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Truncated Lucas sequence and its period

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ABSTRACT

We prove some theorems concerning the truncated Lucas sequences. We also state some conjectures concerning the period (mod m) of truncated Lucas sequences, where the integer m is at least 2. In addition, we present some computer-generated results that tend to support these conjectures.

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1. Introduction

We let s_n denote the n^{th} member of the sequence of integers $s_0 = a, s_1 = b, \ldots$, where $s_{n+1} = s_n + s_{n-1}$. A sequence $\{s_i\}$ is said to be *periodic with period k* if it satisfies $s_i = s_{i+k}$ for $i = 1, 2, \ldots$. The symbol $h(m)$ will denote the length of the period of the sequence resulting from reducing each s_n modulo m. An n-step Fibonacci sequence $\{f_k^{(n)}\}_{k=1}^{\infty}$ is defined by letting $f_k^{(n)} = 0$ for $k \leqslant 0$, $f_1^{(n)} = 1$, $f_2^{(n)} = 1$, and other terms according to the linear recurrence equation $f_k^{(n)} = \sum_{i=1}^{n} f_{k-i}^{(n)}$ for $k > 2$. The basic (2-step) Fibonacci sequence will be given by $f_0 = 0, f_1 = 1, \ldots$ and the Lucas sequence by $g_0 = 2, g_1 = 1, \ldots$. The symbol $k(m)$ will denote the length of the period of the 2- step or 3-step Fibonacci sequences when they are reduced modulo m. We sometimes call it Wall number after a pioneer in this area. We know that $g_n = f_{n-1} + f_{n+1}, n \geqslant 2$. So the Wall number of the Fibonacci sequence is equal to the Wall number of the Lucas sequence.

For example, let f_n be the 2-step Fibonacci sequence and $m = 5$. If we reduce each f_n modulo 5, then we obtain the following periodic sequence.

0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1,... and then repeat; so $k(5) = 20$.

In the recent years, there has been much interest in applications of Fibonacci and Lucas numbers and sequences. Fibonacci and Lucas sequences appear in many branches of mathematics. These include group theory, calculus, applied mathematics, linear algebra, etc. We can see applications of Fibonacci sequences in group theory in [2,5,8,10] and also see some generalized Fibonacci and Lucas sequences in [9,15,1,3,7,13]. Takahashi gives a fast algorithm which is based on the product of Lucas numbers to compute large Fibonacci numbers [14]. Özkan has proved two theorems concerning the Wall number of the 3-step Fibonacci sequences and given some conjectures concerning $k(m)$ of the 3-step Fibonacci sequence [12]. We will give a few of them in Conjecture 1.3. In [6], Choi et al. introduced the truncated Fibonacci sequences and the truncated Lucas sequences. In [11], the author proved some theorems on truncated Fibonacci sequences and stated some conjectures on their Wall number.

Let $k_r(m)$ denote the Wall number of truncated Fibonacci sequences (defined in Section 2). We will let Z and Z_m represent the group of integers and the group of integers modulo m, respectively.

In this paper, we give some interesting properties of the truncated Lucas sequences and we show that the function $k_r(m)$ associated with this sequence is a multiplicative function. Furthermore, we examine the relationship between m and $k_r(m)$ of the truncated Lucas sequences. We also give results similar to the following theorems. We will give them in Section 3.

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For the 2-step Fibonacci sequences, we gather some of the well-known properties of $k(m)$ into the following two theorems.

Theorem 1.1 [16,17].

i. *(Wall): If m has the prime factorization* $m = \prod p_i^{e_i}$ *and if* $h_i = k(p_i^{e_i})$ *then* $k(m)$ *is the least common multiple of the* h_i.
ii. *(Wall): If* $k(p) \neq k(p^2)$, *then* $k(p^e) = p^{e-1}k(p)$. *Also, if n is the largest integer with* $k(p) = k(p^n)$, *then* $k(p^e) = p^{e-1}k(p)$ *for* $e > n$.
iii. *(Wilcox): For* $m > 2$, $k(m) = 2n$, *where* $n = \min(\{n : n \text{ even and } m|f_n\} \cup \{n : n \text{ odd and } m|g_n\})$.

