

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials

Engin Özkan & İpek Altun

To cite this article: Engin Özkan & İpek Altun (2019) Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials, Communications in Algebra, 47:10, 4020-4030, DOI: 10.1080/00927872.2019.1576186

To link to this article: https://doi.org/10.1080/00927872.2019.1576186

Published online: 06 Apr 2019.

Submit your article to this journal

Article views: 349

View related articles

View Crossmark data

Citing articles: 9 View citing articles

Full Terms & Conditions of access and use can be found at
https://www.tandfonline.com/action/journalInformation?journalCode=lagb20

COMMUNICATIONS IN ALGEBRA®
2019, VOL. 47, NO. 10, 4020–4030
https://doi.org/10.1080/00927872.2019.1576186

Taylor & Francis
Taylor & Francis Group

Check for updates

Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials

Engin Özkan[a] and İpek Altun[b]

[a]Department of Mathematics, Faculty of Arts and Sciences, Erzincan University, Erzincan, Turkey; [b]Graduate School of Natural and Applied Sciences, Erzincan University, Erzincan, Turkey

ABSTRACT
In this article, we find elements of the Lucas polynomials by using two matrices. We extend the study to the *n*-step Lucas polynomials. Then the Lucas polynomials and their relationship are generalized in the paper. Furthermore, we give relationships between the Fibonacci polynomials and the Lucas polynomials.

ARTICLE HISTORY
Received 5 April 2018
Revised 31 August 2018
Communicated by Mark Lewis

KEYWORDS
Fibonacci numbers; Lucas numbers; Fibonacci polynomials; Lucas polynomials

2010 MATHEMATICS SUBJECT CLASSIFICATION
11B39; 11B50

1. Introduction

Fibonacci numbers and sequence applications are of great interest. The Fibonacci sequence appears in many branches of mathematics. These include group theory, calculus, applied mathematics, linear algebra, etc. Also, these numbers have many important applications to diverse fields such as computer science, physics, biology and statistics. We can see applications of the Fibonacci sequence in group theory in [3, 8] and also see some generalized Fibonacci and Lucas sequences in [1, 2, 4, 7, 9, 10, 13–15, 17]. We can see some applications of the Fibonacci polynomials in [5, 6, 16, 18, 19]. One of the researches in this area is that Hoggatt and Bicknell study on the Fibonacci polynomials and *n*-step Fibonacci polynomials and give some properties of the polynomials [11].

In this article, we find elements of the Lucas polynomials by aid of a new matrix and the generalized φ-matrix investigated by Ivie [12]. We extend the study to the *n*-step Lucas polynomials. Then the Lucas polynomials and their relationship are generalized in the paper. Also, we give relationships between the Fibonacci polynomials and the Lucas polynomials.

Let us give the Fibonacci polynomials in [11] before we give the Lucas polynomials.

2. Materials and methods

Definition 2.1. The Fibonacci polynomials are defined by the recurrence relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), n \geq 0$$

with $F_0(x) = 0$ and $F_1(x) = 1$.

CONTACT Engin Özkan eozkan@erzincan.edu.tr Department of Mathematics, Faculty of Arts and Sciences, Erzincan University, Erzincan, Turkey.

© 2019 Taylor & Francis Group, LLC

Table 1. The first few Fibonacci polynomials and the array of their coefficients.

Fibonacci polynomials	Coefficient array			
$F_1(x) = 1$	1			
$F_2(x) = x$	1			
$F_3(x) = x^2 + 1$	1	1		
$F_4(x) = x^3 + 2x$	1	2		
$F_5(x) = x^4 + 3x^2 + 1$	1	3	1	
$F_6(x) = x^5 + 4x^3 + 3x$	1	4	3	
$F_7(x) = x^6 + 5x^4 + 6x^2 + 1$	1	5	6	1
$F_8(x) = x^7 + 6x^5 + 10x^3 + 4x$	1	6	10	4

That the Fibonacci polynomials are generated by a matrix $\mathbb{Q}_2$,

$$\mathbb{Q}_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \ \mathbb{Q}_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix}.$$

can be verified by mathematical induction. Also, it is clear that, when $x = 1$, $F_n(1) = F_n$, the n^{th} Fibonacci number, and when $x = 2$, $F_n(2) = P_n$, the n^{th} Pell number.
Moreover, when we write Pascal's triangle in left-justified form, the sums of the elements along the rising diagonals bring about the Fibonacci numbers, and, indeed, those elements are the coefficients of the Fibonacci polynomials. That is,

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1}$$

where $[x]$ is the greatest integer contained in x and $\binom{n}{j}$ is a binomial coefficient.

The first few Fibonacci polynomials are displayed below as well as the array of their coefficients (Table 1).

To form the coefficient of the k^{th} term of $F_n(x)$, we add the coefficients of the k^{th} term of $F_{n-1}(x)$ and the $(k-1)^{th}$ term of $F_{n-2}(x)$, and we can clearly see that the array of coefficients formed has the same rule of formation as Pascal's triangle when it is written in left-justified form if we use the formation of the Fibonacci polynomials, and if we write the polynomials in descending order. The only difference between the array of coefficients and Pascal's triangle is that each column is moved one line lower. By beginning in the left-most column and preceding up one and right one throughout the left-justified Pascal triangle, so-formed coefficients are those elements which appear along the diagonals formed. In this paper, the term 'the rising diagonal of such an array' will be used for this diagonal [11] (Tables 1–8).

