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On Pell Functions and Pell Numbers

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Abstract

Number sequences have been studied by many scientists for many years because they find applications in nature and many branches of science. The most well-known number sequence among these integer sequences is the Fibonacci sequence. Later, these integer sequences were generalized and many number sequences were defined and their properties were examined. One of these is Pell number sequences. Pell numbers can be calculated in a way similar to the recurrence relation of Fibonacci numbers. The Pell numbers P_n are defined by $P_n = 2P_{n-1} + P_{n-2}, n \geq 3$ with $P_1 = 1$ and $P_2 = 2$. The first few terms of Pell numbers are 1,2,5,12,29,… . In mathematics, the Pell numbers are an infinite sequence of integers, known since ancient times, that comprise the denominators of the closest rational approximations to the square root of 2. In addition to being used to approximate the square root of two, Pell numbers can also be used to find square triangle numbers, create integer approximations to the right isosceles triangle, and solve certain combinatorial numbering problems. In this study, we define Pell functions on real numbers R. We give the relations between Pell functions and Pell numbers. We examine Pell functions using the concept of the $p-odd$ and $p-even$ functions. Additionally, if p is a Pell function, then $\lim_{x\to\infty} \frac{p(x+1)}{p(x)} = 1+\sqrt{2}$ which is one of the roots of equation $r^2 - 2r - 1 = 0$.

Keywords: Pell functions, Pell numbers, $p-odd$ functions, $p-even$ functions, silver ratio, characteristic equation, continuous functions.

INTRODUCTION

Number sequences and their polynomials have attracted the attention of many scientists for many years, as they find application in nature and in many sciences. Of course, the Fibonacci numbers are the best known of the sequences of numbers. Number sequences, especially the Fibonacci sequence, have attracted the attention of many researchers because they have applications in almost every field of mathematics, as well as in Physics, Chemistry, Biology, Engineering, Economics and even social sciences. Many generalizations of Fibonacci sequences were then described and studied [1–16].

One of these generalizations is the Pell numbers. Pell numbers arise historically in the rational approximation of $\sqrt{2}$.

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We know well that the Pell numbers P_n are defined by

$$P_n = 2P_{n-1} + P_{n-2}, n \geq 3$$

with $P_1 = 1$ and $P_2 = 2$. [15]

The first few terms of Pell numbers are 1,2,5,12, 29, … .

Ratios of consecutive Pell numbers approach the silver ratio. Thus,

$$\frac{\lim_{n\to\infty} P_n}{P_{n-1}} = \sqrt{2} + 1.$$

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In [17], the authors mentioned some properties and applications of Pell numbers related to the silver ratio.

In arbitrary groupoids, it was studied a Fibonacci norm of positive integers and Fibonacci sequences groupoids [18, 19].

In [20], the authors introduced Fibonacci functions and gave the relationship between Fibonacci functions and Fibonacci numbers.

Tribonacci functions were defined and the relations between these functions and Tribonacci numbers are given in [21].

In this study, we define Pell functions based on studies [20, 21]. We give the connections between Pell numbers and Pell functions. Then, we introduce some examples for these functions We examine them by using the $p-odd$ and $p-even$ functions. Additionally, we investigate the limit of a Pell function. If p is a Pell function, then $\lim_{x\to\infty} \frac{p(x+1)}{p(x)} = 1+\sqrt{2}$ which is one of the roots of equation $r^2 - 2r - 1 = 0$.

MATERIALS AND METHODS

Pell Functions

Let R represent real numbers throughout the paper. If the function p satisfies the following condition for every $x \in R$

$$p(x+2) = 2p(x+1) + p(x) \tag{1}$$

then p is called a Pell function defined over the real numbers.

Example 1.1: For $c > 0$, let $p(x) \coloneqq c^x$ be a Pell function on R.

Then,

$$\begin{aligned} p(x+2) &= 2p(x+1) + p(x) \\ c^{x+2} &= 2c^{x+1} + c^x \\ c^x c^2 &= c^x(2c+1) \\ c^x(c^2 - 2c - 1) &= 0 \end{aligned}$$

Since $c > 0$, we get $c^2 = 2c + 1$ and $c = 1+\sqrt{2}$. Therefore $p(x) = c^x = \left(1+\sqrt{2}\right)^x$ is a Pell function and the unique Pell function of this style.

Similarly, in the other root of the characteristic equation, we can give the following example

Example 1.2: For $d = 1-\sqrt{2}$, let $p(x) \coloneqq d^x$ be a Pell function on R.