Remark. In all known cases, $k(p^2) = pk(p)$ for p a prime, and it appears to be an open question whether the case $k(p^2) = k(p)$ ever occurs.

Theorem 1.2 [16]. *For the 2-step Fibonacci sequences, the following hold:*

i. *If p is a prime and* $p = 10x \pm 1$, *then* $k(p)|(p-1)$.
ii. *If p is a prime and* $p = 10x \pm 3$, *then* $k(p)|(2p+2)$.
iii. *If p is a prime and* $p = 10x \pm 3$, *then* $h(p^e) = k(p^e)$.

Corollary [16]. *If* $p = 10x \pm 3$, *then* $k(p) \equiv 0(mod 4)$.

Let f_n be the 3-step Fibonacci sequence. Then, $f_n(modm)$ forms *a simply periodic sequence*, that is, the sequence is periodic and repeats by returning to its starting values.

The following conjectures on $k(m)$ appears in [12] ((i)–(iii) concern 3-step Fibonacci sequences). (Note that (iv) does not concern 3-step Fibonacci sequences.)

Conjecture 1.3 [12].

i. *If p is a prime number and* $k(p) < p$, *then* $k(p)|(p-1)$.
ii. *If p is a prime number and* $k(p) \equiv 0(mod 2)$, *then* $(p \pm 1)|k(p)$.
iii. *If p is a prime number and* $k(p) \equiv 1(mod 2)$, *then* $(p \pm 1)|(k(p)-1)$ *or* $p|(k(p)-1)$.
iv. *If p is an odd prime number and n is an even number greater than 2, then the Wall number k(p) of the n-step Fibonacci sequence is even.*

2. Truncated Lucas sequence

Throughout this paper, let $\mathbb{R}^m$ denote the infinite dimensional real vector space consisting of all real sequences $(x_0, x_1, x_2, \ldots)^{\mathrm{T}}$, and let P denote The Pascal matrix $[(\binom{i}{j})], (i,j = 0,1,2,\ldots)$.

Associated with the Pascal matrix P, let $P^{\rightarrow}P^{\downarrow}$ denote the matrices defined by

$$P^{\rightarrow} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & . & . \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & . & & . & \\ . & . & . & . & . & . & . & . & & & . \end{bmatrix},$$

where the row i, $(i = 0,1,2,\ldots)$, is that of the Pascal matrix P preceded by 0_i^{T}, where 0_i is the i-vector of zeros, and

$$P^{\downarrow} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & . & . & . \\ 1 & 2 & 0 & 0 & . & . & . \\ 1 & 3 & 1 & 0 & . & . & . \\ 1 & 4 & 3 & 0 & . & . & . \\ 1 & 5 & 6 & 1 & . & . & . \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix},$$

where the column j, $(j = 0, 1, 2, \ldots)$, is that of P preceded by 0_j^{T}.

$$\text{Let} \quad Q = P + \begin{bmatrix} 1 & 0^{\mathrm{T}} \\ 0 & P \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 2 & 0 & 0 & 0 & . & . & . \\ 1 & 3 & 2 & 0 & 0 & . & . & . \\ 1 & 4 & 5 & 2 & 0 & . & . & . \\ 1 & 5 & 9 & 7 & 2 & . & . & . \\ . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & \\ . & . & . & . & . & . & . & . \end{bmatrix} = \begin{bmatrix} q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,4} & . & . & . \\ q_{1,0} & q_{1,1} & q_{1,2} & q_{1,3} & q_{1,4} & . & . & . \\ q_{2,0} & q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} & . & . & . \\ q_{3,0} & q_{3,1} & q_{3,2} & q_{3,3} & q_{3,4} & . & . & . \\ q_{4,0} & q_{4,1} & q_{4,2} & q_{4,3} & q_{4,4} & . & . & . \\ . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & \\ . & . & . & . & . & . & . & . \end{bmatrix}.$$

Associated with Q, let $Q^{\rightarrow}, Q^{\downarrow}$ denote the matrices defined by

$$Q^{\rightarrow} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 3 & 2 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & 4 & 5 & 2 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 1 & 5 & 9 & 7 & 2 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & . & . & . & . & \end{bmatrix},$$

where the row i is that of Q preceded by 0_i^{T}, $(i = 0, 1, 2, \ldots)$ and

$$Q^{\downarrow} = \begin{bmatrix} 2 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 1 & 2 & 0 & 0 & . & . & . \\ 1 & 3 & 0 & 0 & . & . & . \\ 1 & 4 & 2 & 0 & . & . & . \\ 1 & 5 & 5 & 0 & . & . & . \\ 1 & 6 & 9 & 2 & . & . & . \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix},$$

where the column j is that of Q preceded by 0_j^{T}, $(j = 0, 1, 2, \ldots)$.

