Chiang Mai J. Sci. 2017; 44(4) : 1744-1750
http://epg.science.cmu.ac.th/ejournal/
Contributed Paper

On Relationship Among a New Family of *k*-Fibonacci, k-Lucas Numbers, Fibonacci and Lucas Numbers

Engin Özkan*, İpek Altun and Ali Aykut Göçer
Department of Mathematics, Faculty of Sciences and Arts, Erzincan University, Erzincan 24000, Turkey.
*Author for correspondence; e-mail: eozkan@erzincan.edu.tr

Received 4 May 2015
Accepted 18 October 2016

ABSTRACT

We prove some theorems concerning a family of Fibonacci numbers defined in [11]. We give some relationship between the family of Fibonacci and Lucas numbers. Then we define a new family of k-Lucas numbers and prove some properties of it related to the Lucas numbers and the family of Lucas numbers.

Keywords: recurrance relation, Fibonacci numbers, generalized Fibonacci numbers, period, lucas numbers

1. INTRODUCTION

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. They appear in many branches of mathematics. These include group theory, calculus, applied mathematics, linear algebra, etc. Also, these numbers have many important applications to diverse fields such as computer science, physics, biology and statistics. We can see applications of Fibonacci sequences in group theory in [2,3,7,13] and also see some generalized Fibonacci and Lucas sequences in [4,6,8,12,16,17,18]. Özkan has proved two theorems concerning the period of the 3-step Fibonacci sequences and given some conjectures concerning the period of the 3-step Fibonacci sequence in [14]. In [5], Gyoung-Sik et al. introduced the truncated Fibonacci sequences and the truncated Lucas sequences. In [14], Ozkan proved some theorems on truncated Fibonacci sequences and stated some conjectures on their periods.

In the present paper, we give the detail proof of the Binet formula for $\{F_n\}$ and its extension for $\{L_n\}$ in [10]. We shall focus on a new family of these numbers defined in [11]. We will prove some theorems concerning the family and give some relationship between the family and Fibonacci number. Then we will give a new family of *k*-Lucas numbers and prove some properties of it related to the Lucas numbers the family of Lucas numbers.

2. MATERIALS AND METHODS

It is well known that the Fibonacci numbers: F_n for $n = 0,1..$ are defined by the Binet's formula as follows:

$$F_n: \ = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}), \quad n = 0, 1, \ldots, \tag{1}$$

Let α and β be the roots of the quadratic equation

$$x^2 - x - 1 = 0$$

where $\alpha = \frac{(1+\sqrt{5})}{2}$ and $\beta = \frac{(1-\sqrt{5})}{2}$.

The two roots play an important role in studying the Fibonacci and Lucas Sequences.

The number alpha is called the golden ratio. They satisfy the following equations

$$\alpha + \beta = 1,\ \alpha - \beta = \sqrt{5}, \alpha\beta = -1, \alpha^2 = \alpha + 1$$
$$\text{and } \beta^2 = \beta + 1.$$

This explicit formula for F_n is called Binet's formula, after the French mathematician Jacques-Phillipe-Marie Binet (1786-1856), who discovered it in 1843. In fact, it was first discovered in 1718 by the French mathematician Abraham De Moivre (1667-1754) using generating functions, and also arrived at independently in 1844 by the French engineer and mathematician Gabriel Lame (1795-1870). The numbers F_n satisfy the second order linear recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad n = 1,2,\ldots,$$

with the initial conditions $F_{-1} = 0$, $F_0 = 1$. It is also widely know that the F_n is related by the determinant of the special tridiagonal matrix of the form

$$T_n = \begin{pmatrix} 1 & 1 & 0 & & & \\ -1 & 1 & 1 & & & \\ & & \ddots & & & \\ & \ddots & & -1 & 1 & 1 \\ & & & 0 & -1 & 1 \end{pmatrix} \epsilon R^{nXn}$$

We will prove the Binet form for the Fibonacci numbers by using mathematics induction on n

When $n = 1$, it is clear that

$$F_1 = 1 = \frac{\alpha-\beta}{\alpha-\beta}.$$

Hence, $F_n = \frac{\alpha^n - \beta^n}{\alpha-\beta}$ is true for $n = 1$. Let k be an arbitrary positive integer. Suppose, $F_n = \frac{\alpha^n - \beta^n}{\alpha-\beta}$ for any positive integer form 1 to k. Then we must show that equation also holds when $n = k + 1$.