If we let $u_0 = 0$ and $u_1 = 1$, let's note that the full Pell sequence with

$$u_{-n} = (-1)^{n-1}u_n, \text{ for } n > 0$$

where u_n is nth Pell numbers. That is,

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$$\{u_n\}_{-\infty}^{\infty} = \{\dots, -12{,}5, -2{,}1{,}0{,}1{,}2{,}5, \dots\}$$

Example 1.3: Let $\{\vartheta_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ be full Pell sequence. We describe a function

$$p(x) \coloneqq 2\vartheta_{\lfloor x \rfloor} + \sigma_{\lfloor x \rfloor} t$$

where $t = x - \lfloor x \rfloor \in (0,1)$. So, we have

$$p(x+2) = 2\vartheta_{\lfloor x+2 \rfloor} + \sigma_{\lfloor x+2 \rfloor} t$$

since

$$\vartheta_{\lfloor x+2 \rfloor} = \vartheta_{\lfloor x \rfloor + 2} = 2\vartheta_{\lfloor x \rfloor + 1} + \vartheta_{\lfloor x \rfloor}$$

and

$$\sigma_{\lfloor x+2 \rfloor} = \sigma_{\lfloor x \rfloor + 2} = 2\sigma_{\lfloor x \rfloor + 1} + \sigma_{\lfloor x \rfloor}$$

Thus, we can write as follows

$$p(x+2) = 2\vartheta_{\lfloor x+2 \rfloor} + \sigma_{\lfloor x+2 \rfloor} t$$

$$2\left(2\vartheta_{\lfloor x \rfloor + 1} + \sigma_{\lfloor x \rfloor + 1} t\right) + \left(2\vartheta_{\lfloor x \rfloor} + \sigma_{\lfloor x \rfloor} t\right)$$

$$2p(x+1) + p(x).$$

So, $p(x)$ is a Pell functions.

Example 1.4: In Example 1.3, we think the function

$$p(x) \coloneqq 2\vartheta_{\lfloor x \rfloor} + \sigma_{\lfloor x \rfloor} t$$

where $t = x - \lfloor x \rfloor \in (0,1)$. If we define $\sigma_{\lfloor x \rfloor}$ like this $\sigma_{\lfloor x \rfloor} \coloneqq \vartheta_{\lfloor x \rfloor - 1}$, then

$p(x) \coloneqq 2\vartheta_{\lfloor x \rfloor} + \vartheta_{\lfloor x \rfloor - 1} t$ is a Pell function.

$$p(x+2) = 2\vartheta_{\lfloor x+2 \rfloor} + \vartheta_{\lfloor x+2 \rfloor - 1}$$

$$2\vartheta_{\lfloor x \rfloor + 2} + \vartheta_{\lfloor x \rfloor + 2 - 1} t$$

$$2\left(2\vartheta_{\lfloor x \rfloor + 1} + \vartheta_{\lfloor x \rfloor}\right) + \left(2\vartheta_{\lfloor x \rfloor} + \vartheta_{\lfloor x \rfloor - 1}\right) t$$

$$2p(x+1) + p(x)$$

So, $p(x) \coloneqq 2\vartheta_{\lfloor x \rfloor} + \vartheta_{\lfloor x \rfloor - 1} t$ is a Pell function.

Proposition 1.5: For $t, x \in R$, if we describe

$$h(x) \coloneqq p(x+t)$$

then h is also a Pell function.

Proof. We have

$$h(x+2) = p(x+t+2)$$

$$2p(x+t+1) + p(x+t)$$

$$2h(x+1) + h(x)$$

So, h is a Pell function.

Example 1.6: Since $p(x) = \left(1+\sqrt{2}\right)^x$ is a Pell function, in that case for $t \in R$,

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$$\upsilon(x) \coloneqq \left(1+\sqrt{2}\right)^{x+t}$$

is also a Pell function.

We get

$$\upsilon(x+2) = \left(1+\sqrt{2}\right)^{x+t+2}$$
$$\left(2\left(1+\sqrt{2}\right)^{x+1} + \left(1+\sqrt{2}\right)^{x}\right)\left(1+\sqrt{2}\right)^{t}$$
$$2\upsilon(x+1) + \upsilon(x).$$

From here, υ is a Pell function.

Note that $\upsilon(x) = \alpha^t p(x)$.

