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A New Approach to *k*-Oresme and *k*-Oresme-Lucas Sequences

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Abstract: In this study, the *k*-Oresme and *k*-Oresme-Lucas sequences are defined, and some terms of these sequence are given. Then, the relations between the terms of the *k*-Oresme and *k*-Oresme-Lucas sequences are presented. In addition, we give these sequences the Binet formulas, generating functions, Cassini identity, Catalan identity etc. Moreover, the *k*-Oresme and *k*-Oresme-Lucas sequences are associated with Fibonacci, Pell numbers and Lucas, and Pell- Lucas numbers, respectively. Finally, the Catalan transforms of these sequences are given and Hankel transforms are applied to these Catalan sequences and associated with the terms of the sequence.

Keywords: Oresme numbers; sequences; Oresme-Lucas numbers; generating function

1. Introduction

The Fibonacci, Jacobsthal, Pell, and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci and Lucas sequences have been applied in various fields, such as Algebraic Coding Theory, Phylotaxis, Biomathematics, Computer Science, and so on. New sequences were obtained by changing the recurrence relation and initial conditions of the generalized Fibonacci sequence. The known examples of such sequences are the Horadam sequence, *k*-Pell sequence, Oresme Numbers, *k*-Jacobsthal-Lucas, Padovan sequences, and so on (see, for details, [1–5]).

The Lucas numbers L_n and Fibonacci numbers F_n are defined by the recurrence relations, respectively,

$$L_{n+2} = L_{n+1} + L_n, \text{ for } n \geq 0 \text{ and } F_{n+2} = F_{n+1} + F_n, \text{ for } n \geq 0$$

with the initial conditions $L_0 = 2$, $L_1 = 1$ and $F_0 = 0$, $F_1 = 1$.

Binet formula for Lucas numbers L_n and Fibonacci numbers F_n are given by the following relations, respectively:

$$L_n = \alpha^n + \beta^n \text{ and } F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. Here the number α is the known golden ratio.

In addition, the Pell numbers R_n and the Pell-Lucas numbers Q_n are defined by the recurrence relations, respectively,

$$R_{n+2} = 2R_{n+1} + R_n, \text{ for } n \geq 0 \text{ and } Q_{n+2} = 2Q_{n+1} + Q_n, \text{ for } n \geq 0$$

with the initial conditions $R_0 = 0$, $R_1 = 1$ and $Q_0 = 2$, $Q_1 = 2$.

The Binet formula for Pell numbers R_n and Pell-Lucas numbers Q_n are given by relations, respectively:

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } Q_n = \alpha^n + \beta^n$$

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where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ are the roots of the characteristic equation $r^2-2r-1=0$. Here, the number α is the known silver ratio.

The Oresme numbers were introduced by Nicole Oresme. Soykan in [6] defined generalized Oresme sequence $\{O_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$, and Modified Oresme sequence $\{G_n\}_{n\geq 0}$ by the recurrence relations, respectively:

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n,\ H_{n+2} = H_{n+1} - \frac{1}{4}H_n \text{ and } G_{n+2} = G_{n+1} - \frac{1}{4}G_n,$$

with the initial conditions $O_0 = 0$, $O_1 = \frac{1}{2}$, $H_0 = 2$, $H_1 = 1$ and $G_0 = 0$, $G_1 = 1$.

In [7], studies were carried out on the properties of the sum of Oresme numbers. In addition, Horadam [8] obtained many properties of Oresme numbers.

With the help of the recurrence relation of the Fibonacci sequence, k-sequences were introduced and these sequences had an important place in number theory. Falcon [9] introduced the k-Fibonacci sequence, and he obtained many properties related to this sequence. In addition, Falcon applied Hankel transform to the k-Fibonacci sequence, and he obtained the terms of Fibonacci sequences in a different way.

After reviewing these studies, we define the k-Oresme and k-Oresme-Lucas sequences.

We separate the article into three parts. In Section 2, we define the k-Oresme and k-Oresme-Lucas sequence, and we then give the characteristic equation, the Binet formulas, and some properties for these sequences. Then, we examine the relationship between the k-Oresme and k-Oresme-Lucas sequences. In addition, we show the relationship of the k-Oresme and k-Oresme-Lucas sequences for Catalan identity, Cassini identity, Vajda's identity, etc. Moreover, the k-Oresme and k-Oresme-Lucas sequences are associated with Fibonacci, Pell numbers and Lucas, and Pell-Lucas numbers, respectively.

In Section 3, the Catalan transformations of the k-Oresme and k-Oresme-Lucas sequences are defined, and some properties are given. In addition, the Catalan generating functions of these sequences are obtained. Moreover, Hankel transformations are applied to the Catalan transformations of the k-Oresme and k-Oresme-Lucas sequences and the results are associated with the terms of the sequences.

2. Materials and Methods

k-Oresme and k-Oresme-Lucas Sequence

For the k positive real number, the k-Oresme and k-Oresme-Lucas sequences $O_{k,n}$, $P_{k,n}$ for $n \geq 0$ are defined by, respectively:

$$O_{k,n+2} = O_{k,n+1} - \frac{1}{k^2}O_{k,n}\ ,n \geq 0 \tag{1}$$

with $O_{k,0} = 0$ and $O_{k,1} = \frac{1}{k}$,

$$P_{k,n+2} = P_{k,n+1} - \frac{1}{k^2}P_{k,n}\ n \geq 0 \tag{2}$$

with $P_{k,0} = 2$ and $P_{k,1} = 1$.

