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Jacobi–Perron Algorithm and Recurrence Relations

Carlos M. da Fonseca [1,2,*], Engin Özkan [3] and Anthony G. Shannon [4]

[1] Department of Mathematics, Kuwait College of Science and Technology, Doha District, Safat 13133, Kuwait
[2] Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Kliment Ohridski Blvd. 8, 1000 Sofia, Bulgaria
[3] Department of Mathematics, Faculty of Sciences, Marmara University, Istanbul 34722, Türkiye; engin.ozkan@marmara.edu.tr
[4] Warrane College, The University of New South Wales, Kensington, NSW 2033, Australia; t.shannon@warrane.unsw.edu.au
* Correspondence: c.dafonseca@kcst.edu.kw

Abstract

This paper explores connections between multidimensional continued fractions, specifically the Jacobi–Perron algorithm, and generalized Fibonacci and Lucas sequences of arbitrary order r. Recurrence relations are established, a sufficient criterion for the convergence of the associated ratio vectors is proved, and the limit is identified explicitly in terms of the dominant root of the auxiliary equation. When the coefficient of the oldest term of the recurrence equals one, this limit is shown to be the initial vector of a purely periodic Jacobi–Perron algorithm. The results link the work of Bernstein, Williams, and related authors to fundamental recursive sequences.

Keywords: Jacobi–Perron algorithm; multidimensional continued fractions; recurrence relations; generalized Fibonacci sequences

MSC: 11J70; 11B37; 11B39

1. Introduction

Multidimensional continued fractions, introduced by Jacobi and Perron [1], generalize the classical continued fraction algorithm. The Jacobi–Perron algorithm (JPA) was developed extensively by Bernstein [2,3] and connected with recursive sequences and number-theoretic questions in [4]; see also [5–7] and, for more recent work on multidimensional continued fractions and linear recurrences, [8–10].

The purpose of this paper is to relate these results to generalized Fibonacci and Lucas sequences of arbitrary order r, building on the third-order framework of Bell [11] and the general theories of Jarden [12] and Ward [13] as well as on Williams's sequences arising from Pascal-triangle constructions [14].

Building on the classical theory, in Section 2 we recall, in a self-contained way, the auxiliary equation associated with an order-r linear recurrence, the resulting basic and primordial sequences, and their Binet-type representations; we take the opportunity to state explicitly the admissible domains of all parameters involved, since several of these were left implicit in earlier expositions. Section 3 relates these sequences to a further family $H_{s,n}^{(r)}$, following Williams [15], and records an addition formula linking the two families. Section 4 develops matrix recurrence relations governing the basic sequences and relates them to the companion matrix of the auxiliary polynomial and to the elementary symmetric functions of its roots. In Section 5, we derive first-order relations for the basic sequences,

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introduce their ratio vectors, and recall the Jacobi–Perron algorithm in $\mathbb{E}^{r-1}$ together with the associated T-transformation. Section 6 gives a sufficient condition (P-boundedness of the recurrence) under which the ratio vectors converge, identifies the limit explicitly, and shows that when the coefficient of the oldest term of the recurrence equals one, a normalized form of the limit is a fixed point of a T-transformation and hence the initial vector of a purely periodic Jacobi–Perron algorithm. Section 7 illustrates these results in the three-dimensional case $r = 3$.

Although several ingredients of the theory are classical, the synthesis developed here provides a unified perspective on recursive sequences and multidimensional continued fractions together with a fully explicit and self-consistent set of definitions, hypotheses and index conventions.

2. Preliminaries: Basic and Primordial Sequences

Throughout the paper, F denotes a field of characteristic zero (the reader may take $F = \mathbb{R}$ or $F = \mathbb{C}$ without loss of the essential content) and $\bar{F}$ an algebraic closure of F. Fix an integer $r \geq 2$ and consider the auxiliary equation

$$x^r - P_{r,1}x^{r-1} + P_{r,2}x^{r-2} - \cdots + (-1)^r P_{r,r} = 0, \tag{1}$$

with coefficients $P_{r,1}, \ldots, P_{r,r} \in F$. No sign or positivity condition is imposed on the $P_{r,k}$ a priori: they may vanish or be negative. We denote by $\alpha_{r,1}, \ldots, \alpha_{r,r} \in \bar{F}$ the roots of (1), and we assume throughout that

Hypothesis 1 (H1). *The roots are pairwise distinct (they need not lie in F itself; in particular, they may be non-real even when $F = \mathbb{R}$).*

Hypothesis 2 (H2). *The roots are non-zero; equivalently, $P_{r,r} = \prod_{i=1}^{r} \alpha_{r,i} \neq 0$ are pairwise distinct (they need not lie in F itself; in particular, they may be non-real even when $F = \mathbb{R}$).*

Hypothesis (H1) is what makes the Vandermonde-type constructions below nondegenerate, while (H2) is used in Section 6, where it guarantees that the coefficient of the oldest term of the recurrence does not vanish.

Define the Vandermonde determinant

$$d = \prod_{1 \leq i < j \leq r} (\alpha_{r,j} - \alpha_{r,i}) = \begin{vmatrix} 1 & \alpha_{r,1} & \alpha_{r,1}^2 & \ldots & \alpha_{r,1}^{r-1} \\ 1 & \alpha_{r,2} & \alpha_{r,2}^2 & \ldots & \alpha_{r,2}^{r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{r,r} & \alpha_{r,r}^2 & \ldots & \alpha_{r,r}^{r-1} \end{vmatrix}. \tag{2}$$

By the classical Vandermonde identity, $d \neq 0$ if and only if the roots are pairwise distinct; thus, (H1) is exactly equivalent to $d \neq 0$, which we use freely below.

For each $s = 1, 2, \ldots, r$, we define the *basic sequence* $U_{s,n}^{(r)}$ ($n \geq 1$) by the Binet-type formula

$$U_{s,n}^{(r)} = \frac{1}{d} \sum_{j=1}^{r} c_{s,j}\, \alpha_{r,j}^{n-1}, \tag{3}$$

where $c_{s,j}$ is the cofactor of the entry $\alpha_{r,j}^{s-1}$ in the Vandermonde matrix of (2) (this entry lies in row j and column s of that matrix).

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Remark 1. *Formula* (3) *is the unique solution, via Cramer's rule, of the initial-value problem described below. Seeking* $U_{s,n}^{(r)} = \sum_{j=1}^{r} \gamma_{s,j} \alpha_{r,j}^{n-1}$ *subject to* $U_{s,n}^{(r)} = \delta_{s,n}$ *for* $n = 1, \ldots, r$ *(with* $\delta_{s,n}$ *the Kronecker delta) amounts to solving the linear system* $\sum_{j=1}^{r} \gamma_{s,j} \alpha_{r,j}^{n-1} = \delta_{s,n}$, $n = 1, \ldots, r$, *whose coefficient matrix (with rows indexed by n and columns by j) is the transpose of the Vandermonde matrix in* (2) *and therefore has determinant d. By Cramer's rule,* $\gamma_{s,j}$ *equals the determinant obtained by replacing the j-th column of that matrix by the standard basis vector* e_s*, divided by d; expanding this determinant along the replaced column shows that this quantity is precisely* $c_{s,j}/d$*, where* $c_{s,j}$ *is the cofactor of the* (s, j) *entry* $\alpha_{r,j}^{s-1}$ *of the transposed matrix, which coincides with the cofactor of the same entry in the matrix of* (2)*, as claimed. This justifies* (3).

By construction, the sequences $U_{s,n}^{(r)}$ satisfy the linear recurrence of order r

$$U_{s,n}^{(r)} = \sum_{k=1}^{r} (-1)^{k+1} P_{r,k} \, U_{s,n-k}^{(r)}, \qquad n > r, \tag{4}$$

with initial conditions $U_{s,n}^{(r)} = \delta_{s,n}$ for $n = 1, \ldots, r$. Thus, for each fixed s, the sequence starts with a single 1 at position s and zeros elsewhere among the first r terms.

The terminology "basic sequence" is motivated by the third-order recursive sequences of Bell [11], namely recurrences of the form

$$T_n = c_1 T_{n-1} + c_2 T_{n-2} + c_3 T_{n-3},$$

with constant coefficients determined by an associated characteristic equation and by the general treatment of Jarden [12].