Thus, by the inductive hypothesis, we have

$$F_k = \frac{\alpha^k - \beta^k}{\alpha-\beta} \text{ and } F_{k-1} = \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha-\beta}$$

Hence it follows that,

$$F_{k+1} = F_k + F_{k-1}$$
$$= \frac{\alpha^k - \beta^k}{\alpha-\beta} + \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha-\beta}$$
$$= \frac{\alpha^{k-1}(a+1) - \beta^{k-1}(\beta+1)}{\alpha-\beta}$$
$$= \frac{\alpha^{k-1}\alpha^2 - \beta^{k-1}\beta^2}{\alpha-\beta} \qquad (\alpha^2 = \alpha + 1, \beta^2 = \beta + 1)$$
$$= \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha-\beta}$$

Thus, by the principle of mathematical induction, $F_n = \frac{\alpha^n - \beta^n}{\alpha-\beta}$ is true for all positive integer n and the proof is complete.

The Lucas numbers are given by $L_n = L_{n-1} + L_{n-2}$, $n = 2,3,\ldots$ with initial conditions $L_0 = 2$ and $L_1 = 1$. The first few Lucas numbers are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, ... The Binet's formula for the Lucas numbers L_n is $L_n = \alpha^n + \beta^n$, $n = 0,1,\ldots$

Proof. We have prove by using mathematical induction on n.

When $n = 1$, then $L_1 = \alpha + \beta$.

Hence $L_n = \alpha^n + \beta^n$ is true for $n = 1$.

We suppose that $L_n = \alpha^n + \beta^n$ is true for any integer. Than, we will show the above equation also holds when $n = k + 1$.

Thus, we have

$$L_{k+1} = L_k + L_{k-1}$$
$$= \alpha^k + \beta^k + \alpha^{k-1} + \beta^{k-1}$$
$$= \alpha^{k-1}(a+1) + \beta^{k-1}(\beta+1)$$
$$= \alpha^{k-1}\alpha^2 + \beta^{k-1}\beta^2$$
$$= \alpha^{k+1} + \beta^{k+1},$$

Hence, by the principle of mathematical induction, $L_n = \alpha^n + \beta^n$ is true for all positive integers n.

Therefore, we see that the Lucas and Fibonacci numbers are related by

$$L_n = F_n + F_{n-2} = \frac{F_{2n-1}}{F_{n-1}}$$

Now, we give the definition of a generalized k-Fibonacci number in [11] as follows.

Definition 2.1 Let n and k ($\neq 0$) be natural numbers, then there exist unique numbers m

and n such that $n = mk + r. (0 \le r < k)$. Using these parameters, we define the generalized kFibonacci numbers $F_n^{(k)}$ by

$$F_n^{(k)} := \frac{1}{(\sqrt{5})^k}(\alpha^{m+2}\text{-}\beta^{m+2})^r(\alpha^{m+1}\text{-}\beta^{m+1})^{k\text{-}r},$$

$$n = mk + r.$$

The first new numbers of the new family for $k = 2, 3$ are as follows:

$$\left\{F_n^{(2)}\right\}_{n=0}^{10} = \{1, 1,1,2,4,6,9,15,25,40,64\},$$

$$\left\{F_n^{(3)}\right\}_{n=0}^{10} = \{1,1,2,4,8,12,18,27,45\}.$$

From (1) and Definition 2.1, the generalized k-Fibonacci and Fibonacci numbers are related by

$$F_n^{(k)} = (F_m)^{k\text{-}r}(F_{m+1})^r, \quad n = mk + r.$$

Considering the case $k = 1$ in (1), we see that $m = n$ and $r = 0$. Therefore, $F_n^{(1)}$ is the Fibonacci number F_n because we have

$$F_n^{(1)} = (F_n)^{1-0}(F_{n+1})^0$$

$$F_n^{(1)} = (F_n)^1(F_{n+1})^0$$

$$F_n^{(1)} = (F_n)^1 = F_n$$

That is,

$$\left\{F_n^{(1)}\right\}_{n=0}^{10} = \{1, 1,2,3,5,8,13,21,34,55\}.$$

3. RESULTS AND DISCUSSION

Definition 3.1 Let n and $k\ (\neq 0)$ be natural numbers, then there exist unique numbers m and n such that $n = mk + r\ (0 \le r < k)$. Using these parameters, we define the generalized k Lucas numbers $L_n^{(k)}$ by