The sequence $L = (l_0, l_1, l_2, \ldots)^{\mathrm{T}}$ defined by $l_0 = 2, l_1 = 1, l_n = l_{n-1} + l_{n-2}, (n \geqslant 2)$ is called the Lucas sequence.

There are some generalizations of the notion of Fibonacci or Lucas sequences (see [6,4] for example). Furthermore, in [6], Choi et al. present the following generalization of Lucas sequences. We are interested in the generalization of Lucas sequences.

Suppose that we arrange n congruent cubic blocks in such a way that the blocks are arranged in one or two rows and each of the blocks in the top row (top blocks) is placed on a block in the bottom row(base block). They call such an arrangement a *Fibonacci arrangement* of n blocks. Let F_n denote the set of all Fibonacci arrangements of n blocks. It is well known that $|F_n|$, the number of elements of F_n, is equal to $\binom{n}{0} + \binom{n-1}{1} + \ldots + \binom{1}{n-1} + \binom{0}{n} = f_{n+1}$, the $(n+1)^{th}$ Fibonacci number. For a nonnegative integer r, let $F_{r,n}$ denote the set of all Fibonacci arrangements of n blocks with the *'truncation'* condition that the number of top blocks is $\leqslant r$.

In what follows we assume, for an integer i and for a nonnegative integer t, that $\binom{i}{t} = 0$ if $i < t$. Let $F_r = (f_{r,0}, f_{r,1}, f_{r,2}, \ldots)^{\mathrm{T}}$ be defined by $f_{r,0} = 0, f_{r,1} = 1, f_{r,n} = |F_{r,n-1}|$ for $n \geqslant 2$. Let t be the number of top blocks in an arrangement in $F_{r,n}$. Then the number of base blocks in that arrangement is $n - t$, and hence there are $\binom{n-t}{t}$ ways that the t top blocks can be placed. Since $0 \leqslant k \leqslant r$, we have $f_{r,n+1} = \sum_{t=0}^{r} \binom{n-t}{t}$. We see that $f_{r,n+1}$ is equal to the sum of the entries of P lying on the line $i + j = n$ within the range $0 \leqslant j \leqslant r$, and hence that F_r is the row sum vector of $\begin{bmatrix} 0^{\mathrm{T}} \\ \mathrm{P}^{\downarrow}[0, 1, 2, \ldots | 0, 1, \ldots, r] \end{bmatrix}$. They call F_r the r^{th} *truncated Fibonacci sequence*.

Let $Q = [q_{i,j}]$ and let $g_{r,n} = \sum_{t=0}^{r} q_{n-t,t}$. Then $g_{r,n}$ is equal to the sum of the entries of Q lying on the line $i + j = n$ within the range $0 \leqslant j \leqslant r$.

Notice that $q_{i,j} = q_{i-1,j} + q_{i-1,j-1}$. In the sequel, we assume, for integers i,j, that $q_{i,j} = 0$ if either $i < 0$ or $j < 0$.

Let $L_r = (g_{r,0}, g_{r,1}, g_{r,2}, \ldots)^T$, then L_r is the row sum vector of $Q^{\downarrow}[0,1,2,\ldots|0,1,\ldots,r]$. They call L_r the rth *truncated Lucas sequence*.

For some r and n, the values of $L_{r,n}$ are given as follows.

Theorem 2.1. $l_{r,n+2} = l_{r-1,n} + l_{r,n+1}$.