To recover the classical second-order Lucas sequence ($r = 2$) of Lucas [16–19], we also define the *primordial sequence* of order r, which is denoted $U_{0,n}^{(r)}$, by the power-sum values

$$U_{0,n}^{(r)} = \sum_{j=1}^{r} \alpha_{r,j}^{n-1}, \qquad n = 1, \ldots, r,$$

extended by the same recurrence (4). We claim that this extension is again given by the closed form

$$U_{0,n}^{(r)} = \sum_{j=1}^{r} \alpha_{r,j}^{n-1}, \qquad n \geq 1. \tag{5}$$

Remark 2. *The right-hand side of* (5)*, viewed as a function of n, is a linear combination of the fundamental solutions* $n \mapsto \alpha_{r,j}^{n-1}$*, for* $j = 1, \ldots, r$*, of the linear recurrence* (4)*; each such fundamental solution satisfies* (4) *automatically because the* $\alpha_{r,j}$ *are roots of the auxiliary polynomial, so the whole sum does too, for every* $n \geq 1$ *(not merely* $n > r$*). By definition, this sum also agrees with* $U_{0,n}^{(r)}$ *at* $n = 1, \ldots, r$*. Since a linear recurrence of order r has a unique solution once its first r values are prescribed, the two sequences, the one defined by the initial values together with* (4)*, and the closed form on the right of* (5)*, coincide for every* $n \geq 1$*. This proves* (5).

For $r = 2$, (5) reduces (up to an index shift) to the classical Lucas sequence.

We also define, for each $s \in \{0, 1, \ldots, r-1\}$ (a complete residue system modulo r, the natural range since the phase factor below is r-periodic in s), the family

$$L_{s,n}^{(r)} = \sum_{j=1}^{r} \alpha_{r,j}^{n} \, \zeta_r^{s(j-1)}, \qquad \zeta_r = e^{2\pi i/r}. \tag{6}$$

For $s = 0$, we recover the primordial sequence: $L_{0,n}^{(r)} = U_{0,n+1}^{(r)}$. For $r = 2$, (6) gives $L_{0,n}^{(2)} = \alpha_{2,1}^n + \alpha_{2,2}^n$, which is the standard Lucas sequence up to an index shift. The sequences $L_{s,n}^{(r)}$ will be used only incidentally below; the reader interested purely in the JPA connections of Sections 4–7 may skip (6) on first reading.

3. Sequence Connections

Retain the auxiliary Equation (1), the roots $\alpha_{r,1}, \ldots, \alpha_{r,r}$, and Vandermonde determinant d of Section 2. Choose constants $\beta_0, \beta_1, \ldots, \beta_{r-1} \in F$, not all zero, and define

$$\varphi_{r,i} = \sum_{j=0}^{r-1} \beta_j \, \alpha_{r,i}^j, \qquad i = 1, \ldots, r.$$

We assume throughout this section that $\varphi_{r,i} \neq 0$, for $i = 1, \ldots, r$, so that the determinant constructions below are nondegenerate; this is an additional standing hypothesis on the β_j, which is not automatic from (H1) and (H2).

Following Williams [15], we define a family of sequences $H_{s,n}^{(r)}$, with $s = 1, \ldots, r$ and $n \geq 1$, as follows: for $2 \leq s \leq r-1$, $H_{s,n}^{(r)}$ is the determinant obtained from the Vandermonde matrix of (2) by replacing its s-th column (the column of entries $\alpha_{r,i}^{s-1}$, $i = 1, \ldots, r$) by the column of entries $\varphi_{r,i}^{n-1}$, $i = 1, \ldots, r$, and dividing by d:

$$H_{s,n}^{(r)} = \frac{1}{d} \begin{vmatrix} 1 & \alpha_{r,1} & \ldots & \alpha_{r,1}^{s-2} & \varphi_{r,1}^{n-1} & \alpha_{r,1}^{s} & \ldots & \alpha_{r,1}^{r-1} \\ 1 & \alpha_{r,2} & \ldots & \alpha_{r,2}^{s-2} & \varphi_{r,2}^{n-1} & \alpha_{r,2}^{s} & \ldots & \alpha_{r,2}^{r-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_{r,r} & \ldots & \alpha_{r,r}^{s-2} & \varphi_{r,r}^{n-1} & \alpha_{r,r}^{s} & \ldots & \alpha_{r,r}^{r-1} \end{vmatrix}. \tag{7}$$

For $s = 1$, the replaced column is the first column of the Vandermonde matrix (there are no columns $\alpha_{r,i}^j$, $j < 0$, to precede it), so that

$$H_{1,n}^{(r)} = \frac{1}{d} \begin{vmatrix} \varphi_{r,1}^{n-1} & \alpha_{r,1} & \alpha_{r,1}^2 & \ldots & \alpha_{r,1}^{r-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \varphi_{r,r}^{n-1} & \alpha_{r,r} & \alpha_{r,r}^2 & \ldots & \alpha_{r,r}^{r-1} \end{vmatrix}.$$

For $s = r$, the replaced column is the last column of the Vandermonde matrix (there are no columns $\alpha_{r,i}^j$, $j \geq r$, to follow it), so that

$$H_{r,n}^{(r)} = \frac{1}{d} \begin{vmatrix} 1 & \alpha_{r,1} & \ldots & \alpha_{r,1}^{r-2} & \varphi_{r,1}^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_{r,r} & \ldots & \alpha_{r,r}^{r-2} & \varphi_{r,r}^{n-1} \end{vmatrix}.$$

This construction extends Williams's sequences $A_{i,n}$ from [14] (pp. 406–407).

The quantities $H_{s,n}^{(r)}$ are equivalently characterized as the unique solution, in $s = 1, \ldots, r$, of the linear system

$$\sum_{j=1}^{r} \alpha_{r,i}^{j-1} H_{j,n}^{(r)} = \varphi_{r,i}^{n-1}, \qquad i = 1, \ldots, r,$$

equivalently

$$\varphi_{r,i}^{n-1} = \sum_{j=1}^{r} H_{j,n}^{(r)} \, \alpha_{r,i}^{j-1}, \qquad i = 1, \ldots, r. \tag{8}$$

Recall the basic sequences $U_{s,n}^{(r)}$ defined in (3). Replacing n by m in (8), multiplying by $c_{s,i}$, summing over i, and dividing by d, we obtain

$$\frac{1}{d}\sum_{i=1}^{r} c_{s,i}\,\varphi_{r,i}^{m-1} = \sum_{j=1}^{r}\left(\frac{1}{d}\sum_{i=1}^{r} c_{s,i}\,\alpha_{r,i}^{j-1}\right) H_{j,m}^{(r)}. \tag{9}$$

By Cramer's rule and the determinant representation (7), the left-hand side of (9) is precisely $H_{s,m}^{(r)}$, while the inner sum on the right-hand side equals $U_{s,j}^{(r)}$. Hence,

$$H_{s,m}^{(r)} = \sum_{j=1}^{r} U_{s,j}^{(r)}\, H_{j,m}^{(r)}.$$

Since $U_{s,j}^{(r)} = \delta_{s,j}$ for $1 \le j \le r$, this identity expresses the compatibility of the determinant representation (7) with the Binet-type formula (3) and with the initial conditions of the basic sequences. The non-trivial link between the two families is the addition formula below, in which the basic sequences appear at time indices beyond r.

