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Applications of Fibonacci sequences in a finite nilpotent group

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Abstract

We have proved two original theorem concerning Wall number of the the three-step Fibonacci sequences constructed by three elements of any finite p-group of exponent p and nilpotency class 4.
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1. Introduction

Let $s = (s_i)$ denote the ordinary three-step Fibonacci sequence in GF(p) defined by the recurrence $s_{i+3} = s_i + s_{i+1} + s_{i+2}$ and the initial data $s_0 = 0$, $s_1 = 0$ and $s_2 = 1$. This is a bi-infinite periodic sequence or loop indexed by the integers. This sequence or loop must be periodic and we use the letter k to denote the fundamental period of (s_i) that is the shortest period of that sequence. Obviously k depends on p.

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Takahashi gives a fast algorithm which is based on the product of Lucas number to compute large Fibonacci numbers [1]. Fibonacci sequences have been an interesting subject in applied mathematics. West has shown using transfer matrices that the number $|S_n(123, 3214)|$ of permutations avoiding the patterns 123 and 3214 is the Fibonacci number F_{2n} [2].

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The study of Fibonacci sequences in groups began with the earlier work of Wall [3] where the ordinary Fibonacci sequences in cyclic groups were investigated and then Winson was particularly interested in ranks of apparition in ordinary Fibonacci sequences [4]. In the mid 80s, Wilcox extended the problem to abelian groups [5]. Prolific co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups [6]. Ryba constructed and analysed a pair of sequences of representation of the symmetric groups [7]. Aydın and Smith proved in [8] that the lengths of ordinary two-step Fibonacci sequences are equal to the lengths of the two-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and exponent a prime number p. Furthermore, Aydın and Dikici proved in [9] that the length of the two-step general Fibonacci sequences are equal to the length of the two-step general Fibonacci recurrences constructed by two generating elements of finite nilpotent groups of nilpotency class 2 and exponent a prime number p.

One of the latest works in this area is [10] where it is shown that for the three-step general Fibonacci recurrence and any finite p-group of exponent p and nilpotency class 2, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary three-step general Fibonacci sequence in the field $\mathrm{GF}(p)$.

We shall be interested in the shortest period of the three-step Fibonacci sequence the entries of which are taken in any finite group p-group of exponent p and nilpotency class 4.

Definition 1.1. Let H and K be normal in G and $K \leqslant H$. If H/K is contained in the centre of G/K, then H/K is called a central factor of G. A group G is called nilpotent if it has finite series of normal subgroups

$$G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_r = 1$$

such that G_{i-1}/G_i, is a central factor of G for each $i = 1, 2, \ldots, r$. The smallest possible r is called the nilpotency class of G.

Further details about nilpotent groups and related topics can be found in [11].

2. The main theorem

The fundamental period of a sequence satisfying a linear recurrence is sometimes called the Wall number of that sequence. The Wall number with respect to a particular linear recurrence $k(G) = k$ of a group G is the lowest common multiple of the fundamental period of all sequence satisfying the recurrence in the group G. As a different perspective of what it usually used to be, k denotes the fundamental period of the three-step general Fibonacci sequences taken modulo a distinguished prime p. We consider a three-step Fibonacci

sequence $r = (r_i)$ in a finite p-group G, given some initial data r_0, r_1 and r_2. Such a sequence or loop must be periodic and we denote the shortest period of this sequence sometimes called the fundamental period by k_r. From now on k denotes the fundamental period of the standard three-step Fibonacci sequence $0, 0, 1, 1, 2, \ldots$ taken modulo a distinguished prime p.

Theorem 2.1. *Let $p > 3$ be a prime number, then if G is a non-trivial finite p-group of exponent p and nilpotency class 4 then $k(G) = k$, except for finitely bad primes. Of course if G is the trivial group then $k(G) = 1$.*

Theorem 2.2. *Let $p > 3$ be a prime number, then if G is a non-trivial finite p-group of exponent p and nilpotency class 4 then $k(G) = kp$ for all primes.*

3. Some lemmas concerning three-step Fibonacci sequence

Although the proofs of all the following lemmas are not intricate they are omitted here and can be found in [12]. The notation $\sum_{i<j}$ indicates that we are dealing with a double sum taken over all i and j subject to the constraint that $0 \leqslant i < j \leqslant k-1$.