$$L_n^{(k)} := (\alpha^{m+1} + \beta^{m+1})^r(\alpha^m + \beta^m)^{k\text{-}r},$$

$$n = m$$

The first new numbers of the new family for $k = 2, 3$ are as follows:

$$\left\{L_n^{(2)}\right\}_{n=0}^{10} = \{4,2,1,3,9,12,16,28,49,77,121\},$$

$$\left\{L_n^{(3)}\right\}_{n=0}^{10} = \{8,4,2,1,3,9,27,36,48,64,112\}.$$

From Definition 3.1, the generalized k-Lucas and Lucas numbers are related by

$$L_n^{(k)} = (L_m)^{k\text{-}r}(L_{m+1})^r, \quad n = mk + r. \qquad (2)$$

Considering the case $k = 1$ in (2), we see that $m = n$ and $r = 0$. Therefore, $L_n^{(1)}$ is the Lucas number L_n.

$$L_n^{(1)} = (L_n)^{1-0}(L_{n+1})^0$$

$$L_n^{(1)} = (L_n)^1(L_{n+1})^0$$

$$L_n^{(1)} = (L_n)^1 = L_n$$

That is ,

$$\left\{L_n^{(1)}\right\}_{n=0}^{10} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76\}.$$

We give two simple properties of the family of Lucas as follows,

i. $L_{2n+1}^{(2)} = (L_n)(L_{n+1})$

Proof. Let $k = 2$, $r = 1$ in (2) , we see that $n = mk + r = 2m + 1$

$$L_n^{(2)} = (L_m)^{2-1}(L_{m+1})^1$$

$$L_n^{(2)} = (L_m)^1(L_{m+1})^1$$

$$L_{2m+1}^{(2)} = (L_m)(L_{m+1}) \text{ . } \square$$

ii. $L_{2n}^{(2)} = (L_n)^2$

Proof. Let $k = 2$, $r = 0$ in (2), we see that $n = mk + r = 2m$

$$L_n^{(2)} = (L_m)^{2-0}(L_{m+1})^0$$

$$L_n^{(2)} = (L_m)^2(L_{m+1})^0$$

$$L_{2m}^{(2)} = (L_m)^2. \ \square$$

Theorem 3.2 Let $n \in \{1,2, \ldots\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$(F_{2n}^{(2)})^2 - F_{2n-1}^{(2)}.F_{2n+1}^{(2)} = (-1)^{n+1}.F_{2n}^{(2)} \text{ . }$$

Proof. We have the following equalities from [11].

$$F_{2n}^{(2)} = (F_n)^2$$

$$F_{2n-1}^{(2)} = (F_n)(F_{n-1})$$
$$F_{2n+1}^{(2)} = (F_n)(F_{n+1}).$$

From these equalities, we have

$$(F_{2n}^{(2)})^2 - F_{2n-1}^{(2)}.F_{2n+1}^{(2)} = (F_n)^4 - (F_n)(F_{n-1})(F_n)(F_{n+1})$$
$$= (F_n)^4 - (F_n)^2(F_{n-1})(F_{n+1})$$
$$= (F_n)^2\{(F_n)^2 - (F_{n-1})(F_{n+1})\}$$
$$= (F_n)^2.(-1)^{n+1}$$
$$= (-1)^{n+1}F_{2n}^{(2)}. \square$$

Theorem 3.3 Let $n \in \{1,2,...\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$F_{4n-2}^{(2)} + F_{4n-1}^{(2)} - F_{4n}^{(2)} = +1 .$$

Proof. We have the following equalities from [11].

$$F_{4n-2}^{(2)} = (F_{2n-1})^2$$
$$F_{4n-1}^{(2)} = (F_{2n})(F_{2n-1})$$
$$F_{4n}^{(2)} = (F_{2n})^2.$$

From these equalities, we have

$$F_{4n-2}^{(2)} + F_{4n-1}^{(2)} - F_{4n}^{(2)} = (F_{2n-1})^2 + (F_{2n})(F_{2n-1}) - (F_{2n})^2$$
$$= F_{2n-1}\{(F_{2n}) + (F_{2n-1})\} - (F_{2n})^2$$
$$= F_{2n-1}\{F_{2n+1}\} - (F_{2n})^2$$
$$= [(F_{2n-1})(F_{2n+1}) - (F_{2n})^2]$$
$$= [(-1)^{2n}]$$
$$= +1. \square$$

