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Properties of Generalized Bronze Fibonacci Sequences and Their Hyperbolic Quaternions

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Abstract: In this study, we establish some properties of Bronze Fibonacci and Bronze Lucas sequences. Then we find the relationships between the roots of the characteristic equation of these sequences with these sequences. What is interesting here is that even though the roots change, equality is still maintained. Also, we derive the special relations between the terms of these sequences. We give the important relations among these sequences, positive and negative index terms, with the sum of the squares of two consecutive terms being related to these sequences. In addition, we present the application of generalized Bronze Fibonacci sequences to hyperbolic quaternions. For these hyperbolic quaternions, we give the summation formulas, generating functions, etc. Moreover, we obtain the Binet formulas in two different ways. The first is in the known classical way and the second is with the help of the sequence's generating functions. In addition, we calculate the special identities of these hyperbolic quaternions. Furthermore, we examine the relationships between the hyperbolic Bronze Fibonacci and Bronze Lucas quaternions. Finally, the terms of the generalized Bronze Fibonacci sequences are associated with their hyperbolic quaternion values.

Keywords: Bronze Fibonacci number; Bronze Lucas number; quaternions; generating function; Catalan identity

MSC: 11B39; 11B50; 11R52; 05A15

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1. Introduction

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci and Lucas sequences have been applied in various fields such as algebraic coding theory [1], phylotaxis [2], biomathematics [3], computer science [4], etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the Bronze Fibonacci, Bronze Lucas, k-Fibonacci, k-Lucas, Oresme, k-Chebsyhev, k-Jacobsthal-Lucas, Pell, Lenardo, Copper Fibonacci, Narayana, Padovan sequences, etc. (see details in [5–11]).

For $n \in \mathbb{N}$, the Fibonacci numbers F_n, Bronze Fibonacci numbers BF_n, Copper Fibonacci numbers CF_n, Lucas numbers L_n, Bronze Lucas numbers BL_n, and Copper Lucas numbers CL_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n,\ BF_{n+2} = 3BF_{n+1} + BF_n,\ CF_{n+2} = 4CF_{n+1} + CF_n$$

$$L_{n+2} = L_{n+1} + L_n,\ BL_{n+2} = 3BL_{n+1} + BL_n,\ CL_{n+2} = 4CL_{n+1} + CL_n,$$

with the initial conditions $F_0 = 0,\ F_1 = 1, BF_0 = 0,\ BF_1 = 1,\ CF_0 = 0,\ CF_1 = 1,\ L_0 = 2,$ $L_1 = 1, BL_0 = 2,\ BL_1 = 3,$ and $CL_0 = 2,\ CL_1 = 4.$

For $F_n,\ L_n,\ BF_n,\ BL_n,\ CF_n,\ CL_n$ the Binet formulas

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ L_n = \alpha^n + \beta^n,\ BF_n = \frac{\lambda^n - \psi^n}{\lambda - \psi},$$

$$BL_n = \lambda^n + \psi^n,\ CF_n = \frac{\theta^n - \sigma^n}{\theta - \sigma}, \text{ and } CL_n = \theta^n + \sigma^n.$$

Hence, $\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ \lambda = \frac{3+\sqrt{13}}{2},\ \psi = \frac{3-\sqrt{13}}{2},\ \theta = 2 + \sqrt{5},$ and $\sigma = 2 - \sqrt{5}$ are the roots of the characteristic equation $r^2 - r - 1 = 0,\ v^2 - 3v - 1 = 0,\ s^2 - 4s - 1 = 0,$ respectively. Here α and λ numbers are the known golden ratio, and bronze ratio, respectively.

In [12], Hong et al. found among $F_n,\ BF_n,\ L_n,\ BL_n$ sequences the following relations:

i. $BF_{m+n} = BF_{m-1}BF_n + BF_mBF_{n+1}$, **ii.** $BF_{2n+1} = BF_n^2 + BF_{n+1}^2,$

iii. $BF_{2n} = BF_{n-1}BF_n + BF_nBF_{n+1}$, **iv.** $F_{m+n} - (-1)^n F_{m-n} = F_nL_m,$

v. $L_{m+n} - (-1)^n L_{m-n} = 5F_mF_n$, **vi.** $L_{m+n} + (-1)^n L_{m-n} = L_nL_m,$

vii. $\sum_{i=0}^{n} 2^i F_i = \frac{1}{5}(2^{n+1}L_{n+1} - 2).$

In [13], Akbiyik and Alo defined the third-order Bronze Fibonacci numbers and made many applications of these numbers. Alo related the third-order Bronze Fibonacci numbers to quaternions and found out many properties of these quaternions such as the special summation formulas, the special generating functions, the matrix representations, and the important identities [14]. Karaarslan worked on Gaussian Bronze Lucas numbers [15]. In [16], Özkan and Akkuş found many features for the generalized Copper Fibonacci sequences. In addition, they made applications related to these sequences [16]. Akhtamova et al. [17] worked on generating functions.