Lemma 3.1. *For all integers α and β we have*

$$\sum_{i<j} s_{j+\alpha} s_{i+\beta} = 0.$$

Lemma 3.2. *For all integers α, β and c we have*

$$\sum_{i<j} s_{j-i+\alpha} s_{i+\beta} s_{i+c} = 0.$$

Lemma 3.3. *For all integers α, β and γ we have*

$$\sum_{j=0}^{k-1} s_{j+\alpha} s_{j+\beta} s_{-j+\gamma} s_j = 0.$$

Lemma 3.4. *For all integers α, β, c, d and e we have*

$$\sum_{i<j} s_{-j+\alpha} s_{j+\beta} s_{j-i-d} s_{i+e} s_{i+c} = 0.$$

Furthermore, we have proved following lemma using MAPLE. Computer codes and their results available on a request. Moreover, we have lots of data for various values of p for which indeed is proved the following lemmas. That

is, we know whether or not the following lemmas are provided for which primes. The notation $\sum_{n<i<j}$ indicates that we are dealing with a trio sum taken over all n, i and j subject to the constraint that $0 \leqslant n < i < j \leqslant k-1$.

Lemma 3.5. *For all integers a, b, c, d, e, f and g we have*

$$\sum_{n<i<j} s_{-j+a}s_{j+b}s_{j-i-c}s_{i+d}s_{i-n-e}s_{n+f}s_{n+g} = 0.$$

Lemma 3.6. *For all integers a, b, c, d and e we have*

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j-i-c}s_{i+d}s_{i+e}s_i = 0.$$

Lemma 3.7. *For all integers a, b, c, d, e and f we have*

$$\sum_{i<j} s_{-j+a}s_{j+b}s_{j+c}s_{j-i-d}s_{i+e}s_{i+f} = 0.$$

Lemma 3.8. *For all integers a, b, c and d we have*

$$\sum_{j=0}^{k-1} s_{j+a}s_{j+b}s_{j+c}s_{-j+d}s_j = 0.$$

4. The proofs of Theorem 2.1 and Theorem 2.2

Let G be a nilpotent group with nilpotency class 4 and exponent p for a prime number $p > 3$. G has two generators x and y. A presentation of G is

$$G = \langle x, y, z, t, u : (y,x) = z, (z,x) = t, (t,x) = u \rangle,$$

where pairs of generators with unspecified commutator are implicitly deemed to commute. The subgroup $\langle y, z, t, u \rangle$ is abelian and u is in the centre of G. Every element of G has a unique representation as

$$x^a y^b z^c t^d u^e,$$

where the exponents are elements of GF(p). Having distinguished this way of writing elements we can even think of groups elements as vectors of dimension 5 over GF(p), i.e. as (a, b, c, d, e). So triple multiplication is determined by the following rules:

$$
\begin{aligned}
&(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0})(x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1})(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})\\
&\quad=(x^{a_3}y^{b_3}z^{c_3}t^{d_3}u^{e_3})=(x^{a_0}y^{b_0}z^{c_0}x^{a_1}y^{b_1}z^{c_1}t^{d_0+d_1}u^{e_0+e_1+d_0a_1})(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})\\
&\quad=\left(x^{a_0}y^{b_0}x^{a_1}y^{b_1}z^{c_0+c_1}t^{d_0+d_1+c_0a_1}u^{e_0+e_1+d_0a_1+c_0\binom{a_1}{2}}\right)(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})\\
&\quad=\left(x^{a_0+a_1}y^{b_0+b_1}z^{c_0+c_1+b_0a_1}t^{d_0+d_1+c_0a_1+b_0\binom{a_1}{2}}u^{e_0+e_1+d_0a_1+c_0\binom{a_1}{2}+b_0\binom{a_1}{3}}\right)\\
&\qquad\times(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})\\
&\quad=x^{a_0+a_1+a_2}y^{b_0+b_1+b_2}z^{c_0+c_1+c_2+b_0a_1+(b_0+b_1)a_2}\\
&\qquad\times t^{d_0+d_1+d_2+c_0a_1+b_0\binom{a_1}{2}+a_2(c_0+c_1+b_0a_1)+(b_0+b_1)\binom{a_2}{2}}\\
&\qquad\times u^{e_0+e_1+e_3+d_0a_1+c_0\binom{a_1}{2}+b_0\binom{a_1}{3}+a_2\left(d_0+d_1+c_0a_1+b_0\binom{a_1}{2}\right)+(c_0+c_1+b_0a_1)\binom{a_2}{2}+(b_0+b_1)\binom{a_2}{3}}.
\end{aligned}
$$