Theorem 3.4 Let $n \in \{1,2,3...\}$, For fixed n, the generalized 2-Fibonacci numbers satisfy

$$F_{2n+2}^{(2)} = F_{2n-2}^{(2)} + 2F_{2n-1}^{(2)} + F_{2n}^{(2)} .$$

Proof.
$$F_{2n+2}^{(2)} = (F_{n+1})^2$$
$$= (F_{n-1} + F_n)^2$$
$$= (F_{n-1})^2 + 2F_nF_{n-1} + (F_n)^2$$
$$= F_{2n-2}^{(2)} + 2F_{2n-1}^{(2)} + F_{2n}^{(2)} . \square$$

Theorem 3.5 $n \in \{1,2,...\}$. For fixed n, we have a relation among the generalized 2-Fibonacci numbers, Fibonacci and Lucas numbers as follows,

$$L_n.F_{n+1} - L_n.F_{n-1} = F_{2n+2}^{(2)} - F_{2n-2}^{(2)} .$$

Proof.

$$L_n.F_{n+1} - L_n.F_{n-1}$$
$$= [(F_{n+1} + F_{n-1})F_{n+1}] - [(F_{n+1} + F_{n-1})F_{n-1}]$$
$$= F_{n-1}.F_{n+1} + (F_{n+1})^2 - (F_{n-1})^2 - F_{n-1}.F_{n+1}$$
$$= (F_{n+1})^2 - (F_{n-1})^2$$
$$= F_{2n+2}^{(2)} - F_{2n-2}^{(2)}. \square$$

Theorem 3.6 Let $n \in \{1,2,...\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$(L_{n+1})^2 - (L_n)^2 = 5F_{2n+1}^{(2)} + 5F_{2n-1}^{(2)} .$$

Proof. We know $(L_{n+1})^2 - (L_n)^2 = 5(F_{2n})$ from [9]. By using it, we have

$$(L_{n+1})^2 - (L_n)^2 = 5(F_{2n})$$
$$= 5(F_n.F_{n+1} + F_n.F_{n-1})$$
$$= 5(F_{2n+1}^{(2)} + F_{2n-1}^{(2)})$$
$$= 5F_{2n+1}^{(2)} + 5F_{2n-1}^{(2)}. \square$$

Theorem 3.7 Let $n \in \{1,2,...\}$. For fixed n, we have the relation

$$(L_n)^2 = 3F_{2n-2}^{(2)} + 2F_{2n-1}^{(2)} + F_{2n+2}^{(2)} .$$

between the generalized 2-Fibonacci numbers and Lucas numbers.

Proof. $(L_n)^2 = (F_{n-1} + F_{n+1})^2$
$$= (F_{n-1})^2 + 2F_{n-1}.F_{n+1} + (F_{n+1})^2$$
$$= (F_{n-1})^2 + 2F_{n-1}(F_{n-1} + F_n) + (F_{n+1})^2$$
$$= (F_{n-1})^2 + 2(F_{n-1})^2 + 2F_n.F_{n-1} + (F_{n+1})^2$$
$$= 3(F_{n-1})^2 + 2F_n.F_{n-1} + (F_{n+1})^2$$
$$= 3F_{2n-2}^{(2)} + 2F_{2n-1}^{(2)} + F_{2n+2}^{(2)}. \square$$

Theorem 3.8 Let $n \in \{1,2,\ldots\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy the equation

$$F_{2n+2}^{(2)} - F_{2n-2}^{(2)} = F_{2n-1}^{(2)} + F_{2n+1}^{(2)}.$$

Proof. $F_{2n+2}^{(2)} - F_{2n-2}^{(2)} = (F_{n+1})^2 - (F_{n-1})^2$.

$= (F_{n+1} - F_{n-1}).(F_{n+1} + F_{n-1})$

$= (F_n).(F_{n+1} + F_{n-1})$

$= (F_n.F_{n+1} + F_n.F_{n-1})$

$= F_{2n-1}^{(2)} + F_{2n+1}^{(2)}$. □

Theorem 3.9 Let $n \in \{1,2,\ldots\}$. For fixed n, we have a relation among the generalized 2-Fibonacci numbers, Fibonacci and Lucas numbers as follows,

$$(L_n)^2 + 2F_{2n} + F_{2n}^{(2)} = 4F_{2n+2}^{(2)}.$$

Proof. $(L_n)^2 + 2F_{2n} + F_{2n}^{(2)} = (L_n)^2 + 2F_n.L_n + (F_n)^2$

$= (L_n + F_n)^2$

$= (F_{n+1} + F_{n-1} + F_n)^2$

$= (F_{n+1} + F_{n+1})^2$

$= (2F_{n+1})^2$

$= 4F_{2n+2}^{(2)}$. □

Theorem 3.10 Let $n \in \{1,2,\ldots\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy the equation