Theorem 1.7: Let $\{P_n\}$ be a sequence of Pell numbers. Then we have

$$p(x+n) = P_n p(x+1) + P_{n-1}p(x)$$

for any $x \in R$ and $n \geq 2$.

Proof. Let's show the proof by induction. For $n = 2$, it is satisfied as follows

$$p(x+2) = 2p(x+1) + 1p(x)$$
$$P_2 p(x+1) + P_1 p(x)$$

We suppose that it is true for $n, n+1$.

Now, we have to show that it is true for $n+2$. From Pell function definition as follow,

$$p(x+n+2) = 2p(x+n+1) + p(x+n)$$

since it is true for n and $n+1$, we have

$$2\left(P_{n+1}p(x+1) + P_n p(x)\right) + P_n p(x+1)$$
$$+P_{n-1}p(x)$$
$$(2P_{n+1} + P_n)p(x+1) + (2P_n + P_{n-1})p(x)$$
$$p(x+n+2) = P_{n+2}p(x+1) + P_{n+1}p(x)$$

Thus, the proof is completed.

Corollary 1.8: If $\{P_n\}$ is the sequence of Pell numbers with $P_0 = 0$ and $P_1 = 1$ and a is root of the equation $b^2 - 2b - 1 = 0$, $a > 1$. Then,

$$\left(1+\sqrt{2}\right)^{n} = P_n\left(1+\sqrt{2}\right) + P_{n-1}$$

Proof. From Example 1.1, $p(x) = \left(1+\sqrt{2}\right)^{x}$ is a Pell function. If we use Theorem 1.7 then we obtain

$$\left(1+\sqrt{2}\right)^{x+n} = P_n\left(1+\sqrt{2}\right)^{x+1} + P_{n-1}\left(1+\sqrt{2}\right)^{x}$$
$$\left(1+\sqrt{2}\right)^{n} = P_n\left(1+\sqrt{2}\right) + P_{n-1}$$

Theorem 1.9: Let $\{\vartheta_n\}_{n=0}^{\infty}$ be the full Pell sequence. Then,

$$\vartheta_{\lfloor x+n \rfloor} = P_{n-1}\vartheta_{\lfloor x \rfloor} P_n \vartheta_{\lfloor x \rfloor + 1}$$

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and

$$\vartheta_{\lfloor x+n\rfloor-1} = P_n\vartheta_{\lfloor x\rfloor} + P_{n-1}\vartheta_{\lfloor x\rfloor-1}.$$

Proof. The $p(x) \coloneqq 2\vartheta_{\lfloor x\rfloor} + \vartheta_{\lfloor x\rfloor-1}t$ map is a Pell function from Example 1.4 and we have

$$p(x+n) = 2\vartheta_{\lfloor x+n\rfloor} + \vartheta_{\lfloor x+n\rfloor-1}t$$

If we use Theorem 1.7, then

$$p(x+n) = P_n p(x+1) + P_{n-1}p(x).$$

$$P_n\left(2\vartheta_{\lfloor x+1\rfloor} + \vartheta_{\lfloor x+1\rfloor-1}t\right) + P_{n-1}\left(2\vartheta_{\lfloor x\rfloor} + \vartheta_{\lfloor x\rfloor-1}t\right)$$

$$2\left(P\right|\left|n\vartheta_{\lfloor x\rfloor+1} + P_{n-1}\vartheta_{\lfloor x\rfloor}\right) + \left(P_n\vartheta_{\lfloor x\rfloor} + P_{n-1}\vartheta_{\lfloor x\rfloor-1}\right)t.$$

So, we get

$$\vartheta_{\lfloor x+n\rfloor} = P_n\vartheta_{\lfloor x\rfloor+1} + P_{n-1}\vartheta_{\lfloor x\rfloor}$$

$$\vartheta_{\lfloor x+n\rfloor-1} = P_n\vartheta_{\lfloor x\rfloor} + P_{n-1}\vartheta_{\lfloor x\rfloor-1}$$

Thus, the result is obtained.

Corollary 1.10: If $n \geq 2$, then

$$P_{\lfloor x+n\rfloor} = P_n P_{(\lfloor x\rfloor+1)} + P_{n-1}P_{\lfloor x\rfloor} \tag{2}$$

and

$$P_{\lfloor x+n\rfloor-1} = P_n P_{\lfloor x\rfloor} + P_{n-1}P_{(\lfloor x\rfloor-1)} \tag{3}$$

Corollary 1.11: If $n \in Z^+$, we have

$$P_{n+2} = P_n P_3 + P_{n-1}P_2.$$

Proof. If we specially take $x = 2$ in (2) or $x = 3$ in (3), then the desired is found.