That is, $(a_0,b_0,c_0,d_0,e_0)(a_1,b_1,c_1,d_1,e_3)(a_2,b_2,c_2,d_2,e_3)=(a_3,b_3,c_3,d_3,e_3)$ where

$$
\begin{aligned}
a_3&=a_0+a_1+a_2,\\
b_3&=b_0+b_1+b_2,\\
c_3&=c_0+c_1+c_2+b_0a_1+(b_0+b_1)a_2,\\
d_3&=d_0+d_1+d_2+c_0a_1+b_0\binom{a_1}{2}+a_2(c_0+c_1+b_0a_1)+\binom{a_2}{2}(b_0+b_1)
\end{aligned}
$$

and finally

$$
\begin{aligned}
e_3=e_0+e_1+e_2+d_0a_1+c_0\binom{a_1}{2}+b_0\binom{a_1}{3}+a_2\Bigg(d_0+d_1+c_0a_1\\
+b_0\binom{a_1}{2}\Bigg)+\binom{a_2}{2}(c_0+c_1+b_0a_1)+\binom{a_2}{3}(b_0+b_1).
\end{aligned}
$$

We must consider two types of initial data for loops in G. We have a loop v of type I with initial data

$$
\begin{aligned}
v_0&=(0,0,0,0,0),\\
v_1&=(1,0,0,0,0),\\
v_2&=(0,1,0,0,0)
\end{aligned}
$$

and another w of type II with initial data

$$w_0 = (1,0,0,0,0),$$
$$w_1 = (0,1,0,0,0),$$
$$w_2 = (0,0,0,0,0).$$

The analysis of the type II loop is entirely similar to that of type I. Thus the type I loop begins

$$v_0 = (t_0, s_0, 0, 0, 0),$$
$$v_1 = (t_1, s_1, 0, 0, 0),$$
$$v_2 = (t_2, s_2, 0, 0, 0).$$

We focus on the type I loop $v_i = (t_i, s_i, c_i, d_i, e_i)$ where

$$(s_0, s_1, s_2) = (0, 0, 1)$$

and

$$(t_0, t_1, t_2) = (0, 1, 0).$$

It can be easily seen that the sequence t_i can be written in terms of s_i as $t_i = s_{i+1} - s_i$. Now, it follows from [12] that $c_k = c_{k+1} = c_{k+2} = 0$ which corresponds to prove the similar theorem where the nilpotency class of the group reduces to 2 and [13] that $d_k = d_{k+1} = d_{k+2} = 0$ which corresponds to prove the similar theorem where the nilpotency class of the group reduces to 3. In order to prove Theorem 2.1, we are faced with the task of showing that $e_k = e_{k+1} = e_{k+2} = 0$ and we begin with $e_k = 0$, that is, these individual sums actually vanish for all but a finite number of what we call are bad primes for which we do not know whether $e_k = 0$ or not.

We shall need two formulas for c_α and d_α to work out the formula for e_α. By induction it is

$$c_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}))$$

and

$$d_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+1} c_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+1}}{2} s_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+2}(c_i + c_{i+1} + t_{i+1} s_i)$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} \binom{t_{i+2}}{2} (s_i + s_{i+1})$$

for $\alpha \geqslant 0$. These enables us, via a similar process, to describe e_α for $\alpha \geqslant 0$ as

$$e_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}t_{i+1}d_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}c_i\binom{t_{i+1}}{2} + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}s_i\binom{t_{i+1}}{3}$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}t_{i+2}\left(d_i + d_{i+1} + c_it_{i+1} + s_i\binom{t_{i+1}}{2}\right)$$
$$+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}\binom{t_{i+2}}{2}(c_i + c_{i+1} + t_{i+1}s_i) + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i + s_{i+1})\binom{t_{i+2}}{3}.$$

We can break up the expression for e_k as $e_k = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6$.

$$\Delta_1 = \sum s_{k-i-1}t_{i+1}d_i, \quad \Delta_2 = \sum s_{k-i-1}c_i\binom{t_{i+1}}{2}, \quad \Delta_3 = \sum s_{k-i-1}s_i\binom{t_{i+1}}{3},$$

$$\Delta_4 = \sum s_{k-i-1}t_{i+2}\left(d_i + d_{i+1} + c_it_{i+1} + s_i\binom{t_{i+1}}{2}\right),$$

$$\Delta_5 = \sum s_{k-i-1}(c_i + c_{i+1} + s_it_{i+1})\binom{t_{i+2}}{2}$$

and

$$\Delta_6 = \sum s_{k-i-1}(s_i + s_{i+1})\binom{t_{i+2}}{3}$$

and we shall attempt to show that each of these six expressions Δ_i actually vanishes. To this end, we break these expressions up still further.