3. Results

3.1. The Lucas polynomials

Definition 3.1.1. The Lucas polynomials are given by the recurrence relation

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \ n \geq 0$$

with $L_0(x) = 2$ and $L_1(x) = x$.
The Lucas polynomials are obtained by using two matrices

$$\mathbb{Q}_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \varphi_{0,2} = \begin{pmatrix} L_1(x) & L_0(x) \\ L_0(x) & L_{-1}(x) \end{pmatrix} = \begin{pmatrix} x & 2 \\ 2 & -x \end{pmatrix}.$$

That is,

$$\varphi_{n,2} = \varphi_{n-1,2}\mathbb{Q}_2 = \begin{pmatrix} L_n(x) & L_{n-1}(x) \\ L_{n-1}(x) & L_{n-2}(x) \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} L_{n+1}(x) & L_n(x) \\ L_n(x) & L_{n-1}(x) \end{pmatrix}.$$

This can be verified by mathematical induction.

Also, it is apparent that when $x = 1$, $L_n(1) = L_n$, the n^{th} Lucas number.

For $n = 1$, the claim is true, since

$$\varphi_{1,2} = \varphi_{0,2}\mathbb{Q}_2 = \begin{pmatrix} x & 2 \\ 2 & -x \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x^2+2 & x \\ x & 2 \end{pmatrix} = \begin{pmatrix} L_2(x) & L_1(x) \\ L_1(x) & L_0(x) \end{pmatrix}.$$

Suppose the claim is true for some $n = k$, that is

$$\varphi_{k,2} = \varphi_{k-1,2}\mathbb{Q}_2.$$

We have to prove that the claim is true for $n = k + 1$. Hence

$$\begin{aligned} \varphi_{k,2}\mathbb{Q}_2 = \varphi_{k-1,2}\mathbb{Q}_2\mathbb{Q}_2 &= \begin{pmatrix} L_k(x) & L_{k-1}(x) \\ L_{k-1}(x) & L_{k-2}(x) \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} xL_k(x)+L_{k-1}(x) & L_k(x) \\ xL_{k-1}(x)+L_{k-2}(x) & L_{k-1}(x) \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} L_{k+1}(x) & L_k(x) \\ L_k(x) & L_{k-1}(x) \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} xL_{k+1}(x)+L_k(x) & L_{k+1}(x) \\ xL_k(x)+L_{k-1}(x) & L_k(x) \end{pmatrix} \\ &= \begin{pmatrix} L_{k+2}(x) & L_{k+1}(x) \\ L_{k+1}(x) & L_k(x) \end{pmatrix} = \varphi_{k+1,2}. \end{aligned}$$

We can express the Lucas polynomials as follows:

$$L_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k}\binom{n-k}{k}x^{n-2k}$$

where $[x]$ is the greatest integer contained in x, $\binom{n}{j}$ is a binomial coefficient.

The first few Lucas polynomials are displayed below as well as the array of their coefficients (Table 2).

If one observes that, by rule of formation of the Lucas polynomials, if one writes the polynomials in descending order to form the coefficient of the $(k+1)^{th}$ of $L_{n+2}(x)$, one adds the coefficients of the k^{th} term of $L_{n+1}(x)$ and the $(k-1)^{th}$ term of $L_n(x)$.

Table 2. The first few Lucas polynomials and the array of their coefficients.

Lucas polynomials	Coefficient array					
$L_1(x) = x$	1					
$L_2(x) = x^2 + 2$	1	2				
$L_3(x) = x^3 + 3x$	1	3				
$L_4(x) = x^4 + 4x^2 + 2$	1	4	2			
$L_5(x) = x^5 + 5x^3 + 5x$	1	5	5			
$L_6(x) = x^6 + 6x^4 + 9x^2 + 2$	1	6	9	2		
$L_7(x) = x^7 + 7x^5 + 14x^3 + 7x$	1	7	14	7		
$L_8(x) = x^8 + 8x^6 + 20x^4 + 16\ x^2 + 2$	1	8	20	16	2	
$L_9(x) = x^9 + 9x^7 + 27x^5 + 30x^3 + 9x$	1	9	27	30	9	
$L_{10}(x) = x^{10} + 10x^8 + 35x^6 + 50x^4 + 25\ x^2 + 2$	1	10	35	50	25	2

We can write the following interesting determinant identity.

$$\det(\varphi_{n,2}) = \begin{vmatrix} L_{n+1}(x) & L_n(x) \\ L_n(x) & L_{n-1}(x) \end{vmatrix} = (-1)^{n+1}(x^2+4).$$

When $x = 1$ and $k \in \mathbb{Z}$, it is seen that

$$\det(\varphi_{n,2}) = \begin{cases} 5, & n = 2k-1 \\ -5, & n = 2k \end{cases}.$$

Definition 3.1.2. The 3-step Lucas polynomials are defined by the recurrence relation

$$L^*_{n+3}(x) = x^2 L^*_{n+2}(x) + xL^*_{n+1}(x) + L^*_n(x)$$

with $L^*_0(x) = 3$, $L^*_1(x) = 1$ and $L^*_2(x) = x^2 + 2$.
When $x = 1$ and $L^*_n(1) = L^*_n$, the 3-step Lucas number 3, 1, 3, 7, 11, 21, 39, 71, 131, …

$$L^*_{n+3} = L^*_{n+2} + L^*_{n+1} + L^*_n .$$

The 3-step Lucas polynomials are obtained by using two matrices

$$\mathbb{Q}_3 = \begin{pmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \varphi_{0,3} = \begin{pmatrix} L^*_2(x) & L^*_1(x) & L^*_0(x) \\ L^*_1(x) & L^*_0(x) & L^*_{-1}(x) \\ L^*_0(x) & L^*_{-1}(x) & L^*_{-2}(x) \end{pmatrix} = \begin{pmatrix} x^2+2 & 1 & 3 \\ 1 & 3 & 2-3x \\ 3 & 2-3x & 1-2x \end{pmatrix}.$$

That is,

$$\varphi_{n,3} = \varphi_{n-1,3}\mathbb{Q}_3 = \begin{pmatrix} L^*_{n+1}(x) & L^*_n(x) & L^*_{n-1}(x) \\ L^*_n(x) & L^*_{n-1}(x) & L^*_{n-2}(x) \\ L^*_{n-1}(x) & L^*_{n-2}(x) & L^*_{n-3}(x) \end{pmatrix} \begin{pmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} L^*_{n+2}(x) & L^*_{n+1}(x) & L^*_n(x) \\ L^*_{n+1}(x) & L^*_n(x) & L^*_{n-1}(x) \\ L^*_n(x) & L^*_{n-1}(x) & L^*_{n-2}(x) \end{pmatrix}$$

This can be verified by mathematical induction.