Then, let us give some information about the equations of these sequences. The characteristic equation of the k-Oresme and k-Oresme-Lucas sequences is:

$$r^2 - r + \frac{1}{k^2} = 0. \tag{3}$$

The roots of the characteristic equation are as follows:

$$r_1 = \frac{k+\sqrt{k^2-4}}{2k} \tag{4}$$

and

$$r_2 = \frac{k - \sqrt{k^2 - 4}}{2k}. \tag{5}$$

The relationship between these roots is given below:

$$r_1 + r_2 = 1,\ r_1 - r_2 = \frac{\sqrt{k^2-4}}{k}, r_1^2 + r_2^2 = \frac{k^2-2}{k^2} \text{ and } r_1.r_2 = \frac{1}{k^2}.$$

The $O_{k,n}$ and $P_{k,n}$ values for the first seven n natural numbers are given below:

- $O_{k,0} = 0\ P_{k,0} = 2$,
- $O_{k,1} = \frac{1}{k}\ P_{k,1} = 1$,
- $O_{k,2} = \frac{1}{k}\ P_{k,2} = \frac{1}{k^2}\left(k^2-2\right)$,
- $O_{k,3} = \frac{1}{k^3}\left(k^2-1\right)\ P_{k,3} = \frac{1}{k^2}\left(k^2-3\right)$,
- $O_{k,4} = \frac{1}{k^3}\left(k^2-2\right)\ P_{k,4} = \frac{1}{k^4}\left(k^4-4k^2+2\right)$,
- $O_{k,5} = \frac{1}{k^5}\left(k^4-3k^2+1\right)\ P_{k,5} = \frac{1}{k^4}\left(k^4-5k^2+5\right)$,
- $O_{k,6} = \frac{1}{k^5}\left(k^4-4k^2+3\right)\ P_{k,6} = \frac{1}{k^6}\left(k^6-6k^4+9k^2-2\right)$.

Theorem 1. *Let $n \in \mathbb{N}$. Then, the Binet formulas of the k-Oresme and k-Oresme-Lucas sequences are as follows:*

$$O_{k,n} = \frac{r_1^n - r_2^n}{(r_1 - r_2)k} \text{ and } P_{k,n} = r_1^n + r_2^n.$$

Proof. The Binet form of a sequence is as follows

$$O_{k,n} = m{r_1}^n + n{r_2}^n.$$

Here, the scalars m and n can be obtained by substituting the initial conditions. It is obtained by solving the given system of equations. For $n = 0$, $O_{k,0} = 0$, and for $n = 1$, $O_{k,1} = \frac{1}{k}$. Thus, $m = \frac{1}{\sqrt{k^2-4}}$ and $n = \frac{-1}{\sqrt{k^2-4}}$ are obtained. From here, $O_{k,n} = \frac{r_1^n - r_2^n}{(r_1-r_2)k}$. Similarly, we have $P_{k,n} = r_1^n + r_2^n$. □

Theorem 2. *Let $n \geq 2$. Then the terms of sequences k-Oresme and k-Oresme-Lucas sequences with the help of the following relations are obtained.*

i. $O_{k,n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j}(-1)^j k^{-2j-1}$,
ii. $P_{k,n} = k\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j}(-1)^j k^{-2j-1} - \frac{1}{k}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-2-j}{j}(-1)^j k^{-2j-1}$.

Proof. i. Let us show the proof by induction over n.

For $n = 2$, $O_{k,2} = \frac{1}{k}$. For $n = s$ and $n = s-1$, let us assume the equality is true. We have:

$$O_{k,s} = \sum_{j=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s-1-j}{j}(-1)^j k^{-2j-1} \text{ and } O_{k,s-1} = \sum_{j=0}^{\lfloor \frac{s-2}{2} \rfloor} \binom{s-2-j}{j}(-1)^j k^{-2j-1}.$$

For $n = s+1$, let us show that the equality is true. If the definition of the k -Oresme sequence is used, we obtain:

$$\begin{gathered} O_{k,s+1} = O_{k,s} - \frac{1}{k^2} O_{k,s-1} \\ = \sum_{j=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s-1-j}{j}(-1)^j k^{-2j-1} - \frac{1}{k^2}\sum_{j=0}^{\lfloor \frac{s-2}{2} \rfloor} \binom{s-2-j}{j}(-1)^j k^{-2j-1} \\ = \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s-j}{j}(-1)^j k^{-2j-1}. \end{gathered}$$

Thus, the equality is true in $n = s + 1$.

ii. If the relation $P_{k,n} = kO_{k,n+1} - \frac{1}{k}O_{k,n-1}$ is used, we have:

$$kO_{k,n+1} = k\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j}(-1)^j k^{-2j-1}$$

and

$$\frac{1}{k}O_{k,n-1} = \frac{1}{k}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-2-j}{j}(-1)^j k^{-2j-1}.$$

Thus, we obtain $P_{k,n} = k\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j}(-1)^j k^{-2j-1} - \frac{1}{k}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-2-j}{j}(-1)^j k^{-2j-1}$. □

Lemma 1. *We have:*

i. ${r_1}^{2i} = r_1\sqrt{k^2-4}O_{k,2i} + \frac{1}{k^2}P_{k,2i-1}$,
ii. ${r_2}^{2i} = -r_2\sqrt{k^2-4}O_{k,2i} + \frac{1}{k^2}P_{k,2i-1}$,
iii. ${r_1}^{2i+1} = \frac{1}{k^2}\sqrt{k^2-4}O_{k,2i} + r_1P_{k,2i+1}$,
iv. ${r_2}^{2i+1} = -\frac{1}{k^2}\sqrt{k^2-4}O_{k,2i} + r_2P_{k,2i+1}$.