RESULTS AND DISCUSSION

$p-even$ and $p-odd$ Functions

Definition 2.1: Let $b(x)$ be a function with a real value and a real variable. If $b(x)h(x) \equiv 0$ and $l(x)$ is continuous then $l(x) \equiv 0$. If $b(x+1) = b(x)$ for any $x \in R$ then $b(x)$ is called a $p-even$ function.

If $b(x+1) = -b(x)$ for any $x \in R$ then $a(x)$ is called an $p-odd$ function.

Example 2.2: Let $b(x) = x - \lfloor x\rfloor$. Then $b(x)l(x) \equiv 0$ if $x \notin Z$ then it means $l(x) \equiv 0$.

By continuity of $l(x)$ it follows that

$$l(n) = \lim_{x\to n} l(x) = 0$$

for any integer n and hence h(x) ≡ 0. Since

$$b(x+1) = (x+1) - \lfloor x+1\rfloor = (x+1) - (\lfloor x\rfloor + 1) = x - \lfloor x\rfloor = b(x),$$

we see that $b(x)$ is an $p-even$ function.

Theorem 2.3: Let $l(x) = b(x)s(x)$ be a function where $b(x)$ is an $p-even$ function and $s(x)$ is a continuous function. Then $l(x)$ is a Pell function if and only if $s(x)$ is a Pell function.

Proof. Let's $l(x)$ is a Pell function. We prove that $s(x)$ is a Pell function. From (1),

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$$l(x+2) = 2l(x+1) + l(x)$$

$$l(x+2) = b(x+2)s(x+2) = 2b(x+1)s(x+1) + b(x)s(x)$$

Since $b(x)$ is an $p-even$ function, we get

$$b(x+1)s(x+2) = 2b(x)s(x+1) + b(x)s(x)$$

$$b(x)s(x+2) = b(x)\big(2s(x+1) + s(x)\big)$$

$$b(x)\big(s(x+2) - 2s(x+1) - s(x)\big) = 0$$

$$s(x+2) - 2s(x+1) - s(x) = 0$$

$$s(x+2) = 2s(x+1) + s(x)$$

Thus, $s(x)$ is a Pell function.

On the other hand, let $s(x)$ be a Pell function then we prove that $l(x)$ is a Pell function.

$$l(x+2) = b(x+2)s(x+2)$$

$$b(x+2)\big(2r(x+1) + s(x)\big)$$

$$2b(x+1)s(x+1) + b(x)s(x)$$

$$2l(x+1) + l(x)$$

Thus $l(x)$ is a Pell function.

Example 2.4: In Example 2.1,

$$p(x) = \left(1+\sqrt{2}\right)^x$$

is a Pell function.

Since $b(x) = x - \lfloor x \rfloor$ is an $p-even$ function and from Theorem 2.3, we get

$$g(x) = b(x)p(x) = (x - \lfloor x \rfloor)\left(1+\sqrt{2}\right)^x$$

is a Pell function.

Example 2.5: In Example 1.1,

$$p(x) = \left(1-\sqrt{2}\right)^x$$

is a Pell function.

Since $b(x) = x - \lfloor x \rfloor$ is an $p-even$ function and from Theorem 3.3, we have

$$g(x) = b(x)p(x) = (x - \lfloor x \rfloor)\left(1-\sqrt{2}\right)^x$$

is a Pell function.

Example 2.6: If $b(x) = cos(\pi x)$, then $b(x)g(x) \equiv 0$ implies $g(x) \equiv 0$ if $x \neq n\pi$ for any integer n. From continuity of $g(x)$, it follows that

$$g(n\pi) = \lim_{x \to n\pi} g(x)$$

For any integer n and hence $g(x) = 0$.

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Since

$b(x+1) = cos(\pi x + \pi) = -cos(\pi x) = -b(x)$, then $b(x) = cos(\pi x)$ is an $p - odd$ function.

Quotients of Pell Functions

In this chapter, we investigate the limit of a Pell function.

Theorem 3.1: There is a limit of quotient $\frac{p(x+1)}{p(x)}$.