For $n = 1$, it is apparent that

$$\varphi_{0,3}\mathbb{Q}_3 = \varphi_{1,3}$$

$$\begin{pmatrix} x^2+2 & 1 & 3 \\ 1 & 3 & 2-3x \\ 3 & 2-3x & 1-2x \end{pmatrix} \cdot \begin{pmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x^4+2x^2+x+3 & x^2+2 & 1 \\ x^2+2 & 1 & 3 \\ 1 & 3 & 2-3x \end{pmatrix} = \begin{pmatrix} L^*_3 & L^*_2 & L^*_1 \\ L^*_2 & L^*_1 & L^*_0 \\ L^*_1 & L^*_0 & L^*_{-1} \end{pmatrix} = \varphi_{1,3}$$

We can also write the following interesting determinant identity.

$$\det(\varphi_{n,3}) = \begin{vmatrix} L^*_{n+2}(x) & L^*_{n+1}(x) & L^*_n(x) \\ L^*_{n+1}(x) & L^*_n(x) & L^*_{n-1}(x) \\ L^*_n(x) & L^*_{n-1}(x) & L^*_{n-2}(x) \end{vmatrix} = (-1)^{n+1}(9x^4 - 6x^3 + 19x^2 + 4x + 18)$$

When $x = 1$ and $k \in \mathbb{Z}$, it is seen that

Table 3. The first few 3-step Lucas polynomials.

3-step Lucas polynomials
$L^*_1(x) = 1$
$L^*_2(x) = x^2 + 2$
$L^*_3(x) = x^4 + 2x^2 + x + 3$
$L^*_4(x) = x^6 + 2x^4 + 2x^3 + 3x^2 + 2x + 1$
$L^*_5(x) = x^8 + 2x^6 + 3x^5 + 3x^4 + 4x^3 + 3x^2 + 3x + 2$
$L^*_6(x) = x^{10} + 2x^8 + 4x^7 + 3x^6 + 6x^5 + 6x^4 + 6x^3 + 6x^2 + 2x + 3$
$L^*_7(x) = x^{12} + 2x^{10} + 5x^9 + 3x^8 + 8x^7 + 10x^6 + 9x^5 + 12x^4 + 7x^3 + 9x^2 + 4x + 1$
$L^*_8(x) = x^{14} + 2x^{12} + 6x^{11} + 3x^{10} + 2x^9 + 15x^8 + 12x^7 + 20x^6 + 16x^5 + 18x^4 + 16x^3 + 4x^2 + 6x + 2$
$L^*_9(x) = x^{16} + 2x^{14} + 7x^{13} + 3x^{12} + 4x^{11} + 21x^{10} + 15x^9 + 30x^8 + 30x^7 + 30x^6 + 34x^5 + 17x^4 + 21x^3 + 12x^2 + 3x + 3$

Table 4. The array of the first few 3-step Lucas polynomials coefficients.

Coefficient array															
1															
1	2														
1	2	1	3												
1	2	2	3	2	1										
1	2	3	3	4	3	3	2								
1	2	4	3	6	6	6	6	2	3						
1	2	5	3	8	10	9	2	7	9	4	1				
1	2	6	3	2	15	2	20	16	18	16	4	6	2		
1	2	7	3	4	21	15	30	30	30	34	17	21	12	3	3

$$\det\ (\varphi_{n,3}) = \begin{cases} 44 & n = 2k-1 \\ -44 & n = 2k \end{cases}$$

The first few 3-step Lucas polynomials follow (Tables 3 and 4).

Definition 3.1.3. The 4-step Lucas polynomials are defined by the recurrence relation

$$L^{**}{}_{n+4}(x) = x^3 L^{**}{}_{n+3}(x) + x^2 L^{**}{}_{n+2}(x) + x L^{**}{}_{n+1}(x) + L^{**}{}_n$$

with $L^{**}{}_0(x) = 4,\ L^{**}{}_1(x) = 1,\quad L^{**}{}_2(x) = x^3 + 2$ and $L^{**}{}_3(x) = x^6 + 2x^3 + x^2 + 3.$

When $x = 1$ and $L^{**}{}_n(1) = L^{**}_n$, the 4-step Lucas number 4, 1, 3, 7, 15, 26, 51, 99, …,

$$L^{**}{}_{n+4} = L^{**}{}_{n+3} + L^{**}{}_{n+2} + L^{**}{}_{n+1} + L^{**}{}_n\ .$$

The 4-step Lucas polynomials are obtained by using two matrices

$$\mathbb{Q}_4 = \begin{pmatrix} x^3 & 1 & 0 & 0 \\ x^2 & 0 & 1 & 0 \\ x & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\varphi_{0,4} = \begin{pmatrix} L^{**}_3 & L^{**}_2 & L^{**}_1 & L^{**}_0 \\ L^{**}_2 & L^{**}_1 & L^{**}_0 & L^{**}_{-1} \\ L^{**}_1 & L^{**}_0 & L^{**}_{-1} & L^{**}_{-2} \\ L^{**}_0 & L^{**}_{-1} & L^{**}_{-2} & L^{**}_{-3} \end{pmatrix}$$

$$= \begin{pmatrix} x^6 + 2x^3 + x^2 + 3 & x^3 + 2 & 1 & 4 \\ x^3 + 2 & 1 & 4 & 3 - 4x \\ 1 & 4 & 3 - 4x & 4x^2 - 3x - 2 \\ 4 & 3 - 4x & 4x^2 - 3x - 2 & -4x^3 + 3x^2 + 2x - 2 \end{pmatrix}$$

Table 5. The first few 4-step Lucas polynomials.