Proof. i. $r_1\sqrt{k^2-4}O_{k,2i} + \frac{1}{k^2}P_{k,2i-1} = r_1\sqrt{k^2-4}\frac{{r_1}^{2i}-{r_2}^{2i}}{(r_1-r_2)k} + \frac{1}{k^2}\left({r_1}^{2i-1} + {r_2}^{2i-1}\right) = {r_1}^{2i+1} - r_1{r_2}^{2i} + \frac{1}{k^2}{r_1}^{2i-1} + \frac{1}{k^2}{r_2}^{2i-1} = {r_1}^{2i}\left(r_1 + \frac{1}{k^2 r_1}\right) + {r_2}^{2i}\left(-r_1 + \frac{1}{k^2 r_2}\right)$.

Thus, we can write:

$${r_1}^{2i} = r_1\sqrt{k^2-4}O_{k,2i} + \frac{1}{k^2}P_{k,2i-1}.$$

The proofs are shown in the same way as i. □

Theorem 3. *Let us consider $k \in \mathbb{R}, k \geq 2$ and $m, n \in \mathbb{Z}^+$. The following equations are satisfied.*

i. $P_{k,n} = kO_{k,n+1} - \frac{1}{k}O_{k,n-1}$,
ii. $P_{k,n}^2 - \left(k^2-4\right)O_{k,n}^2 = \frac{4}{k^{2n}}$,
iii. $\frac{k^2-4}{k}O_{k,n} = P_{k,n+1} - \frac{P_{k,n-1}}{k^2}$,
iv. $O_{k,n}P_{k,n} = O_{k,2n}$,
v. $kO_{k,n}P_{k,n} = 2kO_{k,n+1}$,
vi. $\sqrt{k^2-4}O_{k,n} + P_{k,n} = 2r_1^n$,
vii. $\sqrt{k^2-4}O_{k,n} - P_{k,n} = -2r_2^n$.

Proof. i. If the Binet formula is used, we have:

$$\begin{aligned} kO_{k,n+1} - \frac{1}{k}O_{k,n-1} &= k\frac{r_1^{n+1}-r_2^{n+1}}{(r_1-r_2)k} - \frac{1}{k}\frac{r_1^{n-1}-r_2^{n-1}}{(r_1-r_2)k} \\ &= \frac{r_1^n\left(r_1 - \frac{1}{k^2 r_1}\right) + r_2^n\left(-r_2 + \frac{1}{k^2 r_2}\right)}{r_1-r_2}. \end{aligned}$$

Thus, we obtain $P_{k,n} = kO_{k,n+1} - \frac{1}{k}O_{k,n-1}$. The proof of the others are shown in the same way as i. using the Binet formulas and Lemma 1. □

Theorem 4. *Let us consider $k \in \mathbb{R}, k \geq 2$, $m, n \in \mathbb{Z}^+$ and $m > n$. We have*

i. $2P_{k,m+n} = kP_{k,m}O_{k,n} + \left(k^2-4\right)O_{k,m}O_{k,n}$,
ii. $2kO_{k,m+n} = kO_{k,m}P_{k,n} + kO_{k,n}P_{k,n}$,
iii. $\frac{2}{k^{2n}}P_{k,m-n} = P_{k,m}P_{k,n} - (k^2-4)O_{k,m}O_{k,n}$,
iv. $\frac{2}{k^{2n-1}}O_{k,m-n} = kO_{k,m}P_{k,n} - kO_{k,n}P_{k,m}$,
v. $2P_{k,n+1} = \frac{k^2-4}{k}O_{k,n} + P_{k,n}$.

Proof. **iv.** If the Binet formulas are used for proof, we obtain:

$$\begin{aligned} kO_{k,m}P_{k,n} - kO_{k,n}P_{k,m} &= k\frac{r_1^m - r_2^m}{(r_1-r_2)k}\left(r_1^n + r_2^n\right) - k\frac{r_1^n - r_2^n}{(r_1-r_2)k}\left(r_1^m + r_2^m\right) \\ &= \frac{r_1^{m+n} + r_1^m r_2^n - r_1^n r_2^m - r_2^{m+n}}{r_1 - r_2} - \frac{r_1^{n+m} + r_1^n r_2^m - r_1^m r_2^n - r_2^{m+n}}{r_1 - r_2}. \end{aligned}$$

Thus, we have

$$\frac{2}{k^{2n-1}}O_{k,m-n} = kO_{k,m}P_{k,n} - kO_{k,n}P_{k,m}.$$

The proofs of the others are shown in the same way as iv. using the Binet formulas and Lemma 1. □

Theorem 5. *Let us consider $k \in \mathbb{R}, k \geq 2$, $m, n \in \mathbb{Z}^+$ and $m > n$. We obtain:*

i. $P_{k,m}P_{k,n} = P_{k,m+n} + \frac{1}{k^{2n}}P_{k,m-n}$,
ii. $kO_{k,m}P_{k,n} = kO_{k,m+n} + \frac{O_{k,m-n}}{k^{2n-1}}$,
iii. $kO_{k,m+n+1} = k^2O_{k,m+1}O_{k,n+1} - O_{k,m}O_{k,n}$,
iv. $P_{k,m+n+1} = kO_{k,m+1}P_{k,n+1} - \frac{1}{k}O_{k,m}P_{k,n}$.

