{t_1}^{n}(x)\,{t_2}^{n}(x)}{(t_2(x) - t_1(x))^2} \\ &= \frac{{t_1}^{n}(x)\,{t_2}^{n}(x)\left(2 - \frac{{t_1}^{p}(x)}{{t_2}^{p}(x)} - \frac{{t_2}^{p}(x)}{{t_1}^{p}(x)}\right)}{(t_1(x) - t_2(x))^2} \\ &= -\left(\frac{{t_1}^{p}(x) - {t_2}^{p}(x)}{t_1(x) - t_2(x)}\right)^2 = -{B_{k,p}}^2(x). \end{aligned}$$

The other part is shown similar to $i)$. □

Theorem 3.5. *(d'Ocagne Identity) For $m \geqslant n$, we have*

$$\begin{aligned}
&i)\ B_{k,m}(x)\,B_{k,n+1}(x) - B_{k,n}(x)\,B_{k,m+1}(x) = B_{k,m-n}(x),\\
&ii)\ C_{k,m}(x)\,C_{k,n+1}(x) - C_{k,n}(x)\,C_{k,m+1}(x)\\
&\quad = c_1(x)\,c_2(x)\,(t_2(x) - t_1(x))\,.\left({t_2}^{n-m}(x) - {t_1}^{n-m}(x)\right).
\end{aligned}$$

Proof. i)
$B_{k,m}(x)\,B_{k,n+1}(x) - B_{k,n}(x)\,B_{k,m+1}(x)$

$$\begin{aligned}
&= \left(\frac{{t_1}^m(x) - {t_2}^m(x)}{t_1(x) - t_2(x)}\right)\left(\frac{{t_1}^{n+1}(x) - {t_2}^{n+1}(x)}{t_1(x) - t_2(x)}\right)\\
&\quad - \left(\frac{{t_1}^n(x) - {t_2}^n(x)}{t_1(x) - t_2(x)}\right)\left(\frac{{t_1}^{m+1} - {t_2}^{m+1}}{t_1(x) - t_2(x)}\right)\\
&= \frac{-{t_2}^m(x)\,{t_1}^{n+1}(x) - {t_1}^m(x)\,{t_2}^{n+1}(x) + {t_2}^n(x)\,{t_1}^{m+1}(x)}{(t_2 - t_1)^2}\\
&= \frac{{t_1}^n(x){t_2}^m(x)\,(t_2(x) - t_1(x)) - {t_2}^n(x){t_1}^m(x)\,(t_2(x) - t_1(x))}{(t_2 - t_1)^2}\\
&= \frac{(t_2(x) - t_1(x))\,({t_1}^n(x){t_2}^m(x) - {t_2}^n(x){t_1}^m(x))}{(t_2 - t_1)^2}\\
&= \frac{{t_1}^n(x){t_2}^n(x)({t_2}^{m-n}(x) - {t_1}^{m-n}(x))}{t_2 - t_1}\\
&= \frac{(t_1(x)t_2(x))^2({t_2}^{m-n}(x) - {t_1}^{m-n}(x))}{t_2(x) - t_1(x)}\\
&= \frac{{t_2}^{m-n}(x) - {t_1}^{m-n}(x)}{t_2(x) - t_1(x)}\\
&= B_{k,m-n}(x).
\end{aligned}$$

The other part is shown similar to i). □

Theorem 3.6. *(Vajda Identity)*

$$\begin{aligned}
&i)\ B_{k,n+i}(x)\,B_{k,n+j}(x) - B_{k,n}(x)\,B_{k,n+i+j}(x) = -B_{k,i}(x)\,B_{k,j}(x),\\
&ii) C_{k,n+i}(x)\,C_{k,n+j}(x) - C_{k,n}(x)\,C_{k,n+i+j}(x)\\
&= c_1(x)\,c_2(x)\left(t_1(x)^i(x) - t_2(x)^i(x)\right)\left(t_1(x)^j(x) - t_2(x)^j(x)\right).
\end{aligned}$$

Proof. i)
$B_{k,n+i}(x)\,B_{k,n+j}(x) - B_{k,n}(x)\,B_{k,n+i+j}(x)$

$$\begin{aligned}
&= \left(\frac{{t_1}^{n+i}(x) - {t_2}^{n+i}(x)}{t_1(x) - t_2(x)}\right)\left(\frac{{t_1}^{n+j}(x) - {t_2}^{n+j}(x)}{t_1(x) - t_2(x)}\right)\\
&\quad - \left(\frac{{t_1}^n(x) - {t_2}^n(x)}{t_1(x) - t_2(x)}\right)\left(\frac{{t_1}^{n+i+j}(x) - {t_2}^{n+i+j}(x)}{t_1(x) - t_2(x)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-{t_1}^{n+i}{t_2}^{n+j} - {t_2}^{n+i}{t_1}^{n+j} + {t_1}^{n}{t_2}^{n+i+j} + {t_2}^{n}{t_1}^{n+i+j}}{\left(t_1(x) - t_2(x)\right)^2} \\
&= \frac{{t_1}^{n}{t_2}^{n+j}\left({t_2}^{i} - {t_1}^{i}\right) + {t_2}^{n}{t_1}^{n+j}\left({t_1}^{i} - {t_2}^{i}\right)}{\left(t_1(x) - t_2(x)\right)^2} \\
&= \frac{\left({t_1}^{i} - {t_2}^{i}\right)\left({t_2}^{n}{t_1}^{n+j} - {t_1}^{n}{t_2}^{n+j}\right)}{\left(t_1\left(x\right) - t_2\left(x\right)\right)^2} \\
&= \frac{\left({t_1}^{i} - {t_2}^{i}\right){t_1}^{n}{t_2}^{n}\left({t_1}^{j} - {t_2}^{j}\right)}{\left(t_1\left(x\right) - t_2\left(x\right)\right)^2} \\
&= -\left(\frac{{t_1}^{i} - {t_2}^{i}}{t_1\left(x\right) - t_2\left(x\right)}\right)\left(\frac{{t_1}^{j} - {t_2}^{j}}{t_1\left(x\right) - t_2\left(x\right)}\right) \\
&= -B_{k,i}\left(x\right)B_{k,j}\left(x\right).
\end{aligned}
$$

The other part is shown similar to i). □

4. Conclusion

We proved some important identities (Cassini, Catalan, Vajda, d'Ocagne, and Honsberger) of Balancing and Balancin-Lucas polynomials. Then, we defined $k-$Balancing and $k-$Balancing-Lucas polynomials. We examined the concepts of recurrence relation, characteristic equation, roots of the characteristic equation, Binet formula, and generator function for polynomials, which are important for number sequences. We gave some identities of $k-$Balancing polynomials.

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