4-Step Lucas Polynomials
$L^{**}{}_1(x) = 1$
$L^{**}{}_2(x) = x^3 + 2$
$L^{**}{}_3(x) = x^6 + 2x^3 + x^2 + 3$
$L^{**}{}_4(x) = x^9 + 2x^6 + 2x^5 + 3x^3 + 2x^2 + x + 4$
$L^{**}{}_5(x) = x^{12} + 2x^9 + 3x^8 + 3x^6 + 4x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$
$L^{**}{}_6(x) = x^{15} + 2x^{12} + 4x^{11} + 3x^9 + 6x^8 + 6x^7 + 4x^6 + 6x^5 + 6x^4 + 4x^3 + 4x^2 + 3x + 2$
$L^{**}{}_7(x) = x^{18} + 2x^{15} + 5x^{14} + 3x^{12} + 8x^{11} + 10x^{10} + 4x^9 + 9x^8 + 12x^7 + 10x^6 + 8x^5 + 9x^4 + 8x^3 + 3x^2 + 4x + 3$

Table 6. The array of the first few 4-step Lucas polynomials coefficients.

Coefficient array															
1															
1	2														
1	2	1	3												
1	2	2	3	2	1	4									
1	2	3	3	4	3	4	3	2	1						
1	2	4	3	6	6	4	6	6	4	4	3	2			
1	2	5	3	8	10	4	9	12	10	8	9	8	3	4	3

That is,

$$\varphi_{n,4} = \varphi_{n-1,4}\mathbb{Q}_4 = \begin{pmatrix} L^{**}_{n+2} & L^{**}_{n+1} & L^{**}_{n} & L^{**}_{n-1} \\ L^{**}_{n+1} & L^{**}_{n} & L^{**}_{n-1} & L^{**}_{n-2} \\ L^{**}_{n} & L^{**}_{n-1} & L^{**}_{n-2} & L^{**}_{n-3} \\ L^{**}_{n-1} & L^{**}_{n-2} & L^{**}_{n-3} & L^{**}_{n-4} \end{pmatrix} \begin{pmatrix} x^3 & 1 & 0 & 0 \\ x^2 & 0 & 1 & 0 \\ x & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} L^{**}_{n+3} & L^{**}_{n+2} & L^{**}_{n+1} & L^{**}_{n} \\ L^{**}_{n+2} & L^{**}_{n+1} & L^{**}_{n} & L^{**}_{n-1} \\ L^{**}_{n+1} & L^{**}_{n} & L^{**}_{n-1} & L^{**}_{n-2} \\ L^{**}_{n} & L^{**}_{n-1} & L^{**}_{n-2} & L^{**}_{n-3} \end{pmatrix}$$

This can be verified by mathematical induction.

The first few 4-step Lucas polynomials follow (Tables 5 and 6).

Definition 3.1.4. *The r-step Lucas polynomials are defined by the recurrence relation*

$$L^{(r)}_{n+r}(x) = x^{r-1}L^{(r)}_{n+r-1}(x) + x^{r-2}L^{(r)}_{n+r-2}(x) + \ldots + L^{(r)}_{n}(x)$$

with $L^{(r)}_{-(r-2)}(x) = L^{(r)}_{-(r-1)}(x) = \ldots L^{(r)}_{-1}(x) = L^{(r)}_{0}(x) = 0$, $L^{(r)}_{1} = 1$ and $L^{(r)}_{2}(x) = x^{r-1} + 2$. For $n = 0, 1, 2, \ldots$, $\binom{n}{j}_r$ is the element in the n^{th} row and j^{th}column of the left-justified r-nomial triangle, and $\binom{n}{j}_r = 0$ when j > n.

$$L^{(r)}_{n}(x) = \sum_{j=0} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj} + 2\sum_{j=0} \binom{n-j-2}{j}_r x^{(r-1)(n-2)-rj}$$
$$+ 3\sum_{j=0} \binom{n-j-3}{j}_r x^{(r-1)(n-3)-rj} + 4\sum_{j=0} \binom{n-j-4}{j}_r x^{(r-1)(n-4)-rj} + \ldots$$
$$+ r\sum_{j=0} \binom{n-j-r}{j}_r x^{(r-1)(n-r)-rj}.$$

We obtain the r-step Lucas polynomials by using two matrices

$$\mathbb{Q}_r = \begin{pmatrix} x^{r-1} & 1 & 0 & 0 & \dots & 0 \\ x^{r-2} & 0 & 1 & 0 & \dots & 0 \\ x^{r-3} & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & 0 & 0 & \dots & \dots & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 \end{pmatrix} \text{ and } \varphi_{0,r} = \begin{pmatrix} L_1^{(r)}(x) & L_0^{(r)}(x) & \dots & \dots & L_{3-r}^{(r)}(x) & L_{2-r}^{(r)}(x) \\ L_0^{(r)}(x) & L_{-1}^{(r)}(x) & \dots & \dots & L_{2-r}^{(r)}(x) & L_{1-r}^{(r)}(x) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{3-r}^{(r)}(x) & L_{2-r}^{(r)}(x) & \dots & \dots & L_{5-2r}^{(r)}(x) & L_{4-2r}^{(r)}(x) \\ L_{2-r}^{(r)}(x) & L_{1-r}^{(r)}(x) & \dots & \dots & L_{4-2r}^{(r)}(x) & L_{3-2r}^{(r)}(x) \end{pmatrix}.$$