Proof. The proofs are shown in the same way as Theorem 3 using the Binet formulas and Lemma 1. □

Theorem 6. *Let us consider $k \in \mathbb{R}, k \geq 2$, $m, n \in \mathbb{Z}^+$ and $m > n$. We have*:

i. $kP_{k,2n}O_{k,m+2n} = kO_{k,m+4n} + \frac{1}{k^{4n-1}}O_{k,m}$,
ii. $P_{k,2n}P_{k,m+2n} = P_{k,m+4n} + \frac{1}{k^{4n}}P_{k,m}$,
iii. $kO_{k,n}P_{k,m+2n} = kO_{k,m+3n} - \frac{1}{k^{2n-1}}O_{k,m+n}$,
iv. $kP_{k,n}O_{k,m+2n} = kO_{k,m+3n} + \frac{1}{k^{2n-1}}O_{k,m+n}$,
v. $\left(k^2-4\right)O_{k,n}O_{k,m+2n} = P_{k,m+3n} - \frac{1}{k^{2n}}P_{k,m+n}$,
vi. $P_{k,n}P_{k,m+2n} = P_{k,m+3n} + \frac{1}{k^{4n}}P_{k,m+n}$.

Proof. The proofs are shown in the same way as Theorem 3 using the Binet formulas and Lemma 1. □

Theorem 7. *The following relations are satisfied. We obtain*:

i. $k^2O_{k,n}^2 + k^2O_{k,n+1}^2 = \frac{k^{2n+2}P_{k,2n}-2k^2}{k^{2n+2}-4k^{2n}} + \frac{k^{2n+4}P_{k,2n+2}-2k^2}{k^{2n+4}-4k^{2n+2}}$,
ii. $P_{k,n}^2 + P_{k,n+1}^2 = P_{k,2n} + P_{k,2n+2} + \frac{2}{k^{2n}} + \frac{2}{k^{2n+1}}$,
iii. $k^2O_{k,n+1}^2 - k^2O_{k,n-1}^2 = \frac{k^{2n+2}P_{k,2n+2}-2}{k^{2n+2}-4k^{2n}} - \frac{k^{2n-2}P_{k,2n-2}-2}{k^{2n-2}-4k^{2n-4}}$,
iv. $P_{k,n+1}^2 - P_{k,n-1}^2 = P_{k,2n+2} - P_{k,2n-2} + \frac{2}{k^{2n+2}} - \frac{2}{k^{2n-2}}$,
v. $O_{k,n}O_{k,n+1} = \frac{k^{2n}P_{k,2n+1}-1}{k^{2n+2}-4k^{2n}}$,
vi. $P_{k,n}P_{k,n+1} = P_{k,2n+1} + \frac{1}{k^{2n}}$,
vii. $O_{k,-n} = -k^{2n}O_{k,n}$,
viii. $P_{k,-n} = k^{2n}P_{k,n}$.

Proof. The proofs are shown in the same way as Theorem 3 using the Binet formulas. □

Theorem 8 (Cassini Identity). *For natural number n, we have:*

i. $O_{k,n+1}O_{k,n-1}$-$O_{k,n}^2$= $-\frac{1}{k^{2n}}$,
ii. $P_{k,n+1}P_{k,n-1}$-$P_{k,n}^2$= $\frac{k^2-4}{k^{2n}}$.

Proof. i. If the Binet formula is used, we obtain

$$O_{k,n-1} - O_{k,n}^2 = \frac{{r_1}^{n+1}-{r_2}^{n+1}}{(r_1-r_2)k}\frac{{r_1}^{n-1}-{r_2}^{n-1}}{(r_1-r_2)k} - \frac{{r_1}^{n}-{r_2}^{n}}{(r_1-r_2)k}\frac{{r_1}^{n}-{r_2}^{n}}{(r_1-r_2)k}$$
$$= \frac{{r_1}^{2n}-r_1{}^{n+1}r_2{}^{n-1}-r_2{}^{n+1}r_1{}^{n-1}+{r_2}^{2n}}{(r_1-r_2)^2k^2} - \frac{{r_1}^{2n}-2r_1{}^{n}r_2{}^{n}+{r_2}^{2n}}{(r_1-r_2)^2k^2} = \frac{(r_1r_2)^n\frac{-r_1}{r_2}}{(r_1-r_2)^2k^2} + \frac{(r_1r_2)^n\frac{-r_2}{r_1}}{(r_1-r_2)^2k^2} + \frac{2(r_1r_2)^n}{(r_1-r_2)^2k^2}.$$

Thus, we obtain

$O_{k,n+1}O_{k,n-1} - O_{k,n}^2 = -\frac{1}{k^{2n}}$.

ii. It is shown in the same way as i. □

Theorem 9 (Catalan Identity). *For n, r natural numbers and $r \leq n$, we have:*

i. $O_{k,n+r}O_{k,n-r} - O_{k,n}^2 = -\frac{1}{k^{2n-2r}}O_{k,r}^2$,
ii. $P_{k,n+r}P_{k,n-r} - P_{k,n}^2 = \frac{k^2-4}{k^{2n-2r}}O_{k,r}^2$.