For completeness, we record the addition formula of Williams [14] (p. 409), which is stated in the conventions of this paper: sequence labels $s = 1, \dots, r$ (rather than the labels $0, \dots, r-1$ used by Williams) and time indices $n \ge 1$, where $H_{s,n}^{(r)}$ corresponds to the power $\varphi_{r,i}^{n-1}$ and $U_{s,N}^{(r)}$ to $\alpha_{r,i}^{N-1}$. For all $n, m \ge 1$ and $s = 1, \dots, r$,

$$H_{s,n+m-1}^{(r)} = \sum_{p=1}^{r}\sum_{q=1}^{r} U_{s,\,p+q-1}^{(r)}\, H_{p,n}^{(r)}\, H_{q,m}^{(r)}. \tag{10}$$

Here, p, q index the sequence labels of the families $H_{p,n}^{(r)}$ and $H_{q,m}^{(r)}$, while $p+q-1 \in \{1, \dots, 2r-1\}$ is a time index of the basic sequence $U_{s,\cdot}^{(r)}$; thus, only quantities already defined in Section 2 occur. For the reader's convenience, we indicate the proof. For each root $\alpha_{r,i}$ and every $N \ge 1$,

$$\alpha_{r,i}^{N-1} = \sum_{s=1}^{r} U_{s,N}^{(r)}\, \alpha_{r,i}^{s-1}, \tag{11}$$

because both sides, as functions of N, satisfy (4) and they agree for $N = 1, \dots, r$. Multiplying (8) for the indices n and m and using (11) with $N = p+q-1$, we obtain

$$\begin{aligned}\varphi_{r,i}^{n+m-2} &= \sum_{p=1}^{r}\sum_{q=1}^{r} H_{p,n}^{(r)} H_{q,m}^{(r)}\, \alpha_{r,i}^{p+q-2} \\ &= \sum_{s=1}^{r}\Big(\sum_{p=1}^{r}\sum_{q=1}^{r} U_{s,p+q-1}^{(r)} H_{p,n}^{(r)} H_{q,m}^{(r)}\Big)\alpha_{r,i}^{s-1}, \qquad i = 1, \dots, r.\end{aligned}$$

Comparing with (8) for the index $n+m-1$, and using that the Vandermonde system has a unique solution by (H1), yields (10).

4. Matrix Equations

We develop matrix recurrence relations associated with the generalized Fibonacci sequences $U_{s,n}^{(r)}$ of Section 2 and relate them to the elementary symmetric functions $P_{r,1}, \dots, P_{r,r}$ of the roots $\alpha_{r,1}, \dots, \alpha_{r,r}$ of the auxiliary equation, that is, the coefficients occurring in the auxiliary Equation (1) itself,

$$P_{r,1} = \sum_{i=1}^{r} \alpha_{r,i}, \quad P_{r,2} = \sum_{1 \le i < j \le r} \alpha_{r,i}\alpha_{r,j}, \quad \dots, \quad P_{r,r} = \prod_{i=1}^{r} \alpha_{r,i}.$$

Throughout this section, we suppress the superscript (r) and write

$$U_{s,n} := U_{s,n}^{(r)}.$$

Consequently, the characteristic polynomial of the companion matrix introduced below is

$$\chi(\lambda) = \lambda^r + \sum_{m=1}^{r} (-1)^m P_{r,m} \lambda^{r-m}.$$

For each positive integer n, the elementary symmetric functions of the n-th powers of the roots are

$$s(r,n,m) = \sum_{1 \le j_1 < \cdots < j_m \le r} \alpha_{r,j_1}^n \cdots \alpha_{r,j_m}^n.$$

In particular, the power sums are $v_n^{(r)} = \sum_{i=1}^r \alpha_{r,i}^n = s(r,n,1)$, and the classical coefficients are recovered at $n = 1$: $s(r,1,m) = P_{r,m}$. The quantities $s(r,n,m)$ are polynomial functions of the power sums $v_k^{(r)}$ via the Newton identities (cf. [20,21]).

For each fixed $s \in \{1, \ldots, r\}$, the sequence $n \mapsto U_{s,n}$ satisfies the linear recurrence induced by the auxiliary polynomial:

$$U_{s,n+r} = P_{r,1} U_{s,n+r-1} - P_{r,2} U_{s,n+r-2} + \cdots + (-1)^{r+1} P_{r,r} U_{s,n}, \qquad n \ge 1. \tag{12}$$

This follows from the Binet representation of $U_{s,n}$ together with the identity

$$\alpha_{r,i}^r = P_{r,1} \alpha_{r,i}^{r-1} - P_{r,2} \alpha_{r,i}^{r-2} + \cdots + (-1)^{r+1} P_{r,r},$$

valid for each root $\alpha_{r,i}$ since it is precisely the auxiliary Equation (1).

To describe the simultaneous evolution of $U_{1,n}, \ldots, U_{r,n}$, define the $r \times r$ matrix

$$R_n = \begin{pmatrix} U_{1,n} & U_{2,n} & \cdots & U_{r,n} \\ U_{1,n+1} & U_{2,n+1} & \cdots & U_{r,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ U_{1,n+r-1} & U_{2,n+r-1} & \cdots & U_{r,n+r-1} \end{pmatrix}, \qquad n \ge 1.$$

The column index corresponds to s, while the row index corresponds to the time shift. Let M be the companion matrix of the auxiliary polynomial,

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ (-1)^{r-1} P_{r,r} & (-1)^{r-2} P_{r,r-1} & \cdots & -P_{r,2} & P_{r,1} \end{pmatrix},$$

so that $\chi_M(\lambda) = \chi(\lambda)$.

Remark 3. *Write the rows of R_n as $\rho_0, \ldots, \rho_{r-1}$, where $\rho_k = (U_{1,n+k}, \ldots, U_{r,n+k})$. For $0 \le k \le r-2$, the corresponding row of MR_n is obtained by acting with the k-th row of M, which is the standard basis vector e_{k+2}; this simply selects ρ_{k+1}, matching row k of R_{n+1}, whose entries are $U_{s,n+k+1}$. For the last row ($k = r-1$), the last row of M applied to R_n gives, in column s, $\sum_{j=1}^r (-1)^{r-j} P_{r,r-j+1} U_{s,n+j-1}$, which after reindexing $\ell = r - j + 1$ is exactly the right-hand side*

of the recurrence (12) *evaluated at this shift, i.e.,* $U_{s,n+r}$. *This is precisely the last row of* R_{n+1}. *Hence,* $R_{n+1} = MR_n$ *for every* $n \geq 1$, *and by induction,*

$$R_n = M^{n-1}R_1, \qquad n \geq 1, \tag{13}$$

where R_1 *is the matrix formed by the initial* r *values of each sequence* $U_{s,n}$; *since* $U_{s,k} = \delta_{s,k}$ *for* $1 \leq k \leq r$, R_1 *is the identity matrix, so that in fact* $R_n = M^{n-1}$. *This verification depends on the convention adopted for* R_n *(time increasing downward) together with the companion matrix acting on the left, which is exactly as set up above; if the sequences are instead arranged in rows, i.e., one works with the transpose* R_n^{T}, *then* $R_{n+1}^{\mathsf{T}} = R_n^{\mathsf{T}} M^{\mathsf{T}}$ *with the transposed companion matrix acting on the right.*

Since powers of a fixed matrix commute, $M^{m+n} = M^m M^n = M^n M^m$. The characteristic polynomial of M is $\chi_M(\lambda) = \lambda^r - P_{r,1}\lambda^{r-1} + \cdots + (-1)^r P_{r,r}$. By the Cayley–Hamilton theorem,

$$M^r = \sum_{j=1}^{r} (-1)^{j+1} P_{r,j} M^{r-j}. \tag{14}$$

Multiplying (14) on the right by M^{n-r}, for $n \geq r$, yields

$$M^n = \sum_{j=1}^{r} (-1)^{j+1} P_{r,j} M^{n-j}, \qquad n \geq r.$$

Thus, the matrix sequence $\{M^n\}_{n\geq 0}$ satisfies the same linear recurrence as each scalar sequence $U_{s,n}$.

Therefore, (13) establishes a direct correspondence between the matrix sequence R_n and the powers of the companion matrix M. This correspondence transfers algebraic identities of the scalar linear recurrence system into the matrix algebra generated by M, providing the structural framework for the analysis of the ratio vectors and of the Jacobi–Perron algorithm in the next sections.

5. Jacobi–Perron Algorithm

Recall that the set of matrices

$$R = \sum_{j=0}^{r-1} W_j M^j, \qquad W_j \in F,$$

is a commutative F-algebra isomorphic to the quotient $F[x]/(f(x))$, where $f(x) = x^r - P_{r,1}x^{r-1} + \cdots + (-1)^r P_{r,r}$ is the auxiliary polynomial (this uses only that M is the companion matrix of f, so that its minimal polynomial is f). Since the roots of f are distinct by (H1), f is squarefree. Consequently, if $f = f_1 \cdots f_t$ is the factorization of f into pairwise distinct monic irreducible factors over F, the Chinese Remainder Theorem gives $F[x]/(f(x)) \cong \prod_{k=1}^{t} F[x]/(f_k(x))$, which is a finite product of fields; in particular, this algebra is a field that is isomorphic to $F(\alpha)$ for a root α of f if and only if f is irreducible over F.