Now we have

$$\Delta_1 = \sum_{i=0}^{\alpha-1} s_{k-i-1}t_{i+1}d_i = \sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}d_j$$
$$= \sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}\left(\sum_{i=0}^{j-1} s_{j-i-1}t_{i+1}c_i + \sum_{i=0}^{j-1} s_{j-i-1}\binom{t_{i+1}}{2}s_i\right.$$
$$\left. + \sum_{i=0}^{j-1} s_{j-i-1}t_{i+2}(c_i + c_{i+1} + t_{i+1}s_i) + \sum_{i=0}^{j-1} s_{j-i-1}\binom{t_{i+2}}{2}(s_i + s_{i+1})\right),$$

$$\Delta_{11} = \sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}\sum_{i=0}^{j-1} s_{j-i-1}t_{i+1}c_i = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+1}c_i$$
$$= \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+1}\left(\sum_{n=0}^{i-1} s_{i-n-1}s_nt_{n+1} + \sum_{n=0}^{i-1} s_{i-n-1}t_{n+2}(s_n + s_{n+1})\right),$$

$$\Delta_{111} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1}\sum_{n=0}^{i-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+1}s_{i-n-1}s_nt_{n+1},$$

$$\Delta_{112} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1}\sum_{n=0}^{i-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+1}s_{i-n-1}t_{n+2}s_n,$$

$$\Delta_{113} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1}\sum_{n=0}^{i-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+1}s_{i-n-1}s_{n+1}t_{n+2},$$

$$\Delta_{11} = \Delta_{111} + \Delta_{112} + \Delta_{113},$$

$$\Delta_{12} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_i\binom{t_{i+1}}{2}$$
$$= \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}s_i\left((t_{i+1})^2 - (t_{i+1})\right)/2,$$

$$\Delta_{13} = \sum_{j=0}^{k-1} s_{k-j-1}t_{j+1}\sum_{i=0}^{j-1} s_{j-i-1}t_{i+2}(c_i + c_{i+1} + s_it_{i+1})$$
$$= \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+2}(c_i + c_{i+1} + s_it_{i+1}),$$

$$\Delta_{131} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+2}c_i,$$

$$\Delta_{132} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+2}c_{i+1},$$

$$\Delta_{133} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+2}s_it_{i+1},$$

$$\Delta_{131} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+2}\left(\sum_{n=0}^{i-1} s_{i-n-1}s_nt_{n+1} + \sum_{n=0}^{i-1} s_{i-n-1}t_{n+2}(s_n + s_{n+1})\right),$$

$$\Delta_{132} = \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_{k-j-1}t_{j+1}s_{j-i-1}t_{i+2}\left(\sum_{n=0}^{i} s_{i-n}s_nt_{n+1} + \sum_{n=0}^{i} s_{i-n}t_{n+2}(s_n + s_{n+1})\right),$$

$$\Delta_{13} = \Delta_{131} + \Delta_{132} + \Delta_{133}$$

and

$$\Delta_{14} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+1} \sum_{i=0}^{j-1} s_{j-i-1} \binom{t_{i+2}}{2} (s_i + s_{i+1})$$

$$= \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1} t_{j+1} s_{j-i-1} ((t_{i+2})^2 - t_{i+2})/2(s_i + s_{i+1}),$$

$$\Delta_1 = \Delta_{11} + \Delta_{12} + \Delta_{13} + \Delta_{14}.$$

Moving to Δ_2, we find that

$$\Delta_2 = \sum_{j=0}^{k-1} s_{k-j-1} \binom{t_{j+1}}{2} c_j,$$

$$\Delta_{21} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} (t_{j+1})^2 c_j,$$

$$\Delta_{22} = -\frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+1} c_j,$$

$$\Delta_{21} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} (t_{j+1})^2 \left(\sum_{i=0}^{j-1} s_{j-i-1} t_{i+1} s_i + \sum_{i=0}^{j-1} s_{j-i-1} t_{i+2} (s_i + s_{i+1}) \right),$$

$$\Delta_{22} = -\frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+1} \left(\sum_{i=0}^{j-1} s_{j-i-1} t_{i+1} s_i + \sum_{i=0}^{j-1} s_{j-i-1} t_{i+2} (s_i + s_{i+1}) \right),$$

