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A New Generalization of Fibonacci Polynomials

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Abstract

The Fibonacci sequences, which have an increasing place in scientific studies and are the most well-known among the number sequences, first appeared in Leonardo Fibonacci's book 'Liber Abaci', which contains many basic problems, in a rabbit problem. Fibonacci sequences were obtained by writing down the monthly total number of new rabbits born by mating only once per month, initially one female and one male rabbit. This sequence, which is obtained by adding the two numbers before it; It continues as 1,1,2,3,5,8,13,21,34,55,89,144,23, 377, . . . In this paper, we define a new generalization of Fibonacci Polynomials. The relationship between known Fibonacci Polynomials and k −Fibonacci Polynomials are given. Some properties of these polynomials have been obtained. Also, matrix presentations of the new family of polynomials are found. Finally, we give Cassini's identity for the polynomials.

Keywords: k-Fibonacci Numbers, k-Fibonacci Polynomials, Cassini Identity, a Matrix Generator.

INTRODUCTION

The Fibonacci numbers, which have an important place today, appear not only in mathematics, but also in every aspect of our daily life. They are numbers, that are seen in nature, in the stock market, in the human body and in all areas of life and is associated with aesthetics. These numbers, which are called the 'Golden Ratio' when proportioned to each other and are thought to give the most aesthetic appearance, are the focus of attention in all areas of the scientific world [1–9].

Some generalizations of these numbers are Lucas number sequences, Jacobsthal number sequences, Pell number sequences. Some examples of these studies are given below.

Thomas Koshy gave important relations of Fibonacci and other number sequences in his book 'Fibonacci and Lucas Numbers with Applications [6].

Thomas Koshy, in his book 'Pell and Pell-Lucas Numbers with Applications', gave important relations about the Pell number sequence. In addition, he gave the connections of this number sequence with the Lucas sequence [7].

Özkan et.all. have obtained many transform, polynomials and identity for Fibonacci and Lucas sequences [10–19].

Mikkawy and Sogabe described a new family of the k-Fibonacci number sequence [8].

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MATERIAL AND METHODS

Definition 2.1. There exist unique numbers m and r such that $n = mk + r(0 \leq r < k)$ and $n, k \in \mathbb{N}$. The k-Fibonacci numbers $F_n^{(k)}$ are defined by

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$$F_n^{(k)} = \frac{1}{(\sqrt{5})^k}(\alpha^{m+2} - \beta^{m+2})^r(\alpha^{m+1} - \beta^{m+1})^{k-r}, n = mk + r\ (0 \le \mathrm{r} < k)$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ ve $\beta = \frac{1-\sqrt{5}}{2}$ [8]

There is a relationship among known Fibonacci numbers and k −Fibonacci numbers [8]:

$$F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$$

We give some values for k −Fibonacci numbers in Table 1.

Table 1. Some values for $F_n^{(k)}$.

$F_n^{(k)}$	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
k=1	1	1	1	1	1	1	1	1	1	1
k=2	1	1	2	4	6	9	15	25	40	64
k=3	1	1	1	2	4	8	12	18	27	45
k=4	1	1	1	1	2	4	8	16	24	36

MAIN RESULTS

Definition 3.1. There exist unique numbers m and r such that $n = mk + r(0 \le r < k)$and $n, k \in \mathbb{N}$. The *k*-Fibonacci numbers $F_n^{(k)}(x)$are defined by

$$F_n^{(k)}(x) = \left(\frac{\alpha^{m+1}(x) - \beta^{m+1}(x)}{\alpha(x) - \beta(x)}\right)^{k-r}\left(\frac{\alpha^{m+2}(x) - \beta^{m+2}(x)}{\alpha(x) - \beta(x)}\right)^r$$

where $\alpha = \frac{\mathrm{x}+\sqrt{\mathrm{x}^2+4}}{2}$ ve $\beta = \frac{\mathrm{x}-\sqrt{\mathrm{x}^2+4}}{2}$.

Also, we can indicate a matrix generator for the k −Fibonacci Polynomials:

$$F_{n+1}^k(x)f_n(x) = \begin{bmatrix} F_{kn+k+1}^{(k)}(x) & F_{kn+k}^{(k)}(x) \\ F_{kn+k}^{(k)}(x) & F_{kn+k-1}^{(k)}(x) \end{bmatrix}$$

where $f_n(x) = \begin{bmatrix} F_{n+1}^{(k)}(x) & F_n^{(k)}(x) \\ F_n^{(k)}(x) & F_{n-1}^{(k)}(x) \end{bmatrix}$.

Relationship between known Fibonacci Polynomials and k −Fibonacci polynomials:

$$F_n^{(k)}(x) = \left(F_m(x)\right)^{k-r}\left(F_{m+1}(x)\right)^r \tag{1}$$

Let's give some values for k −Fibonacci polynomials in Table 2.