By (12), and setting

$$b_j^{(n)} = (-1)^{j+1} P_{r,j}, \qquad j = 1, \ldots, r, \tag{15}$$

(a definition that does not depend on n; the superscript is retained only to match the notation of Bernstein [3], where the corresponding coefficients may vary with n), each basic sequence $U_{s,n}^{(r)}$, $s = 1, \dots, r$, satisfies the order-r linear recurrence

$$U_{s,n+r}^{(r)} = \sum_{j=1}^{r} b_j^{(n)} U_{s,n+r-j}^{(r)}, \qquad n \geq 1, \tag{16}$$

which is simply (12) rewritten with the coefficients (15). In this notation, the auxiliary polynomial reads $f(x) = x^r - \sum_{j=1}^{r} b_j^{(n)} x^{r-j}$. The index range of $b_j^{(n)}$ is $j = 1, \dots, r$ throughout Sections 5–7: $b_1^{(n)}$ is the coefficient of the most recent term $U_{s,n+r-1}^{(r)}$ and $b_r^{(n)}$ that of the oldest term $U_{s,n}^{(r)}$. By (H2), $b_r^{(n)} \neq 0$. Since the $b_j^{(n)}$ do not depend on n, we frequently write simply b_j.

The basic sequences also satisfy the following first-order vector relations, which will be used repeatedly.

Lemma 1. *For every $n \geq 1$,*

$$U_{1,n+1}^{(r)} = b_r \, U_{r,n}^{(r)}, \qquad U_{s,n+1}^{(r)} = U_{s-1,n}^{(r)} + b_{r+1-s} \, U_{r,n}^{(r)} \quad (s = 2, \dots, r). \tag{17}$$

Equivalently, the polynomials $p_n(x) := \sum_{s=1}^{r} U_{s,n}^{(r)} x^{s-1}$ satisfy $p_1(x) = 1$ and $p_{n+1}(x) = x\, p_n(x) - U_{r,n}^{(r)} f(x)$; thus, $p_n(x)$ is the remainder of x^{n-1} upon division by $f(x)$.

Proof. Fix s. As functions of n, both sides of each identity in (17) satisfy the recurrence (16): the left-hand side is a shift of a solution, and the right-hand side is a linear combination of solutions. Hence, it suffices to check the identities for $n = 1, \dots, r$. For $1 \leq n \leq r-1$, we have $U_{r,n}^{(r)} = 0$ and $U_{s,n+1}^{(r)} = \delta_{s,n+1}$, so the identities reduce to $0 = 0$ for $s = 1$ and to $\delta_{s,n+1} = \delta_{s-1,n}$ for $s \geq 2$. For $n = r$, (16) with $n = 1$ gives $U_{s,r+1}^{(r)} = \sum_{j=1}^{r} b_j U_{s,r+1-j}^{(r)} = b_{r+1-s}$, while the right-hand side equals b_r for $s = 1$ and $\delta_{s-1,r} + b_{r+1-s} = b_{r+1-s}$ for $s \geq 2$. Finally, multiplying (17) by x^{s-1} and summing over s gives

$$p_{n+1}(x) = x\, p_n(x) - U_{r,n}^{(r)} x^r + U_{r,n}^{(r)} \sum_{s=1}^{r} b_{r+1-s} x^{s-1} = x\, p_n(x) - U_{r,n}^{(r)} f(x),$$

and since $\deg p_n \leq r-1$ and $p_n \equiv x^{n-1} \pmod{f}$ by induction, p_n is the stated remainder. □

From now on, whenever ratio vectors or the Jacobi–Perron algorithm are considered, we assume $F \subseteq \mathbb{R}$. For $n \geq r$ with $U_{r,n}^{(r)} \neq 0$ (under the hypotheses of Section 6, this holds for every $n \geq r$), we define the *ratio vector*

$$A^{(n)} = \left(A_1^{(n)}, \dots, A_{r-1}^{(n)}\right) \in \mathbb{E}^{r-1}, \qquad A_i^{(n)} := \frac{U_{i,n}^{(r)}}{U_{r,n}^{(r)}}, \quad i = 1, \dots, r-1, \tag{18}$$

where $\mathbb{E}^{r-1}$ denotes $(r-1)$-dimensional Euclidean space.

We now recall the Jacobi–Perron algorithm. For a non-negative integer k and real numbers $a_i^{(k)}$, we write

$$a^{(k)} = \left(a_1^{(k)}, a_2^{(k)}, \dots, a_{r-1}^{(k)}\right) \in \mathbb{E}^{r-1}$$

(following [22], we do not distinguish between row and column vectors here). Following Bernstein [23], we introduce a transformation $T : \mathbb{E}^{r-1} \to \mathbb{E}^{r-1}$ as follows. Let

$$G(a^{(k)}) = g^{(k)} = (g_1^{(k)}, g_2^{(k)}, \ldots, g_{r-1}^{(k)})$$

be a vector function on $\mathbb{E}^{r-1}$ with values in $\mathbb{E}^{r-1}$ such that $a_1^{(k)} \neq g_1^{(k)}$. Then, the T-image of $a^{(k)}$ is

$$a^{(k)}T = \frac{(a_2^{(k)} - g_2^{(k)}, \ \ldots, \ a_{r-1}^{(k)} - g_{r-1}^{(k)}, \ 1)}{a_1^{(k)} - g_1^{(k)}}.$$

A *Jacobi–Perron algorithm* (JPA) *for the initial vector* $a^{(0)}$ is a sequence $(a^{(k)})_{k\geq 0}$ of vectors in $\mathbb{E}^{r-1}$ (see [24]) for which there exists a T-transformation such that $a^{(k)}T = a^{(k+1)}$ for every k.

The ratio vectors (18) are not themselves asserted to form a JPA. Instead, dividing (17) by $U_{r,n+1}^{(r)} = (A_{r-1}^{(n)} + b_1)\, U_{r,n}^{(r)}$ shows that whenever $U_{r,n}^{(r)} U_{r,n+1}^{(r)} \neq 0$, they obey the explicit one-step relations

$$A_1^{(n+1)} = \frac{b_r}{A_{r-1}^{(n)} + b_1}, \qquad A_s^{(n+1)} = \frac{A_{s-1}^{(n)} + b_{r+1-s}}{A_{r-1}^{(n)} + b_1} \quad (s = 2, \ldots, r-1). \tag{19}$$

In Section 6, we prove that $A^{(n)}$ converges under a positivity hypothesis on the coefficients (15) and we identify its limit explicitly (Theorem 1). We then make the relation with the Jacobi–Perron algorithm precise (Proposition 1): after a suitable normalization, and provided $b_r = 1$, consecutive ratio vectors are related by a T-transformation (read in the backward direction), and the limit is a fixed point of this transformation, that is, the initial vector of a purely periodic JPA of period one.

6. Convergence of the Ratio Vectors

In this section, we assume $F \subseteq \mathbb{R}$ and give a sufficient condition for the convergence of the ratio vectors (18) of the recursive sequences of Sections 2–5. We build on Bernstein's framework [23]; the convergence argument is elementary and uses only the convex structure of the recurrence—not the location of the roots of the auxiliary equation.

Definition 1. *The recurrence relation* (12) *(equivalently,* (16)*) is said to be* P-bounded *if, with the coefficients* $b_j^{(k)}$ *defined by* (15) *for* $j = 1, \ldots, r$ *and with the convention* $b_0^{(k)} := 1$ *for all* k*:*

(i) $b_j^{(k)} \geq 0$ *for all* $k \geq 0$ *and* $j = 1, \ldots, r$*, with* $b_1^{(k)} > 0$ *(equivalently,* $(-1)^{j+1} P_{r,j} \geq 0$ *for all* $j = 1, \ldots, r$*, and* $P_{r,1} > 0$*); and*

(ii) There exists a real constant $C \geq 1$ *(independent of* k*) such that*

$$0 \leq \frac{b_j^{(k)}}{b_1^{(k)}} \leq C \quad \text{for all } k \geq 0 \text{ and } j = 0, 2, 3, \ldots, r.$$

Since the coefficients (15) do not depend on k, condition (ii) is automatically satisfied once (i) holds: one may take $C = \max\{1, \ 1/b_1, \ b_2/b_1, \ldots, b_r/b_1\}$. We nevertheless keep (ii) explicit, because the constant C enters the quantitative estimate of Lemma 4. Note also that (i) together with (H2) gives $b_r > 0$.