The quaternions were first described by William Rowan Hamilton in 1843. In addition, quaternions are used to control rotational movements, especially in kinematics [18], 3D games [19], mechanics [20], Eulerian angles [21], and chemistry [22]. In [23], Horadam defined complex Fibonacci quaternions, and various features were found.

The algebra of hyperbolic quaternions is an algebra that is not related to the elements of the form over the real numbers:

$$q = xi_1 + yi_2 + zi_3 + ti_4,\ x, y, z, t \in \mathbb{R}$$

William Rowan Hamilton gave the properties of the q components defined in Table 1.

Table 1. Hyperbolic quaternions units.

.	$\boldsymbol{i_1}$	$\boldsymbol{i_2}$	$\boldsymbol{i_3}$	$\boldsymbol{i_4}$
i_1	i_1	i_2	i_3	i_4
$\boldsymbol{i_2}$	i_2	i_1	i_4	$-i_3$
i_3	i_3	$-i_4$	i_1	i_2
i_4	i_4	i_3	$-i_2$	i_1

In [24], Macfarlane did a lot of research on hyperbolic quaternions and their properties. An expression of the general form of hyperbolic quaternions is as follows:

$$\hbar = \hbar_1 i_1 + \hbar_2 i_2 + \hbar_3 i_3 + \hbar_4 i_4 = (\hbar_1, \hbar_2, \hbar_3, \hbar_4)$$

Here, $\hbar_1, \hbar_2, \hbar_3, \hbar_4$ are the terms of the sequence and i_1, i_2, i_3, i_4 are hyperbolic quaternions. For several recent works on quaternions in relation to number sequences, see [25–31].

We separate the article into three parts.

In Section 2, we obtain some of the properties of Bronze Fibonacci and Bronze Lucas sequences. In Section 3, we present the application of generalized Bronze Fibonacci sequences to hyperbolic quaternions. In addition, we give many features of these applications.

2. Properties of Generalized Bronze Fibonacci Sequences

In this section, we find some properties of Bronze Fibonacci and Bronze Lucas sequences and obtain the relationships between the roots of the characteristic equation of these sequences and these sequences. What is interesting here is that even though the roots change, equality is still maintained. Also, we derive the special relations between the terms of these sequences.

Theorem 1. *The ratio of the biggest to the smallest of two consecutive terms of the Bronze Fibonacci and Bronze Lucas sequences converges to the bronze ratio. We obtain*

$$\lim_{n\to\infty} \frac{BF_{n+1}}{BF_n} = \lambda \text{ and } \lim_{n\to\infty} \frac{BL_{n+1}}{BL_n} = \lambda.$$

Proof. If the Binet formula is used, we have

$$\lim_{n\to\infty} \frac{BF_{n+1}}{BF_n} = \frac{\frac{\lambda^{n+1} - \psi^{n+1}}{\lambda - \psi}}{\frac{\lambda^n - \psi^n}{\lambda - \psi}} = \lim_{n\to\infty} \frac{\lambda^{n+1}\left(1 - \left(\frac{\psi}{\lambda}\right)^{n+1}\right)}{\lambda^n\left(1 - \left(\frac{\psi}{\lambda}\right)^{n}\right)}$$

Thus, we obtain

$$\lim_{n\to\infty} \frac{BF_{n+1}}{BF_n} = \lambda$$

Similarly, we have

$$\lim_{n\to\infty} \frac{BL_{n+1}}{BL_n} = \lambda. \ \square$$

In the following theorems, we examine the relationships between the roots of the characteristic equation of these sequences and these sequences.

Theorem 2. *Let* $n \in \mathbb{N}$. *We obtain*

$$\textbf{i. } \lambda^{2n} = \frac{BF_{2n}}{3}\lambda\sqrt{13} - \frac{BL_{2n-1}}{3}, \textbf{ ii. } \psi^{2n} = \frac{BF_{2n}}{3}(\sqrt{13} - \lambda)\sqrt{13} - \frac{BL_{2n-1}}{3},$$

$$\textbf{iii. } \lambda^{2n+1} = -\frac{BF_{2n}}{3}\sqrt{13} + \lambda\frac{BL_{2n+1}}{3}, \textbf{ iv. } \psi^{2n+1} = \frac{BF_{2n}}{3}\sqrt{13} + \psi\frac{BL_{2n+1}}{3},$$

$$\textbf{v. } \sqrt{13}BF_n + BL_n = 2\lambda^n, \textbf{ vi. } \sqrt{13}BF_n - BL_n = -2\psi^n.$$