Proof. By the definition of the numbers $l_{r,n}$, we have

$$l_{r,n+2} = \sum_{t=o}^{r} q_{n+2-t,t} = q_{n+2,0} + q_{n+1,1} + q_{n,2} + \cdots + q_{n+2-r,r},$$

and

$$\begin{aligned} l_{r-1,n} + l_{r,n+1} &= \sum_{t=o}^{r-1} q_{n-t,t} + \sum_{t=o}^{r} q_{n+1-t,t} = q_{n,0} + q_{n-1,1} + q_{n-2,2} + \cdots + q_{n-r+1,r-1} + q_{n+1,0} + q_{n,1} + q_{n-1,2} + \cdots + q_{n-r+1,r} \\ &= q_{n+1,0} + (q_{n,0} + q_{n,1}) + (q_{n-1,1} + q_{n-1,2}) + (q_{n-2,2} + q_{n-2,3}) + \cdots + (q_{n-r+1,r-1} + q_{n-r+1,r}) \\ &= q_{n+1,0} + q_{n+1,1} + q_{n,2} + \cdots .. + q_{n+2-r,r}, \end{aligned}$$

and since $q_{n+2,0} = q_{n+1,0} = 1$ we obtain

$$l_{r-1,n} + l_{r,n+1} = q_{n+2,0} + q_{n+1,1} + q_{n,2} + \ldots + q_{n+2-r,r} = l_{r,n+2}. \qquad \square$$

Result 1. $l_{r,0} = 2$.

Proof. $l_{r,0} = \sum_{t=o}^{r} q_{-t,t} = q_{0,0} + q_{-1,1} + q_{-2,2} + \ldots + q_{-r,r} = q_{0,0} = 2$. $\square$

Result 2. $l_{1,n} = n + 1, n > 1$.

Proof. $l_{1,n} = \sum_{t=o}^{1} q_{n-t,t} = q_{n,0} + q_{n-1,1} = q_{n,0} + q_{n-2,0} + q_{n-2,1} = q_{n,0} + q_{n-2,0} + q_{n-3,0} + q_{n-3,1}$.

If we continue similarly, then we obtain

$$l_{1,n} = \sum_{t=o}^{1} q_{n-t,t} = \underbrace{q_{n,0} + q_{n-2,0} + q_{n-3,0} + \cdots + q_{1,0}}_{(n-1)terms} + q_{0,0} + q_{0,1} = n - 1 + 2 + 0 = n + 1. \qquad \square$$

Result 3. $l_{0,n} = 1, n > 0$.

Proof. $f_{0,n} = \sum_{t=0}^{0} \binom{n-1-t}{k} = \binom{n-1}{0} = 1$. $\square$

We let $l_{r,n+1} = l^{(0)}_{r,n+1}$ and for $n \geqslant r$, $i \leqslant r-1$ and $i,r,n \in \mathbb{Z}^+$, $l^{(i)}_{r,n+1} = l^{(i-1)}_{r,n+1} - l^{(i-1)}_{r,n}$. For example, $l^{(1)}_{r,n} = l^{(0)}_{r,n} - l^{(0)}_{r,n-1}$, for $r = 2$ $n = 5$ $l^{(1)}_{2,5} = l^{(0)}_{2,5} - l^{(0)}_{2,4} = l_{2,5} - l_{2,4} = 11 - 7 = 4$.

Theorem 2.2. $l^{(r-1)}_{r,n} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} l^{(0)}_{r,n-i} (n \geqslant 2r)$.

Table 1
The values of $L_{r,n}$ for some r and n.

$L_{r,n}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$	$n=15$	$n=16$	$n=17$	$n=18$	$n=19$
$r=0$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$r=1$	2	1	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$r=2$	2	1	3	4	7	11	16	22	29	37	46	56	67	79	92	106	121	137	154	172
$r=3$	2	1	3	4	7	11	18	29	45	67	96	133	179	235	302	381	473	579	700	837
$r=4$	2	1	3	4	7	11	18	29	47	76	121	188	284	417	596	831	888	1133	1514	2093

Proof. Let's prove the theorem by induction on the subscript r.