Assume that the recurrence is P-bounded. Since $U_{s,n}^{(r)} = \delta_{s,n}$ for $n = 1, \dots, r$ and all $b_j \geq 0$, induction on (16) gives $U_{s,n}^{(r)} \geq 0$ for all s and all $n \geq 1$. Moreover, discarding the non-negative terms with $j \geq 2$ in (16) yields

$$U_{r,m+1}^{(r)} \geq b_1 U_{r,m}^{(r)}, \qquad m \geq r, \tag{20}$$

and since $U_{r,r}^{(r)} = 1$ and $b_1 > 0$, it follows that $U_{r,n}^{(r)} > 0$ for all $n \geq r$. Hence, the ratio vectors $A^{(n)}$ of (18) are defined for every $n \geq r$, and their entries are non-negative.

Theorem 1. *Assume that $F \subseteq \mathbb{R}$ and that the recurrence (12) is P-bounded in the sense of Definition 1. Then,*

(a) For each $i = 1, \dots, r-1$ the sequence $(A_i^{(n)})_{n \geq r}$ converges; we write

$$A^* = (A_1^*, \dots, A_{r-1}^*) := \lim_{n \to \infty} A^{(n)} \in \mathbb{E}^{r-1};$$

(b) The auxiliary polynomial f has exactly one positive root ρ, and

$$A_i^* = \sum_{j=r-i+1}^{r} b_j \rho^{r-i-j}, \qquad i = 1, \dots, r-1; \tag{21}$$

equivalently, $A_1^*, \dots, A_{r-1}^*, 1$ are the coefficients of $1, x, \dots, x^{r-1}$ in the polynomial $f(x)/(x-\rho)$. In particular, $A_i^* > 0$ for all i, and $\rho = b_1 + A_{r-1}^*$;

(c) With $\Delta_{i,n}$ as in Lemma 3 and θ as in Lemma 4, for all $n \geq r$ and all $k \geq n$,

$$\left| A_i^{(k)} - A_i^* \right| \leq \Delta_{i,n} \leq \theta^{\lfloor (n-r)/r \rfloor} \Delta_{i,r}.$$

The proof of Theorem 1 rests on the following three lemmas.

Lemma 2. *Assume that the recurrence is P-bounded and let $n \geq r$. Define the weights*

$$T_{r,j}^{(n)} := b_j^{(n)} \frac{U_{r,n+r-j}^{(r)}}{U_{r,n+r}^{(r)}}, \qquad j = 1, \dots, r.$$

Then, $T_{r,j}^{(n)} \geq 0$ for all $j = 1, \dots, r$, $\sum_{j=1}^{r} T_{r,j}^{(n)} = 1$, and for each $i = 1, \dots, r-1$, the ratio sequence satisfies the convex combination

$$A_i^{(n+r)} = \sum_{j=1}^{r} T_{r,j}^{(n)} A_i^{(n+r-j)}.$$

Proof. Since $n \geq r$, all indices $n + r - j$ ($1 \leq j \leq r$) are at least r, so every $U_{r,n+r-j}^{(r)}$ and $U_{r,n+r}^{(r)}$ is positive; together with $b_j^{(n)} \geq 0$, this gives $T_{r,j}^{(n)} \geq 0$. Evaluating (16) for $s = r$ yields $U_{r,n+r}^{(r)} = \sum_{j=1}^{r} b_j^{(n)} U_{r,n+r-j}^{(r)}$, and dividing by $U_{r,n+r}^{(r)}$ gives $\sum_{j=1}^{r} T_{r,j}^{(n)} = 1$. Finally, for $i \in \{1, \dots, r-1\}$, dividing (16) (with $s = i$) by $U_{r,n+r}^{(r)}$ and multiplying and dividing each summand by $U_{r,n+r-j}^{(r)}$, we obtain

$$A_i^{(n+r)} = \sum_{j=1}^{r} \left(b_j^{(n)} \frac{U_{r,n+r-j}^{(r)}}{U_{r,n+r}^{(r)}} \right) \frac{U_{i,n+r-j}^{(r)}}{U_{r,n+r-j}^{(r)}} = \sum_{j=1}^{r} T_{r,j}^{(n)} A_i^{(n+r-j)},$$

which completes the proof. □

The restriction $n \geq r$ in Lemma 2 is essential: for $n < r$, some of the ratios $A_i^{(n+r-j)}$ are undefined, and the identity may fail (for instance, for the example of Section 7 with $n = 1$, one has $T_{3,1}^{(1)} = 1$, $T_{3,2}^{(1)} = T_{3,3}^{(1)} = 0$, but $A_1^{(4)} = 1 \neq 0 = A_1^{(3)}$).

Lemma 3. *Assume that the recurrence is P-bounded. For $n \geq r$ and $i = 1, \ldots, r-1$, define the local minimum, maximum and gap over the sliding window of length r:*

$$m_{i,n} := \min_{1 \leq j \leq r} A_i^{(n+r-j)}, \qquad M_{i,n} := \max_{1 \leq j \leq r} A_i^{(n+r-j)}, \qquad \Delta_{i,n} := M_{i,n} - m_{i,n}.$$

Then, $(m_{i,n})_{n \geq r}$ is non-decreasing, $(M_{i,n})_{n \geq r}$ is non-increasing, $A_i^{(k)} \in [m_{i,n}, M_{i,n}]$ for all $k \geq n \geq r$, and $(\Delta_{i,n})_{n \geq r}$ is non-increasing. Moreover, $m_{i,n}$ and $M_{i,n}$ converge to finite limits m_i and M_i, respectively, with $0 \leq m_i \leq M_i$.

Proof. Let $n \geq r$. By Lemma 2, $A_i^{(n+r)}$ is a convex combination of the r terms $A_i^{(n)}, \ldots, A_i^{(n+r-1)}$ of the current window; hence, $m_{i,n} \leq A_i^{(n+r)} \leq M_{i,n}$. The window for $n+1$ consists of $A_i^{(n+1)}, \ldots, A_i^{(n+r-1)}$, which belong to the window for n, together with the new element $A_i^{(n+r)} \in [m_{i,n}, M_{i,n}]$. Therefore, $m_{i,n+1} \geq m_{i,n}$ and $M_{i,n+1} \leq M_{i,n}$, and consequently, $\Delta_{i,n+1} \leq \Delta_{i,n}$. For $k \geq n$, we have $A_i^{(k)} \in [m_{i,k}, M_{i,k}] \subseteq [m_{i,n}, M_{i,n}]$. Since $m_{i,r} \leq m_{i,n} \leq M_{i,n} \leq M_{i,r}$ and all ratios are non-negative, both monotone sequences are bounded and converge to limits m_i and M_i with $0 \leq m_i \leq M_i$. □

Lemma 4. *Assume that the recurrence is P-bounded, let C be as in Definition 1(ii), and put*

$$L := \sum_{j=2}^{r} C^j, \qquad K := \frac{1}{L+1}, \qquad \theta := 1 - K^r.$$

Then, $0 < K \leq \frac{1}{2}$ and $0 < \theta < 1$, and for all $n \geq r$ and $i = 1, \ldots, r-1$:

(a) $T_{r,1}^{(n)} \geq K$;

(b) $\Delta_{i,n+r} \leq \theta \, \Delta_{i,n}$.

Consequently, $\Delta_{i,n} \to 0$ as $n \to \infty$, that is, $m_i = M_i$.

Proof. Since $r \geq 2$ and $C \geq 1$, we have $L \geq C^2 \geq 1$, so $0 < K \leq \frac{1}{2}$ and $0 < \theta < 1$.