Proof. There are four cases for the proof of the theorem as follows

i. $p(x) > 0, p(x+1) > 0,$
ii. $p(x) > 0, p(x+1) < 0,$
iii. $p(x) < 0, p(x+1) < 0,$
iv. $p(x) < 0, p(x+1) < 0.$

Let's prove ii. If we let $a \coloneqq p(x) > 0,$

$b \coloneqq p(x+1) < 0$, then

$$p(x+3) = 2p(x+2) + p(x+1)$$
$$2a - 5b = P_2 a - P_3 b$$
$$p(x+4) = 2p(x+3) + p(x+2)$$
$$5a - 12b = P_3 a - P_4 b$$

and

$$p(x+5) = 2p(x+4) + p(x+3)$$
$$12a - 29b = P_4 a - P_5 b$$

In this case, we get

$$p(x+n) = P_{n-1}a - P_n b$$

for $n \in N$.

Let $x' \in R$ be given, then there exits $x \in R$ and $n \in Z$ case such that $x' = x + n$. Thus,

$$\lim_{x' \to \infty} \frac{p(x'+1)}{p(x')} = \lim_{n \to \infty} \frac{p(x+n+1)}{p(x+n)}$$

$$= \frac{\lim_{n \to \infty} P_n a - P_{n+1} b}{P_{n-1} a - P_n b}$$

$$= \frac{\lim_{n \to \infty} P_n \left(a - \frac{P_{n+1}}{P_n} b\right)}{P_{n-1}\left(a - \frac{P_n}{P_{n-1}} b\right)}$$

$$= \lim_{n \to \infty} a \left(\frac{a - ab}{a - ab}\right) = a$$

where $\frac{\lim_{n \to \infty} P_n}{P_{n-1}} = a$ such that a is root of the equation $r^2 - 2r - 1 = 0$.

The proof of iii. is done similarly to that of ii.

Now let's examine the case of $i. p(x) > 0, p(x+1) > 0$.

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We can be written $x = \lambda + 2n$ for some $\lambda \in R$ and $n \in N$, because of any real numbers $x > 0$.

From here, we may change $\frac{p(x+1)}{p(x)}$ by $\frac{p(\lambda+2n+1)}{p(\lambda+2n)}$.

$$\frac{p(\lambda+2n+1)}{p(\lambda+2n)} = \frac{2p(\lambda+2n)+p(\lambda+2n+1)}{p(\lambda+2n)}$$

Since

$$\frac{p(\lambda+2n+1)}{p(\lambda+2n)} < 1,$$

$$\frac{p(\lambda+2n+1)}{p(\lambda+2n)} = 2 + \frac{p(\lambda+2n+1)}{p(\lambda+2n)} < 3$$

We claim that $\left\{\frac{p(\lambda+2n+1)}{p(\lambda+2n)}\right\}_{n=1}^{\infty}$ is monotonically increasing.

$$\frac{p(\lambda+2n+3)}{p(\lambda+2n+2)} - \frac{p(\lambda+2n+1)}{p(\lambda+2n)}$$

$$= \frac{p(\lambda+2n+3)p(\lambda+2n)-p(\lambda+2n+2)p(\lambda+2n+1)}{p(\lambda+2n+2)p(\lambda+2n)}$$

We show that the numerator of the equation is positive.

$$\begin{aligned}
&p(\lambda+2n+3)p(\lambda+2n) - p(\lambda+2n+2)p(\lambda+2n+1)\\
&= [2p(\lambda+2n+2) + p(\lambda+2n+1)]p(\lambda+2n) - p(\lambda+2n+2)p(\lambda+2n+1)\\
&= p(\lambda+2n+2)[2p(\lambda+2n) - p(\lambda+2n+1)] + p(\lambda+2n)p(\lambda+2n+1)\\
&> p(\lambda+2n)[2p(\tau+2n) - p(\lambda+2n+1)] + p(\lambda+2n)p(\lambda+2n+1)\\
&= 2\big(p(\lambda+2n)\big)^2\\
&\geq 0.
\end{aligned}$$

This last part shows that the sequence is monotonically increasing. From the monotone convergence theorem, there is the following.

$$\lim_{n\to\infty} \frac{p(\lambda+2n+1)}{p(\lambda+2n)} = \lim_{x\to\infty} \frac{p(x+1)}{p(x)}$$

CONCLUSION

In this study, we introduced Pell functions on R. Using the concepts of $p-even$ and $p-odd$ functions, we called them to Pell functions. We examined the relationships between Pell functions and Pell numbers. Additionally, we showed that $\lim_{x\to\infty} \frac{p(x+1)}{p(x)} = 1+\sqrt{2}$.

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