The case $r = 1$ is obvious.
For $r = k$, we suppose that $l_{k,n}^{(k-1)} = \sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}l_{k,n-i}^{(0)}$.
That is, we have

$$l_{k,n}^{(k-1)} = (-1)^0\binom{k-1}{0}l_{k,n}^{(0)} + (-1)^1\binom{k-1}{1}l_{k,n-1}^{(0)} + (-1)^2\binom{k-1}{2}l_{k,n-2}^{(0)} + \cdots + (-1)^{k-2}\binom{k-1}{k-1}l_{k,n-(k-2)}^{(0)} + (-1)^{k-1}\binom{k-1}{k-1}l_{k,n-(k-1)}^{(0)}.$$

For $r = k + 1$, let's show that

$$l_{k+1,n}^{(k)} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}l_{k+1,n-i}^{(0)} = (-1)^0\binom{k}{0}l_{k+1,n}^{(0)} + (-1)^1\binom{k}{1}l_{k+1,n-1}^{(0)} + \cdots + (-1)^{k-1}\binom{k}{k-1}l_{k+1,n-(k-1)}^{(0)} + (-1)^k\binom{k}{k}l_{k+1,n-k}^{(0)}.$$

By using the equality $l_{k+1,n}^{(k)} = l_{k+1,n}^{(k-1)} - l_{k+1,n-1}^{(k-1)}$, we obtain $l_{k+1,n}^{(k)} = l_{k+1,n}^{(k-1)} - l_{k+1,n-1}^{(k-1)}$

$$\begin{aligned}
&= (-1)^0\binom{k-1}{0}l_{k+1,n}^{(0)} + (-1)^1\binom{k-1}{1}l_{k+1,n-1}^{(0)} + (-1)^2\binom{k-1}{2}l_{k+1,n-2}^{(0)} + \cdots + (-1)^{k-2}\binom{k-1}{k-2}l_{k+1,n-(k-2)}^{(0)}\\
&\quad + (-1)^{k-1}\binom{k-1}{k-1}l_{k+1,n-(k-1)}^{(0)} - (1)^0\binom{k-1}{0}l_{k+1,n-1}^{(0)} - (-1)^1\binom{k-1}{1}l_{k+1,n-2}^{(0)} - (-1)^2\binom{k-1}{2}l_{k+1,n-3}^{(0)}\\
&\quad + \cdots - (-1)^{k-2}\binom{k-1}{k-2}l_{k+1,n-1-(k-2)}^{(0)} - (-1)^{k-1}\binom{k-1}{k-1}l_{k+1,n-1-(k-1)}^{(0)}.
\end{aligned}$$

We can write

$$(-1)^0\binom{k-1}{0}l_{k+1,n}^{(0)}(-1)^0\binom{k}{0}l_{k+1,n}^{(0)}, \quad -(-1)^{k-1}\binom{k-1}{k-1}l_{k+1,n-1-(k-1)}^{(0)} = (-1)^k\binom{k}{k}l_{k+1,n-k}^{(0)},$$

and

$$\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}.$$

Together with the last three equalities, the proof is completed. □

Notice that, if $n < 2r$ the latest term of $l_{r,n}^{(r-1)} = \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}l_{r,n-i}^{(0)}$ is $(-1)^{r-1}\binom{r-1}{r-1}l_{r,n-r+1}^{(0)}$ is not defined. Because, for $n \geqslant r$, $i \leqslant r-1$ and $i,r,n \in \mathbb{Z}^+$, $l_{r,n+1} = l_{r,n+1}^{(0)}$. That is, if $n < 2r$, $n - r + 1 < r + 1$.

Theorem 2.3. *We let* $l_{r,n} = l_{r,n}^{(0)}$ *and* $l_{r,n}^{(i)} = l_{r,n}^{(i-1)} - l_{r,n-1}^{(i-1)}$ *for* $n \geqslant r$, $i \leqslant r-1$ *and* $i,r,n \in \mathbb{Z}^+$. *Then* $(l_{r,2r}^{(r-1)}, l_{r,2r+1}^{(r-1)}, l_{r,2r+2}^{(r-1)}, \ldots, l_{r,2r+s}^{(r-1)}, \ldots) = (3,4,5,\ldots,s+3,\ldots)$.

Proof. Notice that,

$$\left(l_{r,2r}^{(r-1)}, l_{r,2r+1}^{(r-1)}, l_{r,2r+2}^{(r-1)}, \ldots, l_{r,2r+s}^{(r-1)}, \ldots\right) = (3,4,5,\ldots,s+1,\ldots) = \left(l_{1,2}^{(0)}, l_{1,3}^{(0)}, l_{1,4}^{(0)}, \ldots, l_{1,s+2}^{(0)}, \ldots\right).$$

Now, we will show that $l_{r,2r}^{(r-1)} = l_{1,2}^{(0)} = 3$.