Proof. i. If the Binet formulas are used, we obtain

$$\frac{BF_{2n}}{3}\lambda\sqrt{13}-\frac{BL_{2n-1}}{3}=\lambda\sqrt{13}\frac{\lambda^{2n}-\psi^{2n}}{(\lambda-\psi)3}-\frac{\lambda^{2n-1}+\psi^{2n-1}}{3}=\frac{\lambda^{2n+1}-\lambda\psi^{2n}-\lambda^{2n-1}-\psi^{2n-1}}{3}$$

$$=\frac{\lambda^{2n}\left(\lambda-\frac{1}{\lambda}\right)+\psi^{2n}\left(-\lambda-\frac{1}{\psi}\right)}{3}.$$

Since, $\lambda+\psi=3$ and $\lambda\psi=-1$ we obtain

$$\lambda^{2n}=\frac{BF_{2n}}{3}\lambda\sqrt{13}-\frac{BL_{2n-1}}{3}.$$

The proofs of others can also be shown in this way. □

Theorem 3. *Let* $s=\lambda$ *or* $s=\psi$ *and* $x,y,z,t\in\mathbb{N}$. *We obtain*

i. $s^x=sBF_x+BF_{x-1}$, **ii.** $BF_{x(y-z)}=s^{xz}BF_{xy}+s^{xy}BF_{xz}$

iii. $s^{2x}=s^xBL_x-(-1)^x$, **iv.** $s^x=s^yBF_{x-y+1}+s^{y-1}BF_{x-y}$,

v. $s^{xt}=\frac{s^xBF_{xt}}{BF_x}-(-1)^x\frac{BF_{x(t-1)}}{BF_x}$, **vi.** $1+3s+s^{2(2^{x+1}+1)}=s^{2(2^x+1)}BL_{2^{x+1}}$.

Proof. i. For $s=\lambda$, we have

$$sBF_x+BF_{x-1}=\lambda\left(\frac{\lambda^x-\psi^x}{\lambda-\psi}\right)+\left(\frac{\lambda^{x-1}-\psi^{x-1}}{\lambda-\psi}\right)=\frac{\lambda^{x-1}(\lambda^2+1)-\psi^{x-1}(\lambda\psi+1)}{\lambda-\psi}=\lambda^x.$$

For $s=\psi$, we have

$$sBF_x+BF_{x-1}=\psi\left(\frac{\lambda^x-\psi^x}{\lambda-\psi}\right)+\left(\frac{\lambda^{x-1}-\psi^{x-1}}{\lambda-\psi}\right)=\frac{\lambda^{x-1}(\lambda\psi+1)+\psi^x\left(-\psi-\frac{1}{\psi}\right)}{\lambda-\psi}=\psi^x.$$

The proofs of others can also be shown in this way. □

In the following theorems, we give the special relations between the Bronze Fibonacci BF_n and Bronze Lucas BL_n sequences.

Theorem 4. *Let* $n\in\mathbb{N}$. *The following equation is true:*

i. $BF_n=\frac{2}{13}BL_{n+1}-\frac{3}{13}BL_n$, **ii.** $BL_n=2BF_{n+1}-3BF_n$.

Proof. The following relation is used for proofs:

$$BF_n=aBL_{n+1}+bBL_n.$$

For these n values, we obtain

$$BF_0=aBL_1+bBL_0,$$

$$BF_1=aBL_2+bBL_1.$$

We find

$$a=\frac{2}{13}, \text{ and } b=-\frac{3}{13}.$$

The proof of others can also be shown in this way. □

Theorem 5. *Let* $m,n\in\mathbb{N}$, *and* $n>m$. *We obtain*

i. $BF_{n+m+1}=BF_{n+1}BF_{m+1}+BF_nBF_m$, **ii.** $BL_{n+m+1}=BL_{m+1}BF_{n+1}+BL_mBF_n$,

iii. $BF_n^2=\frac{1}{13}BL_{2n}+\frac{2}{13}\frac{BF_{-n}}{BF_n}$, **iv.** $BL_n^2=BL_{2n}+2\frac{BL_{-n}}{BL_n}$,

v. $CL_n^2 + CL_{n+1}^2 = CL_{2n} + CL_{2n+2}$, **vi.** $BF_n^2 + BF_{n+1}^2 = BF_{2n+1}$.