$$F_{2n+1}^{(2)} - F_{2n-3}^{(2)} = F_{2n}^{(2)} + F_{2n-2}^{(2)}.$$

Proof. $F_{2n+1}^{(2)} - F_{2n-3}^{(2)} = F_n.F_{n+1} - F_{n-1}.F_{n-2}$

$= [F_{n-1} + F_{n-2}].F_{n+1} - F_{n-1}.F_{n-2}$

$= F_{n-1}.F_{n+1} + F_{n-2}.F_{n+1} - F_{n-1}.F_{n-2}$

$= F_{n-1}.F_{n+1} + (F_{n+1} - F_{n-1}).F_{n-2}$

$= F_{n+1}.F_{n-1} + F_n.F_{n-2}$

$= F_{n+n-1}$

$= F_{n-1+n}$

$= F_n.F_n + F_{n-1}.F_{n-1}$

$= (F_n)^2 + (F_{n-1})^2$

$= F_{2n}^{(2)} + F_{2n-2}^{(2)}$. □

Theorem 3.11 Let $k \in \{2,3,\ldots\}$. For fixed k, the generalized k-Lucas numbers satisfy the equation

$$L_{nk+k-1}^{(k)} + L_{nk+k}^{(k)} = L_{nk+k+1}^{(k)}.$$

Proof. From (2), we obtain the proof of the theorem as follows

$L_{nk+k-1}^{(k)} + L_{nk+k}^{(k)} = [(L_n)^1.(L_{n+1})^{k-1}]$

$+[(L_{n+1})^k.(L_{n+2})^0]$

$= [(L_n)^1.(L_{n+1})^{k-1}] + (L_{n+1})^k$

$= (L_{n+1})^{k-1}.[(L_n) + (L_{n+1})]$

$= (L_{n+1})^{k-1}.L_{n+2}$:

$= L_{nk+k+1}^{(k)}$ □.

Theorem 3.12 Let $n \in \{1,2,\ldots\}$. For fixed n, the generalized 2-Fibonacci numbers and Lucas numbers satisfy the equation

$$2L_{2n-1}^{(2)} + L_{2n}^{(2)} = 5(F_{2n+1}^{(2)} + F_{2n-1}^{(2)})$$

Proof. $2L_{2n-1}^{(2)} + L_{2n}^{(2)} = 2(L_n)(L_{n-1}) + (L_n)^2$

$= (L_n)(2L_{n-1} + (L_n)$

$= L_n(L_{n-1} + L_{n-1} + L_n)$

$= L_n(L_{n-1} + L_{n+1})$

$= L_n\,5F_n$

$= 5F_n\,L_n$

$= 5F_{2n}$

$= 5(F_n.F_{n+1} + F_{n-1}.F_n)$

$= 5(F_{2n+1}^{(2)} + F_{2n-1}^{(2)})$. □

Theorem 3.13 Let $n \in \{1,2,\ldots\}$. For fixed n, the generalized 2-Lucas numbers and Lucas numbers satisfy

$$L_{2n+2}^{(2)} - L_{2n-2}^{(2)} = L_{2n+1}.$$

Proof. $L_{2n+2}^{(2)} - L_{2n-2}^{(2)} = (L_{n+1})^2 - (L_{n-1})^2$

$= (L_{n+1} - L_{n-1})(L_{n+1} + L_{n-1})$

$= L_n(L_{n+1} + L_{n-1})$

$= L_n L_{n+1} + L_n L_{n-1}$

$= L_{2n+1}$. □

Theorem 3.14 Let $n \in \{1,2,...\}$. For fixed n, the generalized 2-Fibonacci numbers and 2-Lucas numbers satisfy the equation

$$F_{4n+1}^{(2)} = L_{2n+1}^{(2)} F_{2n}^{(2)}.$$

Proof. $F_{4n+1}^{(2)} = F_{2n}.F_{2n+1}$

$= F_{2n}(F_n F_{n+1} + F_{n-1}.F_n)$

$= F_{2n} F_n (F_{n+1} + F_{n-1})$

$= F_{2n} F_n L_{n+1}$

$= F_n L_n F_n L_{n+1}$

$= (F_n)^2 L_n L_{n+1}$

$= L_{2n+1}^{(2)} F_{2n}^{(2)}$. □

Theorem 3.15 Let $n \in \{1,2,...\}$. For fixed n, we have a relation among the generalized 2-Fibonacci numbers, Fibonacci and 2-Lucas numbers as follows