If $r = 1$, then $l_{r,2r}^{(r-1)} = l_{1,2}^{(0)}$ because $l_{1,n}^{(0)} = n + 1$, $l_{1,2}^{(0)} = 3$ by result 1.

For $r = 2$, $l_{r,2r}^{(r-1)} = l_{2,4}^{(1)} = l_{2,4}^{(0)} - l_{2,3}^{(0)} = l_{1,2}^{(0)} = 3$.

For $r = 3$, $l_{r,2r}^{(r-1)} = l_{3,6}^{(2)} = l_{3,6}^{(1)} - l_{3,5}^{(1)} = l_{3,6}^{(0)} - l_{3,5}^{(0)} - l_{3,5}^{(0)} + l_{3,4}^{(0)} = l_{2,4}^{(0)} - l_{2,3}^{(0)} = l_{1,2}^{(0)} = 3$.

For $r = 4$, $l_{r,2r}^{(r-1)} = l_{4,8}^{(3)} = l_{4,8}^{(2)} - l_{4,7}^{(2)} = l_{4,8}^{(1)} - l_{4,7}^{(1)} - l_{4,7}^{(1)} + l_{4,6}^{(1)} = l_{4,8}^{(0)} - l_{4,7}^{(0)} - l_{4,7}^{(0)} + l_{4,6}^{(0)} - l_{4,7}^{(0)} + l_{4,6}^{(0)} + l_{4,6}^{(0)} - l_{4,5}^{(0)} = l_{3,6}^{(0)} - l_{3,5}^{(0)} - l_{3,5}^{(0)} + l_{3,4}^{(0)} = l_{2,4}^{(0)} - l_{2,3}^{(0)} = l_{1,2}^{(0)} = 3$.

Similarly,

$$l_{r,2r}^{(r-1)} = l_{r,2r}^{(r-2)} - l_{r,2r-1}^{(r-2)} = l_{r,2r}^{(r-3)} - l_{r,2r-1}^{(r-3)} - l_{r,2r-1}^{(r-3)} + l_{r,2r-2}^{(r-3)} \vdots = (-1)^0 \binom{r-1}{0} l_{r,2r}^{(0)} + (-1)^1 \binom{r-1}{1} l_{r,2r-1}^{(0)} + (-1)^2 \binom{r-1}{2} l_{r,2r-2}^{(0)}$$

$$+ \cdots + (-1)^{r-2} \binom{r-1}{r-2} l_{r,r+2}^{(0)} + (-1)^{r-1} \binom{r-1}{r-1} l_{r,r+1}^{(0)} = (-1)^0 \binom{r-1}{0} l_{r,2r}^{(0)} + (-1)^1 \left[\binom{r-2}{1} + \binom{r-2}{0} \right] l_{r,2r-1}^{(0)}$$

$$+ (-1)^2 \left[\binom{r-2}{2} + \binom{r-2}{1} \right] l_{r,2r-2}^{(0)} + \cdots + (-1)^{r-2} \left[\binom{r-2}{r-2} + \binom{r-2}{r-3} \right] l_{r,r+2}^{(0)} + (-1)^{r-1} \binom{r-1}{r-1} l_{r,r+1}^{(0)}$$

$$= (-1)^0 \binom{r-1}{0} l_{r,2r}^{(0)} + (-1)^1 \binom{r-2}{0} l_{r,2r-1}^{(0)} + (-1)^1 \binom{r-2}{1} l_{r,2r-1}^{(0)} + (-1)^2 \binom{r-2}{1} l_{r,2r-2}^{(0)} + (-1)^2 \binom{r-2}{2} l_{r,2r-2}^{(0)}$$

$$+ \cdots + (-1)^{r-2} \binom{r-2}{r-3} l_{r,r+2}^{(0)} + (-1)^{r-2} \binom{r-2}{r-2} l_{r,r+2}^{(0)} + (-1)^{r-1} \binom{r-1}{r-1} l_{r,r+1}^{(0)} = (-1)^0 l_{r-1,2r-2}^{(0)} + (-1)^1 \binom{r-2}{1} l_{r-1,2r-3}^{(0)}$$

$$+ \cdots + (-1)^{r-1} l_{r-1,r}^{(0)}$$

a rigorous justification (e.g. indiction).