(a) Taking $j = 0$ in Definition 1(ii) gives $1/b_1 \leq C$, so (20) yields $U_{r,m+1}^{(r)} \geq C^{-1} U_{r,m}^{(r)}$ for $m \geq r$. Applying this $j-1$ times (all indices involved are at least $n \geq r$) gives

$$U_{r,n+r-j}^{(r)} \leq C^{j-1} \, U_{r,n+r-1}^{(r)}, \qquad j = 1, \ldots, r.$$

Since $T_{r,1}^{(n)} > 0$, using (ii) for $j \geq 2$, we obtain

$$\frac{T_{r,j}^{(n)}}{T_{r,1}^{(n)}} = \frac{b_j}{b_1} \cdot \frac{U_{r,n+r-j}^{(r)}}{U_{r,n+r-1}^{(r)}} \leq C \cdot C^{j-1} = C^j, \qquad j = 2, \ldots, r.$$

Summing over $j = 2, \ldots, r$ and using $\sum_{j=1}^{r} T_{r,j}^{(n)} = 1$ (Lemma 2) yields

$$\frac{1 - T_{r,1}^{(n)}}{T_{r,1}^{(n)}} \leq L, \qquad \text{that is,} \qquad T_{r,1}^{(n)} \geq \frac{1}{L+1} = K.$$

(b) One cannot in general expect a strict reduction of the gap in every single step: each step replaces only one element of the window, and an extreme value of the window may persist for up to $r-1$ further steps (an explicit instance is given in Section 7). We therefore follow all r terms that replace the window. Fix $n \geq r$ and i, and write $x_k := A_i^{(k)}$, $m := m_{i,n}$, $M := M_{i,n}$ and $\Delta := \Delta_{i,n}$. If $\Delta = 0$, then $\Delta_{i,n+r} \leq \Delta_{i,n} = 0$ by Lemma 3. Otherwise, write $x_{n+r-1} = m + t\Delta$ with $t \in [0,1]$. By Lemma 3, $x_k \in [m, M]$ for all $k \geq n$. We claim that for $k = 0, 1, \ldots, r-1$,

$$m + K^{k+1} t\,\Delta \;\leq\; x_{n+r+k} \;\leq\; M - K^{k+1}(1-t)\,\Delta. \tag{22}$$

For $k = 0$, Lemma 2 gives $x_{n+r} = T_{r,1}^{(n)} x_{n+r-1} + \sum_{j=2}^{r} T_{r,j}^{(n)} x_{n+r-j}$, where the last $r-1$ terms lie in $[m, M]$. Hence,

$$x_{n+r} \leq T_{r,1}^{(n)}(m + t\Delta) + \big(1 - T_{r,1}^{(n)}\big)M = M - T_{r,1}^{(n)}(1-t)\Delta \leq M - K(1-t)\Delta,$$

by (a), and symmetrically $x_{n+r} \geq T_{r,1}^{(n)}(m + t\Delta) + \big(1 - T_{r,1}^{(n)}\big)m \geq m + Kt\Delta$. Suppose that (22) holds for some $k-1$ with $1 \leq k \leq r-1$. By Lemma 2 applied at $n+k \geq r$, $x_{n+r+k} = T_{r,1}^{(n+k)} x_{n+r+k-1} + \sum_{j=2}^{r} T_{r,j}^{(n+k)} x_{n+r+k-j}$, where again the last $r-1$ terms lie in $[m, M]$. Using the induction hypothesis for $x_{n+r+k-1}$ and (a),

$$x_{n+r+k} \leq T_{r,1}^{(n+k)}\big(M - K^k(1-t)\Delta\big) + \big(1 - T_{r,1}^{(n+k)}\big)M \leq M - K^{k+1}(1-t)\Delta,$$

and likewise $x_{n+r+k} \geq m + K^{k+1} t\Delta$. This proves (22). Since $0 < K \leq 1$, we have $K^{k+1} \geq K^r$ for $k = 0, \ldots, r-1$, so the new window $\{x_{n+r}, \ldots, x_{n+2r-1}\}$ lies in the interval $\big[m + K^r t\Delta,\; M - K^r(1-t)\Delta\big]$, whose length is $(1-K^r)\Delta$. Hence, $\Delta_{i,n+r} \leq \theta\,\Delta_{i,n}$.

Finally, iterating (b) gives $\Delta_{i,r+qr} \leq \theta^q \Delta_{i,r}$ for $q \geq 0$, and since $(\Delta_{i,n})$ is non-increasing by Lemma 3, $\Delta_{i,n} \to 0$. As $\Delta_{i,n} \to M_i - m_i$, we conclude that $m_i = M_i$. □

Proof of Theorem 1. (a) Since $A_i^{(n)}$ belongs to the window for n, we have $m_{i,n} \leq A_i^{(n)} \leq M_{i,n}$ for $n \geq r$. By Lemmas 3 and 4, $m_{i,n}$ and $M_{i,n}$ converge to the common limit $m_i = M_i$, so by the squeeze theorem, $A_i^{(n)}$ converges to $A_i^* := m_i = M_i$.

(b) For $x > 0$, the function $x^{-r} f(x) = 1 - \sum_{j=1}^{r} b_j x^{-j}$ is continuous and strictly increasing (because $b_1 > 0$ and $b_j \geq 0$), tending to $-\infty$ as $x \to 0^+$ and to 1 as $x \to \infty$. Hence, f has exactly one positive root ρ, and $f(x) < 0$ for $0 < x < \rho$, $f(x) > 0$ for $x > \rho$.

Put $\lambda := A_{r-1}^* + b_1 \geq b_1 > 0$. Letting $n \to \infty$ in (19) (all denominators are positive and converge to λ) and setting $A_0^* := 0$, $A_r^* := 1$, we obtain

$$\lambda\, A_s^* = A_{s-1}^* + b_{r+1-s}, \qquad s = 1, \ldots, r, \tag{23}$$

where the case $s = r$ is the definition of λ. Let $q(x) := \sum_{s=1}^{r} A_s^* x^{s-1}$. Using (23),

$$(x-\lambda)\,q(x) = \sum_{s=1}^{r} A_s^* x^s - \sum_{s=1}^{r} A_{s-1}^* x^{s-1} - \sum_{s=1}^{r} b_{r+1-s} x^{s-1} = A_r^* x^r - A_0^* - \sum_{j=1}^{r} b_j x^{r-j} = f(x).$$

Thus, λ is a positive root of f, so $\lambda = \rho$ and $q(x) = f(x)/(x-\rho)$. Solving (23) recursively from $A_0^* = 0$ gives $A_i^* = \sum_{s=1}^{i} b_{r+1-s}\,\rho^{-(i-s+1)}$, which is (21) after the substitution $j = r+1-s$. The term $j = r$ of (21) equals $b_r \rho^{-i} > 0$, so $A_i^* > 0$.

(c) For $n \geq r$, we have $m_{i,n} \leq A_i^* \leq M_{i,n}$ by monotonicity, and $A_i^{(k)} \in [m_{i,n}, M_{i,n}]$ for $k \geq n$ by Lemma 3; hence, $|A_i^{(k)} - A_i^*| \leq \Delta_{i,n}$. With $q := \lfloor (n-r)/r \rfloor$, we have $r + qr \leq n$, so Lemmas 3 and 4 give $\Delta_{i,n} \leq \Delta_{i,r+qr} \leq \theta^q \Delta_{i,r}$. □

Remark 4. *Under P-boundedness, ρ is the dominant root of f: every other root z satisfies $|z| < \rho$. Indeed, $z \neq 0$ by (H2), and $|z|^r = \left|\sum_{j=1}^{r} b_j z^{r-j}\right| \leq \sum_{j=1}^{r} b_j |z|^{r-j}$ means $f(|z|) \leq 0$, whence $|z| \leq \rho$. If $|z| = \rho$, equality holds in the triangle inequality, so every non-zero term $b_j z^{r-j}$ has the same argument as z^r; applying this to $b_1 z^{r-1}$ shows that z is a positive real number; hence, $z = \rho$. Consequently, labeling the roots so that $\alpha_{r,1} = \rho$, part (b) of Theorem 1 can also be read off from the Binet-type formula (3): by Lemma 1 and Lagrange interpolation, $U_{s,n}^{(r)} = \sum_{k=1}^{r} (-1)^{r-s} e_{r-s}\left(\{\alpha_{r,l}\}_{l \neq k}\right) \alpha_{r,k}^{n-1} / f'(\alpha_{r,k})$, where e_m denotes the m-th elementary symmetric function, and therefore*

$$A_i^* = (-1)^{r-i} e_{r-i}(\alpha_{r,2}, \ldots, \alpha_{r,r}), \qquad i = 1, \ldots, r-1,$$

which agrees with (21), since $f(x)/(x-\rho) = \prod_{k=2}^{r}(x - \alpha_{r,k})$. The proof given above avoids any information on the location of the roots.