Proof. If the Binet formula is used, we obtain

$$\textbf{i. } BF_{n+1}BF_{m+1} + BF_nBF_m = \frac{\lambda^{n+1}-\psi^{n+1}}{\lambda-\psi}\frac{\lambda^{m+1}-\psi^{m+1}}{\lambda-\psi} + \frac{\lambda^n-\psi^n}{\lambda-\psi}\frac{\lambda^m-\psi^m}{\lambda-\psi}$$

$$= \frac{\lambda^{n+m+1}(\lambda-\psi)-\theta^{n+m+1}(\lambda-\psi)}{(\lambda-\psi)^2} = \frac{\lambda^{n+m+1}-\psi^{n+m+1}}{\lambda-\psi} = BF_{n+m+1}.$$

$$\textbf{iii. } BF_{-n} = \frac{\lambda^{-n}-\psi^{-n}}{\lambda-\psi} = -\frac{\lambda^n-\psi^n}{(\lambda-\psi)\lambda^n\psi^n} = -\frac{BF_n}{\lambda^n\psi^n}. \text{ So,}$$

$$\lambda^n\psi^n = -\frac{BF_{-n}}{BF_n}.$$

Then, we find

$$BF_n^2 = \frac{\lambda^n-\psi^n}{\lambda-\psi}\frac{\lambda^n-\psi^n}{\lambda-\psi} = \frac{\lambda^{2n}+\psi^{2n}-2\lambda^n\psi^n}{(\lambda-\psi)(\lambda-\psi)} = \frac{1}{13}BL_{2n} - \frac{2}{13}\lambda^n\psi^n.$$

Thus, we obtain

$$BF_n^2 = \frac{1}{13}BL_{2n} + \frac{2}{13}\frac{BF_{-n}}{BF_n}.$$

The proofs of others can also be shown in this way. □

Theorem 6. *Let* $k, m, n \in \mathbb{N}$, *and* $k > m, n$. *We obtain*

$$\textbf{i. } BF_{k+m+n} = \frac{1}{4}BL_kBL_mBF_n + \frac{1}{4}BF_kBL_mBL_n + \frac{1}{4}BL_kBF_mBL_n + \frac{13}{4}BF_kBF_mBF_n,$$

$$\textbf{ii. } BL_{k+m+n} = \frac{1}{4}BL_kBL_mBL_n + \frac{13}{4}BL_kBF_mBF_n + \frac{13}{4}BF_kBL_mBF_n + \frac{13}{4}BF_kBF_mBL_n.$$

The proofs of Theorem 6. are shown using the Binet formulas in a similar way to Theorem 5.

3. Applications

In this section, we give an application of Bronze Fibonacci and Bronze Lucas sequences. We define the hyperbolic Bronze Fibonacci and Bronze Lucas quaternions, and the terms of these quaternions are given. Then we find some properties of these hyperbolic quaternions. İnformation is given about the characteristic equations of hyperbolic Bronze Fibonacci and Bronze Lucas quaternions. Then, we obtain the Binet formulas, generating functions, and the sum of terms of these hyperbolic quaternions. In addition, we examine the relationship of hyperbolic Bronze Fibonacci and Bronze Lucas quaternions. Moreover, we calculate the special identities of these hyperbolic quaternions. Finally, we associate the terms of the generalized Bronze Fibonacci sequences with their hyperbolic quaternion values.

Definition 1. *For* $n \in \mathbb{N}$, *the hyperbolic Bronze Fibonacci* $\breve{H}BF_n$ *and hyperbolic Bronze Lucas* $\breve{H}BL_n$ *quaternions are defined by, respectively,*

$$\breve{H}BF_n = BF_ni_1 + BF_{n+1}i_2 + BF_{n+2}i_3 + BF_{n+3}i_4 = (BF_n, BF_{n+1}, BF_{n+2}, BF_{n+3})$$

and

$$\breve{H}BL_n = BL_ni_1 + BL_{n+1}i_2 + BL_{n+2}i_3 + BL_{n+3}i_4 = (BL_n, BL_{n+1}, BL_{n+2}, BL_{n+3})$$

where BF_n *is the* n^{th} *Bronze Fibonacci number,* BL_n *is the* n^{th} *Bronze Lucas number, and* i_1, i_2, i_3, *and* i_4 *are the hyperbolic quaternion units in Table 1.*

Now, let the first four terms be given of the hyperbolic Bronze Fibonacci $\breve{H}BF_n$ and hyperbolic Bronze Lucas $\breve{H}BL_n$ quaternions, respectively,

- $\breve{H}BF_0 = i_2 + 3i_3 + 10i_4$, $\breve{H}BL_0 = 2i_1 + 3i_2 + 11i_3 + 36i_4$,
- $\breve{H}BF_1 = i_1 + 3i_2 + 10i_3 + 33i_4$, $\breve{H}BL_1 = 3i_1 + 11i_2 + 36i_3 + 119i_4$,
- $\breve{H}BF_2 = 3i_1 + 10i_2 + 33i_3 + 109i_4$, $\breve{H}BL_2 = 11i_1 + 36i_2 + 119i_3 + 393i_4$,
- $\breve{H}BF_3 = 10i_1 + 33i_2 + 109i_3 + 360i_4$, $\breve{H}BL_3 = 36i_1 + 119i_2 + 393i_3 + 1298i_4$,
- $\breve{H}BF_4 = 33i_1 + 109i_2 + 360i_3 + 1189i_4$, $\breve{H}BL_4 = 119i_1 + 393i_2 + 1298i_3 + 4287i_4$.