$$= l_{3,6}^{(0)} - l_{3,5}^{(0)} - l_{3,5}^{(0)} + l_{3,4}^{(0)},$$

$$= l_{2,4}^{(0)} - l_{2,3}^{(0)} = l_{1,2}^{(0)} = 3.$$

Thus, one can show easily that $l_{r,2r+s}^{(r-1)} = l_{1,s+2}^{(0)} = s+3$, and the proof is completed. □

We consider the Lucas sequence $l_n = l_{n-1} + l_{n-2}$, for $l_0 = 2$, $l_1 = 1$ and $n \geqslant 2$ and terms of the truncated Lucas $l_{r,n} = \sum_{t=0}^{r} q_{n-t,t}$. As we see Table 1, $l_{r,n} = l_n$ for $0 \leqslant n \leqslant 2r+1$ but $l_{r,2r+2} \neq l_{2r+2}$. This observation is explained in the following proposition.

Proposition 2.4. *$l_{r,n} = l_n$ if $0 \leqslant n \leqslant 2r+1$ and $l_{r,2r+2} \neq l_{2r+2}$.*

For $r = 2$, $l_{2,6} = 16$ and $l_6 = 18$.

Result. $l_{2r+2} = l_{r,2r+2} + 2$.

3. The Function $k_r(m)$

This section is concerned with determining the length of the recurring sequence obtained by reducing a Fibonacci and Lucas sequences modulo m.

Let $L_r = (l_{r,0}, l_{r,1}, l_{r,2}, \ldots)^T$ be the rth *truncated Lucas sequence*. Then, $L_r(\mathrm{mod\ m})$ does not form a simply periodic sequence. For example, $L_2(\mathrm{mod\ m})$ is that 2,1,0, 1,1,2, 1,1,2, 1,1,2, 1,1,2,... But if we take $L_{r,n} = (l_{r,n}, l_{r,n+1}, l_{r,n+2}, \ldots)^T$, $(n = r+1, r+2, r+3, \ldots)$, then $L_{r,n}(\mathrm{mod\ m})$ forms a simply periodic sequence. For example, $L_{2,3} = (l_{2,3}, l_{2,4}, l_{2,5}, \ldots)^T$ is that 1,1,2, 1,1,2, 1,1,2, 1,1,2, 1,1,2,... Now we can talk about the function $k_r(m)$ of sequence $L_{r,n}(\mathrm{mod\ m})$. We make the following conjectures. Concerning the conjectures, it might be helpful to see whether facts about the so-called *rank of apparition of m* can help to prove any of the conjectures. Such facts can be found [18].

Conjecture 3.2. *$k_r(m.n) = k_r(m).k_r(n)$ on condition that $(m,n) = 1$.*

That is, the function $k_r(m)$ is a multiplicative function.

Conjecture 3.3. *If p is a prime number and $t \in \mathbb{Z}^+$, then $k_r(p^t) = p^{t-1} k_r(p)$.*

Conjecture 3.4. *If p is a prime number and $p > r$, then $k_r(p) = p$.*

Conjecture 3.5. *If m is not a prime number, then $m | k_r(m)$ and $m \neq k_r(m)$.*

For several values of the parameters, these conjectures seem to be validated by a computer program written in MAPLE and FORTRAN.

Table 2
Wall numbers for various values of m and r.