$$\Delta_2 = \Delta_{21} + \Delta_{22}.$$

Next we tackle Δ_3. We have

$$\Delta_3 = \sum_{j=0}^{k-1} s_{k-j-1} s_j \binom{t_{j+1}}{3} = \frac{1}{6} \sum_{j=0}^{k-1} s_{k-j-1} s_j ((t_{j+1})^3 - 3(t_{j+1})^2 + 2t_{j+1}),$$

$$\Delta_4 = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} \left(d_j + d_{j+1} + c_j t_{j+1} + s_j \binom{t_{j+1}}{2} \right),$$

$$\Delta_{41} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} d_j,$$

$$\Delta_{42} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} d_{j+1},$$

$$\Delta_{43} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} c_j t_{j+1},$$

$$\Delta_{44} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} s_j \binom{t_{j+1}}{2},$$

Δ_{41} is similar to Δ_1.

$$\begin{aligned}\Delta_{42} &= \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} d_{j+1} \\ &= \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} \left(\sum_{i=0}^{j} s_{j-i} t_{i+1} c_i + \sum_{i=0}^{j} s_{j-i} \binom{t_{i+1}}{2} s_i + \sum_{i=0}^{j} s_{j-i} t_{i+2} (c_i \right. \\ &\quad \left. + c_{i+1} + t_{i+1} s_i) + \sum_{i=0}^{j} s_{j-i} \binom{t_{i+2}}{2} (s_i + s_{i+1}) \right),\end{aligned}$$

$$\Delta_{421} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} t_{i+1} c_i,$$

$$\Delta_{421} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} t_{i+1} \left(\sum_{n=0}^{i-1} s_{i-n-1} t_{n+1} s_n + \sum_{n=0}^{i-1} s_{i-n-1} t_{n+2} (s_n + s_{n+1}) \right),$$

$$\Delta_{422} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} s_i ((t_{i+1})^2 - t_{i+1})/2,$$

$$\Delta_{423} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} (c_i + c_{i+1} + s_i t_{i+1}),$$

$$\Delta_{4231} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} c_i,$$

$$\Delta_{42311} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \sum_{n=0}^{i-1} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} s_{i-n-1} t_{n+1} s_n,$$

$$\Delta_{42312} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \sum_{n=0}^{i-1} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} s_{i-n-1} t_{n+2} s_n,$$

$$\Delta_{42313} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \sum_{n=0}^{i-1} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} s_{i-n-1} t_{n+2} s_{n+1},$$

$$\Delta_{4231} = \Delta_{42311} + \Delta_{42312} + \Delta_{42313},$$

$$\Delta_{4232} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} c_{i+1},$$

$$\Delta_{42321} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \sum_{n=0}^{i} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} s_{i-n} t_{n+1} s_n,$$

$$\Delta_{42322} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \sum_{n=0}^{i} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} s_{i-n} t_{n+2} s_n,$$

$$\Delta_{42323} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \sum_{n=0}^{i} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} s_{i-n} t_{n+2} s_{n+1},$$

$$\Delta_{4233} = \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} t_{i+2} s_i t_{i+1},$$

$$\Delta_{4232} = \Delta_{42321} + \Delta_{42322} + \Delta_{42323},$$

$$\Delta_{423} = \Delta_{4231} + \Delta_{4232} + \Delta_{4233},$$

$$\Delta_{424} = \sum_{j=0}^{kp-1} \sum_{i=0}^{j} s_{k-j-1} t_{j+2} s_{j-i} \binom{t_{i+2}}{2} (s_i + s_{i+1}),$$

$$\Delta_{42} = \Delta_{421} + \Delta_{422} + \Delta_{423} + \Delta_{424},$$

$$\Delta_{43} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} t_{j+1} c_j,$$

$$\Delta_{431} = \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1} t_{j+2} t_{j+1} s_{j-i-1} t_{i+1} s_i,$$

$$\Delta_{432} = \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1} t_{j+2} t_{j+1} s_{j-i-1} t_{i+2} s_i,$$

$$\Delta_{433} = \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1} t_{j+2} t_{j+1} s_{j-i-1} t_{i+2} s_{i+1},$$

$$\Delta_{43} = \Delta_{431} + \Delta_{432} + \Delta_{433},$$

$$\Delta_{44} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} s_j \binom{t_{j+1}}{2},$$

$$\Delta_5 = \sum_{j=0}^{k-1} s_{k-j-1} \binom{t_{j+2}}{2} (c_j + c_{j+1} + s_j t_{j+1}),$$