Definition 2. *For $n \in \mathbb{N}$, the conjugate of hyperbolic Bronze Fibonacci $\breve{H}BF_n^*$ and conjugate hyperbolic Bronze Lucas $\breve{H}BL_n^*$ quaternions are defined by, respectively,*

$$\breve{H}BF_n^* = BF_n i_1 - BF_{n+1} i_2 - BF_{n+2} i_3 - BF_{n+3} i_4 = (BF_n, -BF_{n+1}, -BF_{n+2}, -BF_{n+3})$$

and

$$\breve{H}BL_n^* = BL_n i_1 - BL_{n+1} i_2 - BL_{n+2} i_3 - BL_{n+3} i_4 = (BL_n, -BL_{n+1}, -BL_{n+2}, -BL_{n+3}).$$

Definition 3. *For $n \in \mathbb{N}$, the norms of the hyperbolic Bronze Fibonacci $\left\|\breve{H}BF_n\right\|$ and the norms of the hyperbolic Bronze Lucas $\left\|\breve{H}BL_n\right\|$ quaternions are defined by, respectively,*

$$\left\|\breve{H}BF_n\right\| = \sqrt{BF_n^2 + BF_{n+1}^2 + BF_{n+2}^2 + BF_{n+3}^2}$$

and

$$\left\|\breve{H}BL_n\right\| = \sqrt{BL_n^2 + BL_{n+1}^2 + BL_{n+2}^2 + BL_{n+3}^2}.$$

In the following theorems, we examine the relations between the $\breve{H}BF_n$, $\breve{H}BF_n^*$, $\breve{H}BL_n$, $\breve{H}BL_n^*$, $\left\|\breve{H}BF_n\right\|$, and $\left\|\breve{H}BL_n\right\|$ quaternions.

Theorem 7. *Let $n \in \mathbb{N}$. The following equations are true:*

$$\textbf{i. } \breve{H}BF_{n+2} = 3\breve{H}BF_{n+1} + \breve{H}BF_n, \ \textbf{ii. } \breve{H}BL_{n+2} = 3\breve{H}BL_{n+1} + \breve{H}BL_n,$$

$$\textbf{iii. } \breve{H}BF_{n+2}^* = 3\breve{H}BF_{n+1}^* + \breve{H}BF_n^*, \ \textbf{iv. } \breve{H}BL_{n+2}^* = 3\breve{H}BL_{n+1}^* + \breve{H}BL_n^*.$$

Proof. i. If the definition is used, we have

$$3\breve{H}BF_{n+1} + \breve{H}BF_n = 3(BF_{n+1}i_1 + BF_{n+2}i_2 + BF_{n+3}i_3 + BF_{n+4}i_4) + (BF_n i_1 + BF_{n+1}i_2 + BF_{n+2}i_3 + BF_{n+3}i_4) = (3BF_{n+1} + BF_n)i_1 + (3BF_{n+2} + BF_{n+1})i_2 + (3BF_{n+3} + BF_{n+2})i_3 + (3BF_{n+4} + BF_{n+3})i_4.$$

Since, $BF_{n+2} = 3BF_{n+1} + BF_n$., we obtain

$$\breve{H}BF_{n+2} = 3\breve{H}BF_{n+1} + \breve{H}BF_n.$$

The proofs of others can also be shown in this way. □

Theorem 8. *We obtain*

$$\textbf{i. } \breve{H}BF_n + \breve{H}BF_n^* = 2BF_n i_1, \ \textbf{ii. } \breve{H}BF_n^2 = 2BF_n\breve{H}BF_n + \left(\left\|\breve{H}BF_n\right\|^2 - 2BF_n^2\right) i_1,$$

$$\textbf{iii. } \breve{H}BL_n + \breve{H}BL_n^* = 2BL_n i_1, \ \textbf{iv. } \breve{H}BL_n^2 = 2BL_n\breve{H}BL_n + \left(\left\|\breve{H}BL_n\right\|^2 - 2BL_n^2\right) i_1.$$

Proof. ii. If the definition is used, we have

$$\breve{H}BF_n^2 = (BF_n i_1 + BF_{n+1} i_2 + BF_{n+2} i_3 + BF_{n+3} i_3)(BF_n i_1 + BF_{n+1} i_2 + BF_{n+2} i_3 + BF_{n+3} i_4)$$
$$= (BF_n^2 + BF_{n+1}^2 + BF_{n+2}^2 + BF_{n+3}^2) i_1 + (2BF_n BF_{n+1}) i_2 + (2BF_n BF_{n+2}) i_3 + (2BF_n BF_{n+3}) i_4$$
$$= 2BF_n (BF_n i_1 + BF_{n+1} i_2 + BF_{n+2} i_3 + BF_{n+3} i_4) + (-BF_n^2 + BF_{n+1}^2 + BF_{n+2}^2 + BF_{n+3}^2) i_1.$$