That is,

$$\varphi_{n,r} = \varphi_{n-1,r}\mathbb{Q}_r = \begin{pmatrix} L_{n+1}^{(r)}(x) & L_n^{(r)}(x) & \dots & \dots & L_{n-r+3}^{(r)}(x) & L_{n-r+2}^{(r)}(x) \\ L_n^{(r)}(x) & L_{n-1}^{(r)}(x) & \dots & \dots & L_{n-r+2}^{(r)}(x) & L_{n-r+1}^{(r)}(x) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{n-r+3}^{(r)}(x) & L_{n-r+2}^{(r)}(x) & \dots & \dots & L_{n-2r+5}^{(r)}(x) & L_{n-2r+4}^{(r)}(x) \\ L_{n-r+2}^{(r)}(x) & L_{n-r+1}^{(r)}(x) & \dots & \dots & L_{n-2r+4}^{(r)}(x) & L_{n-2r+3}^{(r)}(x) \end{pmatrix}.$$

This can be verified by mathematical induction.

For $n = 1$, the claim is true, since

$$\begin{aligned} \varphi_{1,r} &= \varphi_{0,r}\mathbb{Q}_r \\ &= \begin{pmatrix} L_1^{(r)}(x) & L_0^{(r)}(x) & \dots & \dots & L_{3-r}^{(r)}(x) & L_{2-r}^{(r)}(x) \\ L_0^{(r)}(x) & L_{-1}^{(r)}(x) & \dots & \dots & L_{2-r}^{(r)}(x) & L_{1-r}^{(r)}(x) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{3-r}^{(r)}(x) & L_{2-r}^{(r)}(x) & \dots & \dots & L_{5-2r}^{(r)}(x) & L_{4-2r}^{(r)}(x) \\ L_{2-r}^{(r)}(x) & L_{1-r}^{(r)}(x) & \dots & \dots & L_{4-2r}^{(r)}(x) & L_{3-2r}^{(r)}(x) \end{pmatrix} \begin{pmatrix} x^{r-1} & 1 & 0 & 0 & \dots & 0 \\ x^{r-2} & 0 & 1 & 0 & \dots & 0 \\ x^{r-3} & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & 0 & 0 & \dots & \dots & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 \end{pmatrix} \\ &= \begin{pmatrix} x^{r-1}L_1^{(r)}(x) + x^{r-2}L_0^{(r)}(x) + \dots + L_{2-r}^{(r)}(x) & L_1^{(r)}(x) & \dots & \dots & L_{4-r}^{(r)}(x) & L_{3-r}^{(r)}(x) \\ x^{r-1}L_0^{(r)}(x) + x^{r-2}L_{-1}^{(r)}(x) + \dots + L_{1-r}^{(r)}(x) & L_0^{(r)}(x) & \dots & \dots & L_{3-r}^{(r)}(x) & L_{2-r}^{(r)}(x) \\ \dots \quad \dots & & \dots & \dots & \dots & \dots \\ \dots \quad \dots & & \dots & \dots & \dots & \dots \\ x^{r-1}L_{3-r}^{(r)}(x) + x^{r-2}L_{2-r}^{(r)}(x) + \dots + L_{4-2r}^{(r)}(x) & L_{3-r}^{(r)}(x) & \dots & \dots & L_{6-2r}^{(r)}(x) & L_{5-2r}^{(r)}(x) \\ x^{r-1}L_{2-r}^{(r)}(x) + x^{r-2}L_{1-r}^{(r)}(x) + \dots + L_{3-2r}^{(r)}(x) & L_{2-r}^{(r)}(x) & \dots & \dots & L_{5-2r}^{(r)}(x) & L_{4-2r}^{(r)}(x) \end{pmatrix} \\ &= \begin{pmatrix} L_2^{(r)}(x) & L_1^{(r)}(x) & \dots & \dots & L_{4-r}^{(r)}(x) & L_{3-r}^{(r)}(x) \\ L_1^{(r)}(x) & L_0^{(r)}(x) & \dots & \dots & L_{3-r}^{(r)}(x) & L_{2-r}^{(r)}(x) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{4-r}^{(r)}(x) & L_{3-r}^{(r)}(x) & \dots & \dots & L_{6-2r}^{(r)}(x) & L_{5-2r}^{(r)}(x) \\ L_{3-r}^{(r)}(x) & L_{2-r}^{(r)}(x) & \dots & \dots & L_{5-2r}^{(r)}(x) & L_{4-2r}^{(r)}(x) \end{pmatrix} \end{aligned}$$

Suppose the claim is true for some $n = k$, that is

$$\varphi_{k,r} = \varphi_{k-1,r}\mathbb{Q}_r.$$

We have to prove that the claim is true for $n = k + 1$. We have

$$\varphi_{k+1,r} = \varphi_{k,r}\mathbb{Q}_r.$$
$$\varphi_{k,r} = \varphi_{k-1,r}\mathbb{Q}_r$$
$$\varphi_{k,r}\mathbb{Q}_r = \varphi_{k-1,r}\mathbb{Q}_r\mathbb{Q}_r$$