$$\Delta_{51} = \sum_{j=0}^{k-1} s_{k-j-1} \binom{t_{j+2}}{2} c_j,$$

$$\Delta_{52} = \sum_{j=0}^{k-1} s_{k-j-1} \binom{t_{j+2}}{2} c_{j+1},$$

$$\Delta_{53} = \sum_{j=0}^{k-1} s_{k-j-1} \binom{t_{j+2}}{2} s_j t_{j+1},$$

$$\Delta_{51} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1}((t_{j+2})^2 - t_{j+2}) c_j,$$

$$\Delta_{511} = \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1}((t_{j+2})^2 - t_{j+2}) s_{j-i-1} t_{i+1} s_i,$$

$$\Delta_{512} = \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1}((t_{j+2})^2 - t_{j+2}) s_{j-i-1} t_{i+2}(s_i + s_{i+1}),$$

$$\Delta_{51} = \Delta_{511} + \Delta_{512},$$

$$\Delta_{52} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1}((t_{j+2})^2 - t_{j+2}) c_{j+1},$$

$$\Delta_{521} = \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1}((t_{j+2})^2 - t_{j+2}) s_{j-i} t_{i+1} s_i,$$

$$\Delta_{522} = \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=0}^{j} s_{k-j-1}((t_{j+2})^2 - t_{j+2}) s_{j-i} t_{i+2}(s_i + s_{i+1}),$$

$$\Delta_{52} = \Delta_{521} + \Delta_{522}, \quad \Delta_5 = \Delta_{51} + \Delta_{52} + \Delta_{53},$$

$$\begin{aligned}\Delta_{61} &= \sum_{j=0}^{k-1} s_{k-j-1} s_j \binom{t_{j+2}}{3} \\ &= \frac{1}{6} \sum_{j=0}^{k-1} s_{k-j-1} s_j (t_{j+2})^3 - \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} s_j (t_{j+2})^2 + \frac{1}{3} \sum_{j=0}^{k-1} s_{k-j-1} s_j t_{j+2},\end{aligned}$$

$$\begin{aligned}\Delta_{62} &= \sum_{j=0}^{k-1} s_{k-j-1} s_{j+1} \binom{t_{j+2}}{3} \\ &= \frac{1}{6} \sum_{j=0}^{k-1} s_{k-j-1} s_{j+1} (t_{j+2})^3 - \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} s_{j+1} (t_{j+2})^2 + \frac{1}{3} \sum_{j=0}^{k-1} s_{k-j-1} s_{j+1} t_{j+2},\end{aligned}$$

$$\Delta_6 = \Delta_{61} + \Delta_{62}.$$

Thus,

$$e_k = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6.$$

We must show all sums of type Δ actually vanish. Some sums of type Δ are given in Lemmas 3.1–3.4 which was proved in [12].These sums given in Lemmas 3.1–3.4 are sums in the case of nilpotency class 2 and 3.

In case of nilpotency class 4, remaining sums of different type from sums in Lemmas 3.1–3.4 are given in Lemmas 3.5–3.8. These sums are shown to be vanish by using MAPLE, for some primes. For example $p = 5, 11$. These sums do not vanish for finitely primes. For example, for $p = 19, 29, 37$. In fact, Thus we have shown $e_k = 0$ for the type I sequence. It is a matter of algebraic manipulation to show that $e_{k+1} = e_{k+2} = 0$. The analysis of the type II sequence is extremely similar to that of the type I sequence.

To prove Theorem 2.2, we consider following equality

$$\sum_{j=0}^{kp-1} = \sum_{j=0}^{k-1} + \sum_{j=k}^{2k-1} + \cdots + \sum_{j=(p-1)k}^{kp-1}.$$

We write

$$\sum_{j=0}^{k-1} = \sum_{j=k}^{2k-1} = \cdots = \sum_{j=(p-1)k}^{kp-1}.$$

Because the sequence is periodic, its Wall number is k and $s_i = s_{k+i}$ for $i \geqslant 0$.

Hence,

$$\sum_{j=0}^{kp-1} = p \sum_{j=0}^{k-1}.$$

Thus, if we take the latest sum according to modulo p, it vanishes. Similarly, we have shown $e_{kp} = 0$ for the type I sequence. It is a matter of algebraic manipulation to show that $e_{kp+1} = e_{kp+2} = 0$. The analysis of the type II sequence is extremely similar to that of the type I sequence. So the proof is completed.

Conjecture. *The class number can be increased to five. It seems that more complex sums than those of this document are needed to be vanished. Thus, it is likely to put more restrictions on prime numbers in question.*

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