$$L_{2n}^{(2)} - 2F_n.F_{n-2} = F_{2n}^{(2)} + F_{2n-4}^{(2)}.$$

Proof. $L_{2n}^{(2)} - 2F_n.F_{n-2} = (L_n)^2 - 2F_n.F_{n-2}$

$= (F_n + F_{n-2})^2 - 2F_n.F_{n-2}$

$= (F_n)^2 + 2F_n.F_{n-2} + (F_{n-2})^2 - 2F_n.F_{n-2}$

$= (F_n)^2 + (F_{n-2})^2$

$= F_{2n}^{(2)} + F_{2n-4}^{(2)}$. □

4. Conclusion

In this study, it was proved some theorems concerning a new family of Fibonacci numbers defined in [11] and given some relationship between the family of Fibonacci and Lucas numbers. Then it was also given a new family of k-Lucas numbers and proved some properties of it related to the Lucas numbers and the family of Lucas numbers.

References

[1] Akbulak M. and Bozkurt D., *Chaos Soliton Fract.,* 2009; **42(3):** 1347-1355. DOI:10.1016/j.chaos.2009.03.019.

[2] Campbell C. M. and Campbell P. P., *J. Appl. Math. Comput.,* 2005; **19:** 231-240. DOI: 10.1007/BF02935801.

[3] Deveci Ö., Karaduman E. and Campbell C. M., *Algeb. Colloq.,* 2011; **18**: 945-954. DOI: 10.1142/S1005386711000824.

[4] Grabowski A. and Wojtecki P., Lucas numbers and generalized Fibonacci numbers; Available at: http://mizar.org/fm/2004-12/pdf12-3/fib_num3.pdf.

[5] Gyoung-Sik C., Hwang, Suk-Geun; Kim, Ik-Pyo and Shader, Bryan L., *Linear Algebra Appl.*, 2005; **395(1)**: 303-312. DOI:10.1016/j.laa.2004.08.018.

[6] Hartwig R. E., Note on a linear difference equation; Available at: http:/ /www.jstor.org/stable/27641892.

[7] Karaduman E. and Deveci Ö., *Discrete Dyn. Nat. Soc.*, 2009; 453750-10 pages. DOI:10.1155/2009/453750.

[8] Kiliç E. and Tasci D., *Rocky Mountain J. Math.*, 2008; **38**: 1991–2008. DOI:10.1216/RMJ-2008-38-6-1991.

[9] Koshy T., *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience Publication, John Wiley & Sons Inc., 2001.

[10] Mahajan Datta M., *Int. J. Math. Trends Technol.*, 2014; **(10)**: *Number 1 – Jun.* DOI:10.14445/22315373/IJMTT-V10P503.

[11] Mikkawy M. and Sogabe T., *Appl. Math. Comput.*, 2010; **215**: 4456-4461. DOI:10.1016/j.amc.2011.08.014.

[12] Öcal A.A., Tuglu N. and Altinisik E., *Appl. Math. Comput.* 2005; **170**: 584-596. DOI::10.1016/j.amc.2004.12.009.

[13] Özgür N. Y., Generalizations of Fibonacci and Lucas Sequences; Available at: http://siba-ese.unisalento.it/index.php/notemat/article/viewFile/745/589.

[14] Özkan E., On Truncated Fibonacci Sequences; Available at: http://insa.nic.in/writereaddata/UpLoadedFiles/IJPAM/20012fbb_241.pdf.

[15] Özkan E., Aydın H. and Dikici R., *Appl. Math. Comput.*, 2003; **143**: 165-172. DOI: 10.1016/S0096-3003(02)00360-0.

[16] Stanimirovic P.S., Nikolov J. and Stanimirovic I., *Discrete Appl. Math.*, 2008; **156**: 2606-2619. DOI:10.1016/j.dam.2007.09.028.

[17] Taher R. B. and Rachidi M., *Linear Algebra Appl.* 2003; **370**: 341-353. DOI:10.1016/S0024-3795(03)00418-X.

[18] Tasci D. and Kilic E., *Appl. Math. Comput.* 2004; **155**: 637-641. DOI:10.1016/S0096-3003(03)00804-X.