Hence

$$= \begin{pmatrix} L_k^{(r)}(x) & L_{k-1}^{(r)}(x) & \cdots & \cdots & L_{k-r+2}^{(r)}(x) & L_{k-r+1}^{(r)}(x) \\ L_{k-1}^{(r)}(x) & L_{k-2}^{(r)}(x) & \cdots & \cdots & L_{k-r+1}^{(r)}(x) & L_{k-r}^{(r)}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{k-r+2}^{(r)}(x) & L_{k-r+1}^{(r)}(x) & \cdots & \cdots & L_{k-2r+4}^{(r)}(x) & L_{k-2r+3}^{(r)}(x) \\ L_{k-r+1}^{(r)}(x) & L_{k-r}^{(r)}(x) & \cdots & \cdots & L_{k-2r+3}^{(r)}(x) & L_{k-2r+2}^{(r)}(x) \end{pmatrix} \begin{pmatrix} x^{r-1} & 1 & 0 & 0 & \cdots & 0 \\ x^{r-2} & 0 & 1 & 0 & \cdots & 0 \\ x^{r-3} & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x^{r-1}L_k^{(r)}(x) + x^{r-2}L_{k-1}^{(r)}(x) + \ldots + L_{k-r+1}^{(r)}(x) & L_k^{(r)}(x) & \cdots & \cdots & L_{k-r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) \\ x^{r-1}L_{k-1}^{(r)}(x) + x^{r-2}L_{k-2}^{(r)}(x) + \ldots + L_{k-r}^{(r)}(x) & L_{k-1}^{(r)}(x) & \cdots & \cdots & L_{k-r+2}^{(r)}(x) & L_{k-r+1}^{(r)}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{r-1}L_{k-r+2}^{(r)}(x) + x^{r-2}L_{k-r+1}^{(r)}(x) + \ldots + L_{k-2r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) & \cdots & \cdots & L_{k-2r+5}^{(r)}(x) & L_{k-2r+4}^{(r)}(x) \\ x^{r-1}L_{k-r+1}^{(r)}(x) + x^{r-2}L_{k-r}^{(r)}(x) + \ldots + L_{k-r+2}^{(r)}(x) & L_{k-r+1}^{(r)}(x) & \cdots & \cdots & L_{k-2r+4}^{(r)}(x) & L_{k-2r+3}^{(r)}(x) \end{pmatrix}$$

$$\begin{pmatrix} x^{r-1} & 1 & 0 & 0 & \cdots & 0 \\ x^{r-2} & 0 & 1 & 0 & \cdots & 0 \\ x^{r-3} & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} L_{k+1}^{(r)}(x) & L_k^{(r)}(x) & \cdots & \cdots & L_{k-r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) \\ L_k^{(r)}(x) & L_{k-1}^{(r)}(x) & \cdots & \cdots & L_{k-r+2}^{(r)}(x) & L_{k-r+1}^{(r)}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{k-r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) & \cdots & \cdots & L_{k-2r+5}^{(r)}(x) & L_{k-2r+4}^{(r)}(x) \\ L_{k-r+2}^{(r)}(x) & L_{k-r+1}^{(r)}(x) & \cdots & \cdots & L_{k-2r+4}^{(r)}(x) & L_{k-2r+3}^{(r)}(x) \end{pmatrix} \begin{pmatrix} x^{r-1} & 1 & 0 & 0 & \cdots & 0 \\ x^{r-2} & 0 & 1 & 0 & \cdots & 0 \\ x^{r-3} & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x^{r-1}L_{k+1}^{(r)}(x) + x^{r-2}L_k^{(r)}(x) + \ldots + L_{k-r+2}^{(r)}(x) & L_{k+1}^{(r)}(x) & \cdots & \cdots & L_{k-r+4}^{(r)}(x) & L_{k-r+3}^{(r)}(x) \\ x^{r-1}L_k^{(r)}(x) + x^{r-2}L_{k-1}^{(r)}(x) + \ldots + L_{k-r+1}^{(r)}(x) & L_k^{(r)}(x) & \cdots & \cdots & L_{k-r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{r-1}L_{k-r+3}^{(r)}(x) + x^{r-2}L_{k-r+2}^{(r)}(x) + \ldots + L_{k-2r+4}^{(r)}(x) & L_{k-r+3}^{(r)}(x) & \cdots & \cdots & L_{k-2r+6}^{(r)}(x) & L_{k-2r+5}^{(r)}(x) \\ x^{r-1}L_{k-r+2}^{(r)}(x) + x^{r-2}L_{k-r+1}^{(r)}(x) + \ldots + L_{k-r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) & \cdots & \cdots & L_{k-2r+5}^{(r)}(x) & L_{k-2r+4}^{(r)}(x) \end{pmatrix}$$

$$= \begin{pmatrix} L_{k+2}^{(r)}(x) & L_{k+1}^{(r)}(x) & \cdots & \cdots & L_{k-r+4}^{(r)}(x) & L_{k-r+3}^{(r)}(x) \\ L_{k+1}^{(r)}(x) & L_k^{(r)}(x) & \cdots & \cdots & L_{k-r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{k-r+4}^{(r)}(x) & L_{k-r+3}^{(r)}(x) & \cdots & \cdots & L_{k-2r+6}^{(r)}(x) & L_{k-2r+5}^{(r)}(x) \\ L_{k-r+3}^{(r)}(x) & L_{k-r+2}^{(r)}(x) & \cdots & \cdots & L_{k-2r+5}^{(r)}(x) & L_{k-2r+4}^{(r)}(x) \end{pmatrix} = \varphi_{k+1,r}.$$

The first few r-step Lucas polynomials follow (Tables 7 and 8).

Table 7. The first few r-step Lucas polynomials.