Thus, we obtain

$$\breve{H}BF_n^2 = 2BF_n \breve{H}BF_n + \left(\left\| \breve{H}BF_n \right\|^2 - 2BF_n^2 \right) i_1.$$

The proofs of others can also be shown in this way. □

Theorem 9. *We obtain*

i. $i_1 \breve{H}BF_n - i_2 \breve{H}BF_{n+1} - i_3 \breve{H}BF_{n+2} - i_4 \breve{H}BF_{n+3} = 90BF_{n+2} i_1.$

ii. $i_1 \breve{H}BL_n - i_2 \breve{H}BL_{n+1} - i_3 \breve{H}BL_{n+2} - i_4 \breve{H}BL_{n+3} = 90BL_{n+2} i_1.$

Proof. i. If the definition is used, we have

$$i_1 \breve{H}BF_n - i_2 \breve{H}BF_{n+1} - i_3 \breve{H}BF_{n+2} - i_4 \breve{H}BF_{n+3}$$
$$= (BF_n - BF_{n+2} - BF_{n+4} + BF_{n+6}) i_1 + (BF_{n+1} - BF_{n+1} - BF_{n+5} + BF_{n+5}) i_2$$
$$+ (BF_{n+2} + BF_{n+4} - BF_{n+2} - BF_{n+4}) i_3 + (BF_{n+3} - BF_{n+3} + BF_{n+3} - BF_{n+3}) i_4$$
$$= (BF_n - BF_{n+2} - BF_{n+4} + BF_{n+6}) i_1 = 90BF_{n+2} i_1.$$

The proof of others can also be shown in this way. □

In the following theorem, we express the Binet formulas of the $\breve{H}BF_n$, $\breve{H}BF_n^*$, $\breve{H}BL_n$, and $\breve{H}BL_n^*$ quaternions.

Theorem 10. *Let* $n \in \mathbb{N}$. *We obtain*

i. $\breve{H}BF_n = \frac{\overline{\lambda}\lambda^n - \overline{\psi}\psi^n}{\lambda - \psi}$, **ii.** $\breve{H}BL_n = \overline{\lambda}\lambda^n + \overline{\psi}\psi^n,$

iii. $\breve{H}BF_n^* = \frac{\overline{\lambda}^* \lambda^n - \overline{\psi}^* \psi^n}{\lambda - \psi}$, **iv.** $\breve{H}BL_n^* = \overline{\lambda}^* \lambda^n + \overline{\psi}^* \psi^n,$

where

$$\overline{\lambda} = i_1 + \lambda i_2 + \lambda^2 i_3 + \lambda^3 i_4 = (1, \lambda, \lambda^2, \lambda^3),\ \overline{\lambda}^* = (1, -\lambda, -\lambda^2, -\lambda^3),$$

$$\overline{\psi} = i_1 + \psi i_2 + \psi^2 i_3 + \psi^3 i_4 = (1, \psi, \psi^2, \psi^3), \text{ and } \overline{\psi}^* = (1, -\psi, -\psi^2, -\psi^3).$$

Proof. i. With the help of the characteristic equation, the following results are obtained: $(\breve{H}BF_{n+2} = 3\breve{H}BF_{n+1} + \breve{H}BF_n)$

$$s^2 - 3s - 1 = 0\,, \lambda = \frac{3+\sqrt{13}}{2}, \psi = \frac{3-\sqrt{13}}{2},\ \lambda + \psi = 3,$$

$$\lambda - \psi = \sqrt{13},\ \lambda^2 + \psi^2 = 11, \text{ and } \lambda\psi = -1.$$

The Binet form of the hyperbolic Bronze Fibonacci quaternions is

$$\breve{H}BF_n = x\lambda^n + y\psi^n.$$

With the initial conditions, the following equations are obtained:

$$\breve{H}BF_0 = i_2 + 3i_3 + 10i_4 = x + y$$

and

$$\breve{H}BF_1 = i_1 + 3i_2 + 10i_3 + 33i_4 = x\lambda + y\psi.$$

Thus, we have

$$x = \frac{\breve{H}BF_1 - \psi\breve{H}BF_0}{\lambda - \psi} = \frac{i_1 + \lambda i_2 + \lambda^2 i_3 + \lambda^3 i_4}{\lambda - \psi} = \frac{\overline{\lambda}}{\lambda - \psi}$$

$$y = \frac{\breve{H}BF_1 - \lambda\breve{H}BF_0}{-\lambda + \psi} = \frac{i_1 + \psi i_2 + \psi^2 i_3 + \psi^3 i_4}{-(\lambda - \psi)} = \frac{-\overline{\psi}}{\lambda - \psi}.$$

so, we obtain

$$\breve{H}BF_n = \frac{\overline{\lambda}\lambda^n - \overline{\psi}\psi^n}{\lambda - \psi}.$$

The proofs of others can also be shown in this way. □

In the following theorems, we give special summation formulas and binomial sum formulas of the $\breve{H}BF_n$ and $\breve{H}BL_n$ quaternions.