Table 2. Some values for $F_n^{(k)}(x)$.

$F_n^{(k)}(x)$	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
k=1	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
k=2	1	x	x^2	$x^3 + x$	$x^4 + 2x^2 + 1$	$x^5 + x^3 + 4x$	$x^6 + 4x^4 + 4x^2$	$x^7 + 5x^5 + 7x^3 + 2x$	$x^8 + 6x^6 + 11x^4 + 6x^2 + 1$	$x^9 + 7x^7 + 16x^5 + 13x^3 + 3x$	$x^{10} + 16x^9 + 8x^8 + 6x^6 + 24x^4 + 9x^2$
k=3	1	x	x^2	x^3	$x^4 + x^2$	$x^5 + 2x^3 + x$	$x^6 + 3x^4 + 3x^2 + 1$	$x^7 + 4x^5 + 5x^3 + 2x$	$x^8 + 5x^6 + 8x^4 + 4x^2$	$x^9 + 6x^7 + 12x^5 + 8x^3$	$x^{10} + 7x^8 + 17x^6 + 16x^4 + 4x^2$
k=4	1	x	x^2	x^3	x^4	$x^5 + x^3$	$x^6 + 2x^4 + x^2$	$x^7 + 3x^5 + 3x^3 + x$	$x^8 + 4x^6 + 6x^4 + 4x^2 + 1$	$x^9 + 5x^7 + 9x^5 + 7x^3 + 2x$	$x^{10} + 6x^8 + 13x^6 + 12x^4 + 4x^2$

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Theorem 3.2. (Cassini): For$n, k \geq 2$, we have

$$F_{nk+t}^{(k)}(x)F_{nk+t-2}^{(k)}(x) - \left(F_{nk+t-1}^{(k)}(x)\right)^2 = \begin{cases} F_n^{2k-2}(x)(-1)^n, & t = 1 \\ 0, & t \neq 1 \end{cases}$$

Proof. From (1), we get

$$\begin{aligned} F_{nk+t}^{(k)}(x)F_{nk+t-2}^{(k)}(x) - \left(F_{nk+t-1}^{(k)}(x)\right)^2 \\ = \left(F_n^{k-1}(x)F_{n+t}(x)\right)\left(F_n^{k-1}(x)F_{n+t-2}(x)\right) - (F_n^{k-1}(x)F_{n+t-1}(x))^2 \\ = F_n^{2k-2}(x)(F_{n+t}(x)F_{n+t-2}(x) - (F_{n+t-1}(x))^2) \end{aligned}$$

For $t = 1$;

$$\begin{aligned} = F_n^{2k-2}(x)(F_{n+1}(x)F_{n-1}(x) - (F_n(x))^2) \\ = F_n^{2k-2}(x)(-1)^n \end{aligned}$$

For $\neq 1, t = m, (m \in \mathbb{N})$;

$$\begin{aligned} = F_n^{2k-2}(x)(F_{n+m}(x)F_{n+m-2}(x) - (F_{n+m-1}(x))^2) \\ = F_n^{2k-2}(x)\left(F_{2n+2m-2}^{(k)}(x) - F_{2n+2m-2}^{(k)}(x)\right) \end{aligned}$$

$= 0.$

Theorem 3.3. For $k, m \in \{1,2, \ldots\}$, we have

$$(F_{m+1}(x))^k - (F_m(x))^k = F_{(m+1)k}^{(k)}(x) - F_{mk}^{(k)}(x)$$

Proof. By using (1), we find

$$\begin{aligned} F_{(m+1)k}^{(k)}(x) - F_{mk}^{(k)}(x) = (F_m(x))^{k-k}(F_{m+1}(x))^k - (F_m(x))^{k-0}(F_{m+1}(x))^0 \\ = (F_{m+1}(x))^k - (F_m(x))^k \end{aligned}$$

For $k = 2, 3, 4, \ldots, n$, the properties between k–Fibonacci polynomials and known Fibonacci polynomials are given below.

$$\begin{gathered} F_{2n}^{(2)}(x) = F_n^2(x), \\ F_{2n+1}^{(2)}(x) = F_n(x)F_{n+1}(x), \\ F_{2n+1}^{(2)}(x) = xF_{2n}^{(2)}(x) + F_{2n-1}^{(2)}(x), \\ F_{3n}^{(3)}(x) = F_n^3(x), \\ F_{3n+1}^{(3)}(x) = F_n^2(x)F_{n+1}(x), \\ F_{3n+2}^{(2)}(x) = F_n^2(x)F_{n+1}(x), \\ F_{3n+1}^{(3)}(x) = xF_{3n}^{(3)}(x) + F_{3n-1}^{(3)}(x), \\ F_{4n+1}^{(4)}(x) = xF_{4n}^{(4)}(x) + F_{4n-1}^{(4)}(x) \end{gathered}$$