r-Step Lucas Polynomials
$L_1^{(r)} = 1$
$L_2^{(r)} = x^{r-1} + 2$
$L_3^{(r)} = x^{2r-2} + x^{r-2} + 2x^{r-1} + 3$
$L_4^{(r)} = x^{3r-3} + 2x^{2r-3} + 2x^{2r-2} + 2x^{r-2} + 3x^{r-1} + 4$
$L_5^{(r)} = x^{4r-4} + 3x^{3r-4} + x^{2r-4} + 2x^{3r-3} + 4x^{2r-3} + 3x^{2r-2} + 3x^{r-2} + 4x^{r-1} + 5$
$L_6^{(r)} = x^{5r-5} + 4x^{4r-5} + 3x^{3r-5} + 2x^{4r-4} + 6x^{3r-4} + 2x^{2r-4} + 3x^{3r-3} + 6x^{2r-3} + 4x^{2r-2} + 4x^{r-2} + 5x^{r-1} + 6$
$L_7^{(r)} = x^{6r-6} + 5x^{5r-6} + 6x^{4r-6} + x^{3r-6} + 2x^{5r-5} + 8x^{4r-5} + 6x^{3r-5} + 3x^{4r-4} + 9x^{3r-4} + 3x^{2r-4} + 4x^{3r-3} + 8x^{2r-3} + 5x^{2r-2} + 5x^{r-2} + 6x^{r-1} + 7$
$L_8^{(r)} = x^{7r-7} + 6x^{6r-7} + 10x^{5r-7} + 4x^{4r-7} + 2x^{6r-6} + 10x^{5r-6} + 12x^{4r-6} + 2x^{3r-6} + 3x^{5r-5} + 12x^{4r-5} + 9x^{3r-5} + 4x^{4r-4} + 12x^{3r-4} + 4x^{2r-4} + 5x^{3r-3} + 10x^{2r-3} + 6x^{2r-2} + 6x^{r-2} + 7x^{r-1} + 8$

Table 8. The array of the first few r-step Lucas polynomials coefficients.

Coefficient array																			
1																			
1	2																		
1	1	2	3																
1	2	2	2	3	4														
1	3	1	2	4	3	3	4	5											
1	4	3	2	6	2	3	6	4	4	5	6								
1	5	6	1	2	8	6	3	9	3	4	8	5	5	6	7				
1	6	10	4	2	10	12	2	3	12	9	4	12	4	5	10	6	6	7	8

3.2. Relationships between the Fibonacci polynomials and the Lucas polynomials

Theorem 3.2.1. *For any positive integer n, we have a relation between the Fibonacci polynomials and Lucas polynomials as follows,*

$$F_{2n+1}(x) + F_{2n-1}(x) = L_{2n}(x).$$

Proof. We obtain the following equalities from the definition of the Fibonacci polynomials.

$$F_{2n+1}(x) = F_{2n-1}(x) + xF_{2n}(x).$$
$$F_{2n-1}(x) = F_{2n-3}(x) + xF_{2n-2}(x).$$

From these equalities, we have

$$\begin{aligned} F_{2n+1}(x) + F_{2n-1}(x) &= F_{2n-1}(x) + xF_{2n}(x) + F_{2n-3}(x) + xF_{2n-2}(x) \\ &= F_{2n-1}(x) + F_{2n-3}(x) + x(F_{2n}(x) + F_{2n-2}(x)) \\ &= L_{2n-2}(x) + xL_{2n-1}(x) \\ &= L_{2n}(x). \end{aligned}$$

□

Theorem 3.2.2. *For any integer n, we have a relation between the Fibonacci polynomials and Lucas polynomials as follows,*

$$F_{2n+2}(x) + F_{2n}(x) = L_{2n+1}(x).$$

Proof. We obtain the following equalities from the definition of the Fibonacci polynomials.

$$F_{2n+2}(x) = F_{2n}(x) + xF_{2n+1}(x).$$
$$F_{2n}(x) = F_{2n-2}(x) + xF_{2n-1}(x).$$

From these equalities, we have

$$\begin{aligned} F_{2n+2}(x) + F_{2n}(x) &= F_{2n}(x) + xF_{2n+1}(x) + F_{2n-2}(x) + xF_{2n-1}(x) \\ &= F_{2n}(x) + F_{2n-2}(x) + x(F_{2n-1}(x) + F_{2n+1}(x)) \\ &= L_{2n-1}(x) + xL_{2n}(x) = L_{2n+1}(x). \end{aligned}$$

□

Theorem 3.2.3. *For any positive integer n, we have the relation*

$$L_n(x) = xF_n(x) + 2F_{n-1}(x).$$

Proof. We have

$$\begin{aligned} xF_n(x) + 2F_{n-1}(x) = \ & xF_n(x) + F_{n-1}(x) + F_{n-1}(x) \\ &= F_{n+1}(x) + F_{n-1}(x) \\ &= L_n(x). \end{aligned}$$

□

Theorem 3.2.4. *For $x = 1$, r-step Fibonacci(r-bonacci) polynomials* $R_{-(r-2)}(x) = R_{-(r-1)}(x) = \ldots R_{-1}(x) = R_0(x) = 0$ *and* $R_1(x) = 1$ *with* $R_{n+r}(x) = x^{r-1}R_{n+r-1}(x) + x^{r-2}R_{n+r-2}(x) + \ldots + R_n(x)$ and for $n + r = k$, *we have the relation*

$$R_k = 2^{k-2}, k \geq 3.$$

Proof. We proceed by mathematical induction on k.

For $k = 3$, the claim is true, since $R_3 = 2^{3-2} = 2^1 = 2$.

Suppose the claim is true for some $k = m$, that is

$$R_m = 2^{m-2}.$$

We have to prove that the claim is true for $k = m + 1$. We have

$$R_{n+r} = R_{n+r-1} + R_{n+r-2} + \ldots + R_n$$

$R_{n+r-1} = R_{n+r-2} + R_{n+r-3} + \ldots + R_n + R_{n-1}$ and for $n + r = k$

then, $R_k = R_{k-1} + R_{k-1} - R_{n-1} = 2R_{k-1} - R_{n-1} = 2R_{k-1} - 0 = 2R_{k-1}$

$$\begin{aligned} R_{m+1} &= 2R_m \\ &= 2(2^{m-2}) = 2^{m-1}. \end{aligned}$$

□

Theorem 3.2.5. *For $x = 1$, r-step Lucas polynomials* $L^{(r)}_{-(r-2)}(x) = L^{(r)}_{-(r-1)}(x) = \ldots L^{(r)}_{-1}(x) = -1$, $L^{(r)}_0(x) = r$ *and* $L^{(r)}_1(x) = 1$ *with* $L^{(r)}_{n+r}(x) = x^{r-1}L^{(r)}_{n+r-1}(x) + x^{r-2}L^{(r)}_{n+r-2}(x) + \ldots + L^{(r)}_n(x)$ *and for* $n + r = k$ *we have the relation*

$$L^{(n)}_k = 2^k - 1, \ k \geq 1.$$

Proof. We proceed by mathematical induction on k.