We can now state precisely how the limit A^* is related to the Jacobi–Perron algorithm of Section 5.

Proposition 1. *Assume that $F \subseteq \mathbb{R}$, that the recurrence (12) is P-bounded, and that $b_r = 1$, i.e., $(-1)^{r+1} P_{r,r} = 1$. Let T be the T-transformation of Section 5 associated with the constant vector function*

$$G \equiv g := (b_{r-1}, b_{r-2}, \ldots, b_1) \in \mathbb{E}^{r-1}, \qquad \text{i.e., } g_k = b_{r-k}, \quad k = 1, \ldots, r-1,$$

and, for $n \geq r+1$, define the normalized ratio vectors

$$\widehat{A}^{(n)} := \left(\frac{U_{2,n}^{(r)}}{U_{1,n}^{(r)}}, \frac{U_{3,n}^{(r)}}{U_{1,n}^{(r)}}, \ldots, \frac{U_{r,n}^{(r)}}{U_{1,n}^{(r)}} \right) \in \mathbb{E}^{r-1}.$$

Then, the following apply:

(a) *$\widehat{A}^{(n)}$ is well defined for every $n \geq r+1$, $\widehat{A}_1^{(n+1)} \neq g_1$, and $\widehat{A}^{(n+1)} T = \widehat{A}^{(n)}$ for every $n \geq r+1$;*

(b) *$\widehat{A}^{(n)} \to \widehat{A}^* := \left(A_2^*/A_1^*, \ldots, A_{r-1}^*/A_1^*, 1/A_1^*\right) = A^* + g$, one has $\widehat{A}_1^* \neq g_1$, and $\widehat{A}^* T = \widehat{A}^*$.*

Consequently, the constant sequence $a^{(k)} := \widehat{A}^*$ $(k \geq 0)$ is a Jacobi–Perron algorithm for the initial vector $\widehat{A}^*$ in the sense of Section 5; it is purely periodic with period one.

Proof. (a) By Lemma 1 with $b_r = 1$, $U_{1,n}^{(r)} = U_{r,n-1}^{(r)} > 0$ for $n \geq r+1$, so $\widehat{A}^{(n)}$ is well defined. Again, by Lemma 1, for $n \geq r$ and $k = 1, \ldots, r-1$,

$$\widehat{A}_k^{(n+1)} = \frac{U_{k+1,n+1}^{(r)}}{U_{1,n+1}^{(r)}} = \frac{U_{k,n}^{(r)} + b_{r-k} U_{r,n}^{(r)}}{U_{r,n}^{(r)}} = A_k^{(n)} + g_k. \tag{24}$$

In particular, $\widehat{A}_1^{(n+1)} - g_1 = A_1^{(n)} = U_{1,n}^{(r)}/U_{r,n}^{(r)} > 0$ for $n \geq r+1$, and

$$\widehat{A}^{(n+1)} T = \frac{(A_2^{(n)}, \ldots, A_{r-1}^{(n)}, 1)}{A_1^{(n)}} = \frac{(U_{2,n}^{(r)}, \ldots, U_{r-1,n}^{(r)}, U_{r,n}^{(r)})}{U_{1,n}^{(r)}} = \widehat{A}^{(n)}.$$

(b) Since $\widehat{A}^{(n)} = \left(A_2^{(n)}, \ldots, A_{r-1}^{(n)}, 1\right)/A_1^{(n)}$ and $A_1^{(n)} \to A_1^* > 0$ by Theorem 1, we obtain $\widehat{A}^{(n)} \to \left(A_2^*, \ldots, A_{r-1}^*, 1\right)/A_1^*$, while (24) shows that the same limit equals $A^* + g$. Hence, $\widehat{A}_1^* - g_1 = A_1^* > 0$, and

$$\widehat{A}^* T = \frac{\left(\widehat{A}_2^* - g_2, \ldots, \widehat{A}_{r-1}^* - g_{r-1}, 1\right)}{\widehat{A}_1^* - g_1} = \frac{\left(A_2^*, \ldots, A_{r-1}^*, 1\right)}{A_1^*} = \widehat{A}^*.$$

The final assertion is immediate from the definition of a JPA. □

Remark 5. *(1) For $r = 2$ and $b_2 = 1$, Proposition 1 reduces to the classical fact that $\widehat{A}^* = 1/A_1^* = \rho$ satisfies $\rho = 1/(\rho - b_1)$, i.e., ρ has the purely periodic continued fraction expansion $[b_1; b_1, b_1, \ldots]$; for $b_1 = 1$, this is the golden ratio.*

(2) If, in addition, the b_j values are non-negative integers and $0 \le A_k^* < 1$ for $k = 1, \ldots, r-1$, then the identity $\widehat{A}^* = A^* + g$ shows that g is the vector of integer parts of the coordinates of $\widehat{A}^*$. In this case, the algorithm of Proposition 1 is the classical Jacobi–Perron algorithm of $\widehat{A}^*$ with integer parts as partial quotients. This is the situation of the example in Section 7.

(3) The hypothesis $b_r = 1$ reflects the fact that in the recurrence satisfied by the convergents of the Jacobi–Perron algorithm, the coefficient of the oldest term equals 1 (see [1,3]). If $b_r \neq 1$, the same computation with $g_k = b_{r-k}/b_r$ shows that $\widehat{A}^{(n+1)}T$ coincides with $\widehat{A}^{(n)}$ except that its last coordinate is multiplied by b_r, so that the ratio vectors are no longer related by a T-transformation in the strict sense.

7. Numerical Illustration: The Three-Dimensional Case ($r = 3$)

To illustrate Theorem 1 and Proposition 1 explicitly, consider the three-dimensional case $r = 3$ with the generalized Tribonacci sequence. Take the auxiliary equation parameters $P_{3,1} = 1$, $P_{3,2} = -1$, $P_{3,3} = 1$, i.e.,

$$x^3 - x^2 - x - 1 = 0.$$

Its discriminant equals $-44 \neq 0$ and $P_{3,3} \neq 0$, so (H1) and (H2) hold. By (15),

$$b_1 = P_{3,1} = 1, \qquad b_2 = -P_{3,2} = 1, \qquad b_3 = P_{3,3} = 1,$$

so that all $b_j = 1 > 0$ (condition (i) of Definition 1) and, with $b_0 := 1$, $b_j/b_1 = 1 \le C$ for $j = 0, 2, 3$ with $C = 1$ (condition (ii)). Thus, the recurrence is P-bounded and Theorem 1 applies; since, moreover, $b_3 = 1$, Proposition 1 applies as well.

By (4), the basic sequences $U_{s,n}^{(3)}$, $s \in \{1, 2, 3\}$, satisfy

$$U_{s,n+3}^{(3)} = U_{s,n+2}^{(3)} + U_{s,n+1}^{(3)} + U_{s,n}^{(3)},$$

with $U_{s,n}^{(3)} = \delta_{s,n}$ for $n \in \{1, 2, 3\}$. Direct computation gives Table 1.

As a consistency check, the sums $S_n := U_{1,n}^{(3)} + U_{2,n}^{(3)} + U_{3,n}^{(3)}$ satisfy the same order-3 recurrence (since it is linear); for instance, $S_7 = 17 = S_4 + S_5 + S_6 = 3 + 5 + 9$ and $S_{13} = 653 = S_{10} + S_{11} + S_{12} = 105 + 193 + 355$. One also checks the one-step relations (17) directly; e.g., for $n = 12$, $U_{1,13}^{(3)} = 149 = U_{3,12}^{(3)}$, $U_{2,13}^{(3)} = 230 = 81 + 149$ and $U_{3,13}^{(3)} = 274 = 125 + 149$.