Theorem 10 (D'ocagne's Identity). *For n, r natural numbers and $r \leq n$, we have:*

i. $O_{k,n+1}O_{k,r} - O_{k,n}O_{k,r+1} = -\frac{1}{k^{2n-1}}O_{k,n-r}$,
ii. $P_{k,n+1}P_{k,r} - P_{k,n}P_{k,r+1} = \frac{k^2-4}{k^{2r+1}}O_{k,n-r}$.

Theorem 11 (Vajda's Identity). *For n, i and j natural numbers, we have:*

i. $O_{k,n+i}O_{k,n+j} - O_{k,n}O_{k,n+i+j} = \frac{1}{k^{2n}}O_{k,i}O_{k,j}$,
ii. $P_{k,n+i}P_{k,n+j} - P_{k,n}P_{k,n+i+j} = \frac{4-k^2}{k^{2n}}O_{k,i}O_{k,j}$.

Theorem 12 (Melham's Identity). *For natural number n, we have:*

i. $O_{k,n+1}O_{k,n+2}O_{k,n+6} - O_{k,n}^3 = \frac{1}{k^2-4}\left(O_{k,3n+9} - O_{k,3n} + \frac{3O_{k,n}}{k^{2n}} - \frac{O_{k,n-3}}{k^{2n+12}} - \frac{O_{k,n+5}}{k^{2n+4}} - \frac{O_{k,n+7}}{k^{2n+2}}\right)$,
ii. $P_{k,n+1}P_{k,n+2}P_{k,n+6} - P_{k,n}^3 = P_{k,3n+9} - \frac{3P_{k,n}}{k^{2n}} + \frac{P_{k,n-3}}{k^{2n+12}} + \frac{P_{k,n+5}}{k^{2n+4}} + \frac{P_{k,n+7}}{k^{2n+2}} - P_{k,3n}$.

Theorem 13 (Gelin-Cesaro's Identity). *For natural number n, we have:*

i. $O_{k,n+2}O_{k,n+1}O_{k,n-1}O_{k,n-2} - O_{k,n}^4 = \frac{-\frac{1}{k^{2n-4}}P_{k,2n+4} - \frac{1}{k^{2n-2}}P_{k,2n+2} + \frac{1}{k^{4n-6}}P_{k,6}}{(k^2-4)^2}$
$+ \frac{-\frac{1}{k^{2n+2}}P_{k,2n-2} - \frac{1}{k^{4n-2}}P_{k,2} - \frac{1}{k^{2n+4}}P_{k,2n-4} + \frac{4}{k^{2n}}P_{k,2n} - \frac{5}{k^{4n}}}{(k^2-4)^2}$,
ii. $P_{k,n+2}P_{k,n+1}P_{k,n-1}P_{k,n-2} - P_{k,n}^4 = \frac{1}{k^{2n-4}}P_{k,2n+4} + \frac{1}{k^{2n-2}}P_{k,2n+2} + \frac{1}{k^{4n-6}}P_{k,6}\frac{1}{k^{2n+2}}P_{k,2n-2}$
$+ \frac{1}{k^{4n-2}}P_{k,2} + \frac{1}{k^{2n+4}}P_{k,2n-4} - \frac{4}{k^{2n}}P_{k,2n} - \frac{5}{k^{4n}}$.

The proofs of Theorems 9–13 are shown using the Binet formulas in a similar way to Theorem 8.

Theorem 14. *Let $k > 2$ and $|r_2| < r_1$. Then, the ratio of the bigger to the smallest of the two consecutive terms of the k-Oresme and k-Oresme-Lucas sequences are as follows:*

$$\lim_{n\to\infty}\frac{O_{k,n+1}}{O_{k,n}} = r_1 \text{ and } \lim_{n\to\infty}\frac{P_{k,n+1}}{P_{k,n}} = r_1.$$

Proof. If the Binet formula is used, we have $\lim_{n\to\infty}\frac{O_{k,n+1}}{O_{k,n}} = \frac{\frac{{r_1}^{n+1}-{r_2}^{n+1}}{(r_1-r_2)k}}{\frac{{r_1}^{n}-{r_2}^{n}}{(r_1-r_2)k}} = \lim_{n\to\infty}\frac{{r_1}^{n+1}(1-(\frac{r_2}{r_1})^{n+1})}{{r_1}^{n}(1-\left(\frac{r_2}{r_1}\right)^{n+1})}$

Thus, we obtain $\lim_{n\to\infty}\frac{O_{k,n+1}}{O_{k,n}} = r_1$. Similarly, we have $\lim_{n\to\infty}\frac{P_{k,n+1}}{P_{k,n}} = r_1$. □

Theorem 15 (Summation Formulas). *The sum of the first n terms of the k-Oresme and k-Oresme-Lucas sequences are as follows:*

i. $\sum_{s=0}^{n} O_{k,s} = O_{k,n-1} + (1-k^2)O_{k,n}$,
ii. $\sum_{s=0}^{n} P_{k,s} = 2k^2 + P_{k,n-1} + (1-k^2)P_{k,n}$.