$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$k(2) = 4$	$k(2) = 4$	$k(2) = 8$	$k(2) = 8$	$k(2) = 8$	$k(2) = 8$
$k(3) = 3$	$k(3) = 9$	$k(3) = 9$	$k(3) = 9$	$k(3) = 9$	$k(3) = 9$
$k(4) = 8$	$k(4) = 8$	$k(4) = 16$	$k(4) = 16$	$k(4) = 16$	$k(4) = 16$
$k(5) = 5$	$k(5) = 5$	$k(5) = 5$	$k(5) = 25$	$k(5) = 25$	$k(5) = 25$
$k(6) = 12$	$k(6) = 36$	$k(6) = 72$	$k(6) = 72$	$k(6) = 72$	$k(6) = 72$
$k(7) = 7$	$k(7) = 7$	$k(7) = 7$	$k(7) = 7$	$k(7) = 7$	$k(7) = 49$
$k(8) = 16$	$k(8) = 16$	$k(8) = 32$	$k(8) = 32$	$k(8) = 32$	$k(8) = 32$
$k(9) = 9$	$k(9) = 27$	$k(9) = 27$	$k(9) = 27$	$k(9) = 27$	$k(9) = 27$
$k(10) = 20$	$k(10) = 20$	$k(10) = 40$	$k(10) = 200$	$k(10) = 200$	$k(10) = 200$
$k(11) = 11$	$k(11) = 11$	$k(11) = 11$	$k(11) = 11$	$k(11) = 11$	$k(11) = 11$
$k(12) = 24$	$k(12) = 72$	$k(12) = 144$	$k(12) = 144$	$k(12) = 144$	$k(12) = 144$
$k(13) = 13$	$k(13) = 13$	$k(13) = 13$	$k(13) = 13$	$k(13) = 13$	$k(13) = 13$
$k(14) = 28$	$k(14) = 28$	$k(14) = 56$	$k(14) = 56$	$k(14) = 56$	$k(14) = 392$
$k(15) = 15$	$k(15) = 45$	$k(15) = 45$	$k(15) = 225$	$k(15) = 225$	$k(15) = 225$
$k(16) = 32$	$k(16) = 32$	$k(16) = 64$	$k(16) = 64$	$k(16) = 64$	$k(16) = 64$
$k(17) = 17$	$k(17) = 17$	$k(17) = 17$	$k(17) = 17$	$k(17) = 17$	$k(17) = 17$
$k(18) = 36$	$k(18) = 108$	$k(18) = 216$	$k(18) = 216$	$k(18) = 216$	$k(18) = 216$
$k(19) = 19$	$k(19) = 19$	$k(19) = 19$	$k(19) = 19$	$k(19) = 19$	$k(19) = 19$
$k(20) = 40$	$k(20) = 40$	$k(20) = 80$	$k(20) = 400$	$k(20) = 400$	$k(20) = 400$
$k(21) = 21$	$k(21) = 63$	$k(21) = 63$	$k(21) = 63$	$k(21) = 63$	$k(21) = 441$
$k(22) = 44$	$k(22) = 44$	$k(22) = 88$	$k(22) = 88$	$k(22) = 88$	$k(22) = 88$
$k(23) = 23$	$k(23) = 23$	$k(23) = 23$	$k(23) = 23$	$k(23) = 23$	$k(23) = 23$
$k(24) = 48$	$k(24) = 144$	$k(24) = 288$	$k(24) = 288$	$k(24) = 288$	$k(24) = 288$

In addition, Table 2 shows the Wall number of 2,3,...,7. truncated Lucas sequences for some numbers. The above conjectures can be checked with Table 2 for some numbers. We have lots of data for various values of m and r.

Notice that by using Conjecture 3.4, We can now rewrite some items of Theorem 1.1, Theorem 1.2 and Conjecture 1.3. So we can see a relationship between $k(p)$ and $k_r(p)$.

Result 3.6. *If $k(p) \neq k(p^2)$, then $k(p^e) = k_r(p)^{e-1}k(p)$. Also, if n is the largest integer with $k(p) = k(p^n)$, then $k(p^e) = k_r(p)^{e-1}k(p)$ for $e > n$.*

Result 3.7. *The following hold:*

i. *If p is a prime and $p = 10x \pm 1$, then $k(p)|(k_r(p) - 1)$.*
ii. *If p is a prime and $p = 10x \pm 3$, then $k(p)|(2k_r(p) + 2)$.*

Result 3.8.

i. *If p is a prime number and $k(p) < p$, then $k(p)|(k_r(p) - 1)$.*
ii. *If p is a prime number and $k(p) \equiv 0(mod 2)$, then $(k_r(p) \pm 1)|k(p)$.*
iii. *If p is a prime number and $k(p) \equiv 1(mod 2)$,* then $(k_r(p) \pm 1)|(k(p) - 1)$ *or* $k_r(p)|(k(p) - 1)$.

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