Theorem 3.4.

i. $F_{4n}^{(4)}(x) = F_n^4(x),$

ii. $F_{4n+1}^{(4)}(x) = F_n^3(x)F_{n+1}(x),$

iii. $F_{4n+2}^{(4)}(x) = F_n^2(x)F_{n+1}^2(x),$

iv. $F_{4n+3}^{(4)}(x) = F_n(x)F_{n+1}^3(x),$

Proof.

i. Let $k = 4, r = 0$ in (1), we see that

$$\begin{gathered} Fn^{(4)}(x) = (F_m(x))^{4-0}(F_{m+1}(x))^0 \\ F_{4m}^{(4)}(x) = F_m^4(x) \end{gathered}$$

Let $k = 4, r = 1$ in (1), we see that

$$\begin{gathered} F_n^{(4)}(x) = \left(F_m(x)\right)^{4-1}\left(F_{m+1}(x)\right)^1 \\ F_{4m+1}^{(4)}(x) = \left(F_m(x)\right)^3\left(F_{m+1}(x)\right)^1 \end{gathered}$$

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ii. Let $k = 4, r = 2$ in (1), we see that $n = mk + r = 4m + 2$

$$F_n^{(4)}(x) = \left(F_m(x)\right)^{4-2}\left(F_{m+1}(x)\right)^2$$
$$F_{4m+2}^{(4)}(x) = \left(F_m(x)\right)^2\left(F_{m+1}(x)\right)^2$$

iii. Let $k = 4, r = 3$ in (1), we see that $n = mk + r = 4m + 3$

$$F_n^{(4)}(x) = \left(F_m(x)\right)^{4-3}\left(F_{m+1}(x)\right)^3$$
$$F_{4m+3}^{(4)}(x) = \left(F_m(x)\right)^1\left(F_{m+1}(x)\right)^3$$

Theorem 3.5.
For any integer$n$,

$$F_{kn+1}^{(k)}(x) = xF_{kn}^{(k)}(x) + F_{kn-1}^{(k)}(x)$$

Proof. From (1), we have

$$xF_{kn}^{(k)}(x) + F_{kn-1}^{(k)}(x) = xF_n^k(x) + \left(F_{n-1}(x)F_n^{k-1}(x)\right)$$
$$= F_n^{k-1}(x)(xF_n(x) + F_{n-1}(x))$$
$$= F_{kn+1}^{(k)}(x)$$

Theorem 3.6. It is satisfied the following equation.

$$F_{2(n-1)}^{(2)}(x) - F_n(x)F_{n-2}(x) = (-1)^{n+1}$$

Proof.

$$F_{2(n-1)}^{(2)}(x) - F_n(x)F_{n-2}(x) = F_{n-1}^2(x) - F_n(x)F_{n-2}(x) = -(F_n(x)F_{n-2}(x) - F_{n-1}^2(x))$$
$$= (-1)^{n+1}$$

Theorem 3.7. For any integer n, we have

$$(x^2+4)F_{2n}^{(2)}(x) + 4(-1)^n = \begin{cases}(\alpha^n(x) + \beta^n(x))^2, n\ çift\ ise\\ (\alpha^n(x) - \beta^n(x))^2, n\ tek\ ise\end{cases}$$

Proof. From $F_n(x) = \frac{\alpha^n(x)+\beta^n(x)}{\alpha(x)-\beta(x)}$, we have

$$F_{2n}^{(2)}(x) = F_n^2(x) = \frac{1}{(\alpha(x)-\beta(x))^2}\left\{\alpha^{2n}(x) - 2\left(\alpha(x)\beta(x)\right)^n + \beta^{2n}(x)\right\}$$
$$= \frac{\alpha^{2n}(x) - 2(-1)^n + \beta^{2n}(x) + 4(-1)^n - 4(-1)^n}{\left(\sqrt{x^2+4}\right)^2}$$
$$(x^2+4)F_{2n}^{(2)}(x) + 4(-1)^n = \alpha^{2n}(x) - 2(-1)^n + \beta^{2n}(x) + 4(-1)^n$$

If n is even;

$$(x^2+4)F_{2n}^{(2)}(x) + 4 = (\alpha^n(x) + \beta^n(x))^2,$$

If n is odd;
$(x^2+4)F_{2n}^{(2)}(x) - 4 = (\alpha^n(x) - \beta^n(x))^2$.

CONCLUSION

In this paper, we defined a new generalization of $k-$ Fibonacci polynomials. We gave some relations between these polynomials and the known Fibonacci polynomials. In addition, we found the matrix generator of these polynomials. And we demonstrated Cassini's Identities for these polynomials.

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