Proof. **i.** From the definition of the *k*-Oresme sequence, we get: $O_{k,2} = O_{k,1} - \frac{1}{k^2}O_{k,0}$, $O_{k,3} = O_{k,2} - \frac{1}{k^2}O_{k,1}, \vdots O_{k,n} = O_{k,n-1} - \frac{1}{k^2}O_{k,n-2}$. So, we have $-\frac{1}{k} + \sum_{s=0}^{n} O_{k,s} = \sum_{s=1}^{n-1} O_{k,s} - \frac{1}{k^2}\sum_{s=0}^{n-2} O_{k,s}$, $-\frac{1}{k} + \sum_{s=0}^{n} O_{k,s} = (-O_{k,n} - O_{k,1} + \sum_{s=0}^{n} O_{k,s}) - \frac{1}{k^2}(-O_{k,n} - O_{k,n-1} + \sum_{s=0}^{n} O_{k,s})$, Thus, we obtain $\sum_{s=0}^{n} O_{k,s} = k + O_{k,n-1} + (1-k^2)O_{k,n}$.
ii. The proof is shown in the same way as i. □

Theorem 16 (Generating Functions). *The generating functions for k-Oresme and k-Oresme-Lucas sequences are given as follows, respectively:*

$$\mathcal{Q}(x) = \frac{kx}{k^2 - k^2x - x^2} \text{ and } \mathcal{P}(x) = \frac{2k^2 - k^2x}{k^2 - k^2x - x^2}.$$

Proof. By the definition of the *k*-Oresme numbers, we obtain:

$$\begin{aligned}\mathcal{Q}(x) = \sum_{n=0}^{\infty} O_{k,n}x^n &= \tfrac{1}{k}x + \sum_{n=2}^{\infty} O_{k,n}x^n \\ &= \tfrac{1}{k}x + \sum_{n=2}^{\infty} O_{k,n-1}x^n - \tfrac{1}{k^2}\sum_{n=2}^{\infty} O_{k,n-2}x^n \\ &= \tfrac{1}{k}x + x\sum_{n=1}^{\infty} O_{k,n}x^n - \tfrac{1}{k^2}x^2\sum_{n=0}^{\infty} O_{k,n}x^n.\end{aligned}$$

Thus, we have:

$$\mathcal{Q}(x) = \frac{kx}{k^2 - k^2x - x^2}.$$

By the definition of the *k*-Oresme-Lucas numbers, we obtain:

$$\begin{aligned}\mathcal{P}(\mathrm{x}) = \sum_{\mathrm{n}=0}^{\infty} \mathrm{P_{k,n}x^n} &= 2 + \mathrm{x} + \sum_{\mathrm{n}=2}^{\infty} \mathrm{P_{k,n}x^n} \\ &= 2 + \mathrm{x} + \sum_{\mathrm{n}=2}^{\infty} \mathrm{P_{k,n-1}x^n} - \tfrac{1}{\mathrm{k}^2}\sum_{\mathrm{n}=2}^{\infty} \mathrm{P_{k,n-2}x^n} \\ &= 2 + \mathrm{x} + \mathrm{x}(-2 + \sum_{\mathrm{n}=0}^{\infty} \mathrm{P_{k,n}x^n}) - \tfrac{1}{\mathrm{k}^2}\mathrm{x}^2\sum_{\mathrm{n}=0}^{\infty} \mathrm{P_{k,n}x^n}.\end{aligned}$$

So, we obtain:

$$\mathcal{P}(x) = \frac{2k^2 - k^2x}{k^2 - k^2x - x^2}.$$

□

Corollary 1. *For the $k = 3$ value, the following relation can be written between the k-Oresme sequence and the Fibonacci sequence:*

$$O_{3,n} = \frac{F_{2n}}{3^n}.$$

Proof. The Binet formula of the *k*-Oresme sequence is:

$$O_{k,n} = \frac{\left(\frac{\mathrm{k}+\sqrt{k^2-4}}{2\mathrm{k}}\right)^n - \left(\frac{\mathrm{k}-\sqrt{k^2-4}}{2\mathrm{k}}\right)^n}{\sqrt{k^2-4}}.$$

For $k = 3$, the following relation can be written:

$$O_{k,n} = \frac{\left(\frac{3+\sqrt{5}}{6}\right)^n - \left(\frac{3-\sqrt{5}}{6}\right)^n}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}}{3^n\sqrt{5}}.$$

Thus, we obtain:

$$O_{3,n} = \frac{F_{2n}}{3^n}.$$

□

Corollary 2. *For the $k = 3$ value, the following relation can be written between the k-Oresme-Lucas sequence and the Lucas sequence:*

$$P_{3,n} = \frac{L_{2n}}{3^n}.$$

Proof. The Binet formula of the *k*-Oresme-Lucas sequence is:

$$P_{k,n} = \left(\frac{\mathrm{k} + \sqrt{k^2-4}}{2\mathrm{k}}\right)^n + \left(\frac{\mathrm{k} - \sqrt{k^2-4}}{2\mathrm{k}}\right)^n.$$

For $k = 3$, the following relation can be written:

$$P_{k,n} = \left(\frac{3+\sqrt{5}}{6}\right)^n + \left(\frac{3-\sqrt{5}}{6}\right)^n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n}}{3^n}$$

Thus, we obtain:

$$P_{3,n} = \frac{L_{2n}}{3^n}.$$

□

Corollary 3. *For the $k = 6$ value, the following relations can be written between the k-Oresme and k-Oresme-Lucas sequences and the Pell sequence R_n and Pell-Lucas sequence Q_n, respectively,*

i. $O_{6,n} = \frac{R_{2n}}{26^n}$,
ii. $P_{6,n} = \frac{Q_{2n}}{6^n}$.