Theorem 11. *Let* $n \in \mathbb{N}$. *We obtain*

$$\textbf{i}.\ S\breve{H}BF_n = \sum_{i=0}^{n} \breve{H}BF_j = \frac{1}{3}(4\breve{H}BF_n + \breve{H}BF_{n-1} - i_1 - i_2 - 4i_3 - 13i_4).$$

$$\textbf{ii}.\ S\breve{H}BL_n = \sum_{i=0}^{n} \breve{H}BL_j = \frac{1}{3}(4\breve{H}BL_n + \breve{H}BL_{n-1} + i_1 - 5i_2 - 14i_3 - 47i_4).$$

Proof. i. Using the definition, we have

$$S\breve{H}BF_n = \sum_{j=0}^{n} \breve{H}BF_j = i_1 \sum_{j=0}^{n} BF_j + i_2 \sum_{j=0}^{n} BF_{j+1} + i_3 \sum_{j=0}^{n} BF_{j+2} + i_4 \sum_{j=0}^{n} BF_{j+3}.$$

Since,

$$\sum_{j=0}^{n} BF_j = \frac{4BF_n + BF_{n-1} - 1}{3}.$$

so, we get

$$S\breve{H}BF_n = \left(\frac{4BF_n + BF_{n-1} - 1}{3} - 1\right) i_1 + \left(\frac{4BF_{n+1} + BF_n - 1}{3} - 1\right) i_2 + \left(\frac{4BF_{n+2} + BF_{n+1} - 1}{3} - 1\right) i_3 + \left(\frac{4BF_{n+3} + BF_{n+2} - 1}{3} - 4\right) i_4.$$

and so, we obtain

$$S\breve{H}BF_n = \tfrac{1}{3}(4\breve{H}BF_n + \breve{H}BF_{n-1} - i_1 - i_2 - 4i_3 - 13i_4).$$

The proof of others can also be shown in this way □

Theorem 12. *Let* $x, y, z \in \mathbb{N}$. *We obtain*

$$\textbf{i}.\ \sum_{y=0}^{n} \breve{H}BF_{xy} = \frac{(-1)^x \breve{H}BF_{nx} + \breve{H}BF_0 - \breve{H}BF_{nx+x} + \breve{H}BL_0 BF_x - \breve{H}BF_x}{1 - BL_x + (-1)^x},$$

$$\textbf{ii}.\ \sum_{y=0}^{n} \breve{H}BL_{xy} = \frac{(-1)^x \breve{H}BL_{nx} + \breve{H}BL_0 - \breve{H}BF_{nx+x} - \breve{H}BL_0 BL_x - \breve{H}BL_x}{1 - BL_x + (-1)^x},$$

$$\textbf{iii}.\sum_{y=0}^{n}\breve{H}BF_{xy+z}=\begin{cases}\dfrac{(-1)^x\breve{H}BF_{nx+z}-\breve{H}BF_{nx+x+z}-\breve{H}BF_z-(-1)^x\breve{H}BF_{x-z}}{1-BL_x+(-1)^x}, if\ z<x\\ \dfrac{(-1)^x\breve{H}BF_{nx+z}-\breve{H}BF_{nx+x+z}-\breve{H}BF_z-(-1)^x\breve{H}BF_{z-x}}{1-BL_x+(-1)^x}, otherwise\end{cases},$$

$$\textbf{iv}.\sum_{y=0}^{n}\breve{H}BL_{xy+z}=\begin{cases}\dfrac{(-1)^x\breve{H}BL_{nx+z}-\breve{H}BL_z-\breve{H}BL_{nx+x+z}-(-1)^x\breve{H}BL_{z-x}}{1-BL_x+(-1)^x}, if\ z<x\\ \dfrac{(-1)^x\breve{H}BL_{nx+z}-\breve{H}BL_z-\breve{H}BL_{nx+x+z}-(-1)^x\breve{H}BL_{z-x}}{1-BL_x+(-1)^x}, otherwise\end{cases}.$$