For $k = 1$, the claim is true, since

$$L^{(1)}_1 = 2^1 - 1 = 1$$

Suppose the claim is true for some $k = m - 1$, that is,

$$L^{(r)}_{m-1} = 2^{m-1} - 1.$$

We have to prove that the claim is true for $k = m$. We have

$L_{n+r}^{(r)} = L_{n+r-1}^{(r)} + L_{n+r-2}^{(r)} + \ldots + L_n^{(r)}$, $L_{n+r-1}^{(r)} = L_{n+r-2}^{(r)} + L_{n+r-3}^{(r)} + \ldots + L_n^{(r)} + L_{n-1}^{(r)}$ and for $n + r = k$ then,

$$L_k^{(r)} = L_{k-1}^{(r)} + L_{k-1}^{(r)} - L_{n-1}^{(r)} = 2L_{k-1}^{(r)} - L_{n-1}^{(r)} = 2L_{k-1}^{(r)} - (-1)$$
$$L_m^{(r)} = 2L_{m-1}^{(r)} + 1$$
$$= 2(2^{m-1} - 1) + 1 = 2^m - 1.$$ □

Theorem 3.2.6. *For any positive integer n, the Lucas polynomials satisfy*

$$L_{n+1}(x)L_{n-1}(x) + (-1)^n L_2(x) = L_{2n}(x).$$

Proof. A proof could be made by mathematical induction.

Acknowledgments

The authors would like to thank the anonymous reviewers for their useful comments and suggestions.

References

[1] Aires, A. P., Catarino, P., Campos, H., Borges, A., Vasco, P. (2014). s-Generalized Fibonacci numbers: some identities, a generating function and pythagorean triples. *Int. J. Math. Anal.* 8(38):1757–1766. DOI: 10.12988/ijma.2014.47203.
[2] Borges, A., Catarino, P., Aires, A. P., Vasco, P., Campos, H. (2014). Two-by-two matrices involving k-Fibonacci and k-Lucas sequences. *Appl. Math. Sci.* 8(34):1659–1666. DOI: 10.12988/ams.2014.4280.
[3] Campbell, C. M., Campbell, P. P. (2005). The Fibonacci length of certain centro-polyhedral groups. *J. Appl. Math. Comp.* 19(1–2):231–240. DOI: 10.1007/BF02935801.
[4] Catarino, P. (2014). On some identities for k-Fibonacci sequence. *Int. J. Contemp. Math. Sci.* 9(1):37–42. DOI: 10.12988/ijcms.2014.311120.
[5] Catarino, P. (2018). Diagonal functions of k-Pell and k-Pell-Lucas polynomials and some identities. *Acta Math. Univ. Comenianae* 87(1):147–159.
[6] Catarino, P. (2017). A note on certain matrices with h(x)-Fibonacci polynomials. *Acta Math. Univ. Comenianae* 86(2):263–270.
[7] Catarino, P., Vasco, P., Borges, A., Campos, H., Aires, A. P. (2014). Sums, products and identities involving k-Fibonacci and k-Lucas sequences. *JP Algebra Number Theory Appl.* 32(1):63–77. DOI: 10.17654/JPANTAFeb2014_063_077.
[8] Deveci, Ö., Karaduman, E., Campbell, C. M. (2011). On the k-Nacci sequences in finite binary polyhedral groups. *Algebra Colloq.* 18(spec01):945–954. DOI: 10.1142/S1005386711000824.
[9] Firengiz, M. C., Tasci, D. (2010). Incomplete Fibonacci and Lucas p-numbers. *Math. Comput. Model.* 52(9–10):1763–1770.
[10] Gulec, H. H., Taskara, N., Uslu, K. (2013). A new approach to generalized Fibonacci and Lucas numbers with binomial coefficients. *Appl. Math. Comput.* 220:482–486. DOI: 10.1016/j.amc.2013.05.043.
[11] Hoggatt Jr, V. E., Bicknell, M. (1973). Generalized Fibonacci polynomials. *Fibonacci Quarterly* 11:457–465.
[12] Ivie, J. (1972). A general Q-matrix. *Fibonacci Quarterly* 10(3):255–261.
[13] Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications*. New York, NY: Wiley.
[14] Özkan, E. (2007). On truncated Fibonacci sequences. *Indian J. Pure Appl. Math.* 38(4):241–251.
[15] Özkan, E., Aydın, H., Dikici, R. (2003). 3-step Fibonacci series modulo m. *Appl. Math. Comput.* 143(1): 165–172. DOI: 10.1016/S0096-3003(02)00360-0.
[16] Ramírez, J. L. (2014). On convolved generalized Fibonacci and Lucas polynomials. *Appl. Math. Comput.* 229:208–213. DOI: 10.1016/j.amc.2013.12.049.
[17] Stanimirovic, P. S., Nikolov, J., Stanimirovic, I. (2008). A generalization of Fibonacci and Lucas matrices. *Discrete Appl. Math.* 156:2606–2619. DOI: 10.1016/j.dam.2007.09.028.
[18] Tuglu, N., Kocer, E. G., Stakhov, A. (2011). Bivariate Fibonacci like p-polynomials. *Appl. Math. Comput.* 217(24):10239–10246. DOI: 10.1016/j.amc.2011.05.022.
[19] Ye, X., Zhang, Z. (2017). A common generalization of convolved generalized Fibonacci and Lucas polynomials and its applications. *Appl. Math. Comput.* 306:31–37.