Proof. For $k = 6$, the following relation can be written:

i. $O_{6,n} = \frac{\left(\frac{6+4\sqrt{2}}{6}\right)^n - \left(\frac{6-4\sqrt{2}}{6}\right)^n}{4\sqrt{2}} = \frac{\left(1+2\sqrt{2}\right)^{2n} - \left(1-2\sqrt{2}\right)^{2n}}{26^n 2\sqrt{2}}$. Thus, we obtain $O_{6,n} = \frac{R_{2n}}{26^n}$.

ii. $P_{6,n} = \left(\frac{6+4\sqrt{2}}{6}\right)^n + \left(\frac{6-4\sqrt{2}}{6}\right)^n = \frac{\left(1+2\sqrt{2}\right)^{2n} + \left(1-2\sqrt{2}\right)^{2n}}{6^n}$ Thus, we have $P_{6,n} = \frac{Q_{2n}}{6^n}$. □

3. Catalan Number

For $n \in \mathbb{N}$, the n^{th} Catalan numbers are $C_n = \frac{C(2n,n)}{n+1}$, and the generating function is obtained as $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. The Catalan numbers C_n values were obtained as follows (see for details in [10]): $\{C_n\}_{n\in\mathbb{N}} = 1, 1, 2, 5, 14, 132, 429, 1430, \ldots$

3.1. Catalan Transformation of the k-Oresme and k-Oresme-Lucas Sequences

Using the Catalan transform, we defined the Catalan transform of the k-Oresme and k-Oresme-Lucas sequences as follows, respectively:

$$CO_{k,n}=\sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}O_{k,i}, n\geq 1 \text{ with } CO_{k,0}=0$$

and

$$CP_{k,n}=\sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}P_{k,i}, n\geq 1 \text{ with } CP_{k,0}=0$$

Now we can provide the Catalan transformation of the first elements of the k-Oresme and the k-Oresme-Lucas sequences. The $CO_{k,n}$ and $CP_{k,n}$ values for the first four n natural numbers are given below:

- $CO_{k,0}=0\ CP_{k,0}=0$,
- $CO_{k,1}=\frac{1}{k}\ CP_{k,1}=1$,
- $CO_{k,2}=\frac{2}{k}\ CP_{k,2}=\frac{1}{k^2}\left(2k^2-2\right)$,
- $CO_{k,3}=\frac{1}{k^3}\left(5k^2-1\right)\ CP_{k,3}=\frac{1}{k^2}\left(5k^2-8\right)$,
- $CO_{k,4}=\frac{1}{k^3}\left(14k^2-5\right)\ CP_{k,4}=\frac{1}{k^4}\left(14k^4-23k^2+2\right)$,
- $CO_{k,5}=\frac{1}{k^5}\left(42k^4-20k^2+1\right)\ CP_{k,5}=\frac{1}{k^4}\left(42k^4-76k^2+13\right)$.

We can write $CO_{k,n}$ and $CP_{k,n}$ as the product of $nx1$ type $O_{k,n}$ and $P_{k,n}$'s of the lower triangular Catalan matrix C:

$$\begin{bmatrix}CO_{k,1}\\CO_{k,2}\\CO_{k,3}\\CO_{k,4}\\\vdots\end{bmatrix}=\begin{bmatrix}1 & \dots\\1\ 1 & \dots\\2\ 2\ 1 & \dots\\5\ 5\ 3\ 1 & \dots\\\vdots\ \vdots\ \vdots\ \vdots & \ddots\end{bmatrix}\begin{bmatrix}O_{k,1}\\O_{k,2}\\O_{k,3}\\O_{k,4}\\\vdots\end{bmatrix} \text{ and } \begin{bmatrix}CP_{k,1}\\CP_{k,2}\\CP_{k,3}\\CP_{k,4}\\\vdots\end{bmatrix}=\begin{bmatrix}1 & \dots\\1\ 1 & \dots\\2\ 2\ 1 & \dots\\5\ 5\ 3\ 1 & \dots\\\vdots\ \vdots\ \vdots\ \vdots & \ddots\end{bmatrix}\begin{bmatrix}P_{k,1}\\P_{k,2}\\P_{k,3}\\P_{k,4}\\\vdots\end{bmatrix}$$