The ratio vectors are $A^{(n)} = (A_1^{(n)}, A_2^{(n)})$ with $A_1^{(n)} = U_{1,n}^{(3)}/U_{3,n}^{(3)}$ and $A_2^{(n)} = U_{2,n}^{(3)}/U_{3,n}^{(3)}$, $n \ge 3$. Table 2 lists them, in lowest terms, together with the window gaps $\Delta_{i,n}$ of Lemma 3, which are computed over the windows $\{A_i^{(n)}, A_i^{(n+1)}, A_i^{(n+2)}\}$ for $3 \le n \le 11$.

Table 1. Values of $U_{1,n}^{(3)}, U_{2,n}^{(3)}, U_{3,n}^{(3)}$, for $1 \le n \le 13$.

n	$U_{1,n}^{(3)}$	$U_{2,n}^{(3)}$	$U_{3,n}^{(3)}$
1	1	0	0
2	0	1	0
3	0	0	1
4	1	1	1
5	1	2	2
6	2	3	4
7	4	6	7
8	7	11	13
9	13	20	24
10	24	37	44
11	44	68	81
12	81	125	149
13	149	230	274

Table 2. Ratio vectors $A^{(n)}$ and window gaps $\Delta_{i,n}$ over the windows $\{A_i^{(n)}, A_i^{(n+1)}, A_i^{(n+2)}\}$ for the Tribonacci example.

n	$A_1^{(n)}$	$\Delta_{1,n}$	$A_2^{(n)}$	$\Delta_{2,n}$
3	0	1	0	1
4	1	1/2	1	1/4
5	1/2	1/14	1	1/4
6	1/2	1/14	3/4	3/28
7	4/7	3/91	6/7	1/42
8	7/13	1/143	11/13	1/78
9	13/24	1/264	5/6	1/132
10	6/11	2/891	37/44	13/6556
11	44/81	13/22,194	68/81	7/12,069
12	81/149	—	125/149	—
13	149/274	—	115/137	—

With $C = 1$ and $r = 3$, Lemma 4 gives $L = C^2 + C^3 = 2$, $K = 1/3$ and $\theta = 1 - K^3 = 26/27$. Table 2 shows that the gap need not decrease in every single step: $\Delta_{1,5} = \Delta_{1,6} = 1/14$ and $\Delta_{2,4} = \Delta_{2,5} = 1/4$, whereas a one-step contraction with factor $1 - K$ would force $\Delta_{1,6} \le \frac{2}{3}\Delta_{1,5}$. The reason is visible in the windows: for $i = 1$, the windows $\{1/2, 1/2, 4/7\}$ (for $n = 5$) and $\{1/2, 4/7, 7/13\}$ (for $n = 6$) share both extreme values $1/2 = A_1^{(6)}$ and $4/7 = A_1^{(7)}$, and the new element $A_1^{(8)} = 7/13$ lies strictly between them, so replacing $A_1^{(5)}$ by $A_1^{(8)}$ leaves the gap unchanged. In contrast, the r-step estimate $\Delta_{i,n+3} \le \frac{26}{27}\Delta_{i,n}$ of Lemma 4 holds throughout; for instance,

$$\Delta_{1,6} = \tfrac{1}{14} \le \tfrac{26}{27}\Delta_{1,3} = \tfrac{26}{27}, \qquad \Delta_{1,7} = \tfrac{3}{91} \le \tfrac{26}{27}\Delta_{1,4} = \tfrac{13}{27},$$
$$\Delta_{1,8} = \tfrac{1}{143} \le \tfrac{26}{27}\Delta_{1,5} = \tfrac{13}{189}, \qquad \Delta_{2,8} = \tfrac{1}{78} \le \tfrac{26}{27}\Delta_{2,5} = \tfrac{13}{54}.$$

In fact, $\Delta_{i,n+3}/\Delta_{i,n} \le 0.11$ for $i = 1, 2$ and $3 \le n \le 8$, so the constant $\theta = 26/27$ is far from sharp.

The unique positive root of $x^3 - x^2 - x - 1$ is the Tribonacci constant $\tau \approx 1.839287$; by Remark 4, it is dominant, the other two roots being complex conjugates of modulus $\tau^{-1/2} \approx 0.737353$ (their product with τ equals $P_{3,3} = 1$). Formula (21) gives

$$A_1^* = \frac{b_3}{\tau} = \frac{1}{\tau} \approx 0.543689, \qquad A_2^* = \frac{b_2}{\tau} + \frac{b_3}{\tau^2} = \frac{1}{\tau} + \frac{1}{\tau^2} = \tau - 1 \approx 0.839287,$$

where the last equality follows from $\tau^3 = \tau^2 + \tau + 1$. Thus, A_1^* is the reciprocal of τ, whereas $A_2^* = \tau - 1$ is not a single power of τ^{-1}. Numerically, $A_1^{(13)} = 149/274 \approx 0.543796$ and $A_2^{(13)} = 115/137 \approx 0.839416$, which differ from the limits by approximately 1.07×10^{-4} and 1.29×10^{-4}, respectively, which is in accordance with the bound $|A_i^{(13)} - A_i^*| \le \Delta_{i,11}$ of Theorem 1(c) (here $\Delta_{1,11} \approx 5.86 \times 10^{-4}$ and $\Delta_{2,11} \approx 5.80 \times 10^{-4}$).

Finally, Proposition 1 applies with $g = (b_2, b_1) = (1, 1)$. The normalized limit is

$$\widehat{A}^* = \left(\frac{A_2^*}{A_1^*}, \frac{1}{A_1^*} \right) = (\tau^2 - \tau,\ \tau) \approx (1.543689,\ 1.839287),$$

and, since $\tau^2 - \tau - 1 = \tau^{-1}$, one verifies directly that

$$\widehat{A}^* T = \frac{(\tau - 1,\ 1)}{\tau^2 - \tau - 1} = (\tau^2 - \tau,\ \tau) = \widehat{A}^*.$$

Moreover, $\widehat{A}^* = A^* + g$ with $0 < A_1^*, A_2^* < 1$, so $g = (1, 1)$ is the vector of integer parts of $\widehat{A}^*$, and, by Remark 5(2), the classical Jacobi–Perron algorithm of $(\tau^2 - \tau, \tau)$ is purely periodic with period one. At a finite stage, $\widehat{A}^{(13)} = (230/149,\ 274/149)$ and

$$\widehat{A}^{(13)} T = \frac{(274/149 - 1,\ 1)}{230/149 - 1} = \left(\frac{125}{81}, \frac{149}{81} \right) = \widehat{A}^{(12)},$$

in agreement with Proposition 1(a).

The example thus exhibits all ingredients of Section 6 explicitly: the P-boundedness hypotheses, the failure of a one-step contraction together with the validity of the r-step contraction, the explicit limit, and its interpretation as the initial vector of a purely periodic Jacobi–Perron algorithm.

8. Conclusions

In this paper, we have explored the algebraic and structural connections between multidimensional continued fractions, specifically the Jacobi–Perron algorithm, and generalized Fibonacci and Lucas sequences of arbitrary order r. Using Vandermonde determinants and Binet-type formulas, we established linear recurrences and matrix representations elucidating the mechanics of these recursive sequences with the domains of all parameters and the index conventions relating the various families of sequences stated explicitly.

The matrix recurrence relations of Section 4 transfer the scalar evolutionary properties of the sequences into the matrix algebra generated by the companion matrix M. Under the P-boundedness hypothesis of Definition 1, Theorem 1 shows that the ratio vectors of the basic sequences converge, with a geometric contraction of their oscillation over blocks of length r, and identifies the limit explicitly in terms of the unique positive root of the auxiliary equation. Proposition 1 makes the connection with the Jacobi–Perron algorithm precise: when the coefficient of the oldest term of the recurrence equals one, suitably normalized ratio vectors are related by a T-transformation, and their limit is the initial vector of a purely periodic Jacobi–Perron algorithm of period one. The three-dimensional example of Section 7 illustrates each of these statements numerically.

For future research, the algebraic coupling of companion matrices and generalized recurrences could be extended to stability bounds in discrete dynamical systems, and the spectral properties of these sequences within the theory of k-circulant matrices (cf. [25,26]) could yield further insight into higher-order matrix determinants and block-matrix structures.

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