So,

$$\begin{bmatrix}\frac{1}{k}\\\frac{2}{k}\\\frac{1}{k^3}\left(5k^2-1\right)\\\frac{1}{k^3}\left(14k^2-5\right)\\\vdots\end{bmatrix}=\begin{bmatrix}1 & \dots\\1\ 1 & \dots\\2\ 2\ 1 & \dots\\5\ 5\ 3\ 1 & \dots\\\vdots\ \vdots\ \vdots\ \vdots & \ddots\end{bmatrix}\begin{bmatrix}\frac{1}{k}\\\frac{1}{k}\\\frac{1}{k^3}\left(k^2-1\right)\\\frac{1}{k^3}\left(k^2-2\right)\\\vdots\end{bmatrix}$$

and

$$\begin{bmatrix}1\\\frac{1}{k^2}\left(2-2k^2\right)\\\frac{1}{k^2}\left(5-8k^2\right)\\\frac{1}{k^4}\left(2-22k^2+14k^4\right)\\\vdots\end{bmatrix}=\begin{bmatrix}1 & \dots\\1\ 1 & \dots\\2\ 2\ 1 & \dots\\5\ 5\ 3\ 1 & \dots\\\vdots\ \vdots\ \vdots\ \vdots & \ddots\end{bmatrix}\begin{bmatrix}1\\\frac{1}{k^2}\left(k^2-2\right)\\\frac{1}{k^2}\left(k^2-3\right)\\\frac{1}{k^4}\left(k^4-3k^2+2\right)\\\vdots\end{bmatrix}$$

3.2. The Generating Functions of the Catalan Transform k-Oresme and k-Oresme-Lucas Sequences

Let $O(x)$ and $c(x)$ be generating functions of the k-Oresme and Catalan sequences, respectively. $O_{k,n}(x*c(x))$ shows the generating function of the Catalan k-Oresme sequence. The following equations are written for the generating function of the Catalan transform of the k-Oresme sequence:

$$o(x)=CO_{k,n}(x)=O_{k,n}(x*c(x))=\frac{2}{4-4x-3k+(3k-2)\sqrt{1-4x}}.$$

Similarly, the generating function of the Catalan k-Oresme-Lucas sequence is:

$$p(x) = CP_{k,n}(x) = P_{k,n}(x * c(x)) = \frac{2 - 12k + 12k\sqrt{1-4x}}{4 - 4x - 3k + (3k-2)\sqrt{1-4x}}.$$

3.3. Hankel Transform of the Catalan k-Oresme and the k-Oresme-Lucas Sequences

Let the terms of a sequence be $A = \{v_1, v_2, v_3, \ldots\}$. In [11], the Hankel transform H_n of the terms of this sequence was defined as follows:

$$H_n = \begin{vmatrix} v_1 & v_2 & v_3 & v_4 & \ldots \\ v_2 & v_3 & v_4 & v_5 & \ldots \\ v_3 & v_4 & v_5 & v_6 & \ldots \\ v_4 & v_5 & v_6 & v_7 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Let us apply Hankel's work to the Catalan k-Oresme and Catalan k-Oresme-Lucas sequences, respectively. Its results are associated with the terms of the sequences. We obtain:

$HCO_1 = det[CO_{k,1}] = det\left[\frac{1}{k}\right] = \frac{1}{k} = O_{k,1}$,

$HCO_2 = det\begin{bmatrix} CO_{k,1} & CO_{k,2} \\ CO_{k,2} & CO_{k,3} \end{bmatrix} = det\begin{bmatrix} \frac{1}{k} & \frac{2}{k} \\ \frac{2}{k} & \frac{1}{k^3}\left(5k^2-1\right) \end{bmatrix} = \frac{1}{k^4}\left(k^2-1\right) = \frac{1}{k}O_{k,3}$,

$HCO_3 = det\begin{bmatrix} CO_{k,1} & CO_{k,2} & CO_{k,3} \\ CO_{k,2} & CO_{k,3} & CO_{k,4} \\ CO_{k,3} & CO_{k,4} & CO_{k,5} \end{bmatrix} = \frac{1}{k^7}\left(k^4 - 3k^2 + 1\right) = \frac{1}{k^2}O_{k,5}$,

$HCP_1 = det[CP_{k,1}] = 1 = P_{k,1}$,

$HCP_2 = det\begin{bmatrix} CP_{k,1} & CP_{k,2} \\ CP_{k,2} & CP_{k,3} \end{bmatrix} = \frac{k^4-4}{k^4} = P_{k,4} - P_{k,2} + P_{k,1}$,

$HCP_3 = det\begin{bmatrix} CP_{k,1} & CP_{k,2} & CP_{k,3} \\ CP_{k,2} & CP_{k,3} & CP_{k,4} \\ CP_{k,3} & CP_{k,4} & CP_{k,5} \end{bmatrix} = 4P_{k,8} + 48P_{k,7} - 234P_{k,6} + 324P_{k,5} - 141P_{k,4}$.

4. Discussion

Based on this study, other number sequences can be associated with Oresme num-bers as an application of these numbers.

5. Conclusions

In this article, we first defined the k-Oresme and the k-Oresme-Lucas. We then defined the main features of these sequences. We also examined the relationships between the terms of these sequences. Moreover, we associated the k-Oresme and k-Oresme-Lucas sequences with Fibonacci, Pell numbers and Lucas, and Pell-Lucas numbers, respectively. Finally, we defined the Catalan transformation of these sequences. Moreover, Hankel transformations were applied to the Catalan transformations of the k-Oresme and k-Oresme-Lucas sequences and the results associated with the terms of the sequences. If this study is examined, such features can be found in other sequences, such as Horadam and Mersenne sequences.

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