Proof. With the help of definitions, Binet formulas, and geometric series, we have

$$\textbf{i}.\sum_{y=0}^{n}\breve{H}BF_{xy}=\sum_{y=0}^{n}\frac{\overline{\lambda}\lambda^n-\overline{\psi}\psi^n}{\lambda-\psi}=\frac{\overline{\lambda}}{\lambda-\psi}\sum_{y=0}^{n}(\lambda^x)^y-\frac{\overline{\psi}}{\lambda-\psi}\sum_{y=0}^{n}(\psi^x)^y$$

$$=\frac{1}{\lambda-\psi}\left(\frac{\overline{\lambda}\lambda^{nx+x}-\overline{\lambda}}{\lambda^x-1}-\frac{\overline{\psi}\psi^{nx+x}-\overline{\psi}}{\psi^x-1}\right)$$

$$=\frac{1}{\lambda-\psi}\frac{\overline{\lambda}\lambda^{nx+x}\psi^x-\overline{\lambda}\psi^x-\overline{\lambda}\lambda^{nx+x}+\overline{\lambda}-\overline{\psi}\psi^{nx+x}\lambda^x+\overline{\psi}\lambda^x+\overline{\psi}\psi^{nx+x}-\overline{\psi}}{1-BL_x+(-1)^x}.$$

Thus, we obtain

$$\sum_{y=0}^{n}\breve{H}BF_{xy}=\frac{(-1)^x\breve{H}BF_{nx}+\breve{H}BF_0-\breve{H}BF_{nx+x}+\breve{H}BL_0BF_x-\breve{H}BF_x}{1-BL_x+(-1)^x}.$$

The proofs of others can also be shown in this way □

Theorem 13. *Let* $n,r\in\mathbb{N}$. *We obtain*

$$\textbf{i}.\sum_{y=0}^{n}\binom{n}{i}3^i\breve{H}BF_i=\breve{H}BF_{2n},\ \textbf{ii}.\sum_{y=0}^{n}\binom{n}{i}3^i\breve{H}BL_i=\breve{H}BL_{2n},$$

$$\textbf{iii}.\sum_{y=0}^{n}\binom{n}{i}(-1)^i\frac{\breve{H}BF_{n+r+i}}{3^i}=(-1)^n\frac{\breve{H}BF_r}{3^n},\ \textbf{iv}.\sum_{y=0}^{n}\binom{n}{i}(-1)^i\frac{\breve{H}BL_{n+r+i}}{3^i}=(-1)^n\frac{\breve{H}BL_r}{3^n}.$$

Proof. i. With the Binet formula, we have

$$\sum_{y=0}^{n}\binom{n}{i}3^i\breve{H}BF_i=\sum_{i=0}^{n}\binom{n}{i}3^i\left(\frac{\overline{\lambda}\lambda^i-\overline{\psi}\psi^i}{\lambda-\psi}\right)=\frac{\overline{\lambda}}{\lambda-\psi}\sum_{i=0}^{n}\binom{n}{i}(3\lambda)^i-\frac{\overline{\psi}}{\lambda-\psi}\sum_{i=0}^{n}\binom{n}{i}(3\psi)^i$$

$$=\frac{\overline{\lambda}}{\lambda-\psi}(1+3\lambda)^n-\frac{\overline{\psi}}{\lambda-\psi}(1+3\psi)^n=\frac{\overline{\lambda}\lambda^{2n}-\overline{\psi}\psi^{2n}}{\lambda-\psi}=\breve{H}BF_{2n}.$$

The proofs of others can also be shown in this way. □

In the following theorems, we give special generating functions of the $\breve{H}BF_n$ and $\breve{H}BL_n$ quaternions. In addition, we obtain Binet formulas of $\breve{H}BF_n$ and $\breve{H}BL_n$ quaternions with the help of the generating functions.

Theorem 14. *The special generating functions for hyperbolic Bronze Fibonacci and Bronze Lucas quaternions are given as follows, respectively:*

$$\textbf{i}.f(s)=\sum_{n=0}^{infty}\breve{H}BF_ns^n=\frac{(1-3s)\breve{H}BF_0+\breve{H}BF_1s}{1-3s-s^2},\textbf{ii}.l(s)=\sum_{n=0}^{infty}\breve{H}BL_ns^n=\frac{(1-3s)\breve{H}BL_0+\breve{H}BL_1s}{1-3s-s^2},$$

$$\textbf{iii}.\ \sum_{n=0}^{infty}\frac{\breve{H}BF_n}{n!}s^n=\frac{\overline{\lambda}e^{\lambda s}-\overline{\psi}e^{\psi s}}{\lambda-\psi},\ \textbf{iv}.\ \sum_{n=0}^{infty}\frac{\breve{H}BL_n}{n!}s^n=\overline{\lambda}e^{\lambda s}+\overline{\psi}e^{\psi s}.$$

Proof. i. The following equations are written for the hyperbolic Bronze Fibonacci quaternions:

$$f(s)=\sum_{n=0}^{infty}\breve{H}BF_ns^n=\breve{H}BF_0+\breve{H}BF_1s+\sum_{n=2}^{infty}\breve{H}BF_ns^n$$
$$=\breve{H}BF_0+\breve{H}BF_1s+\sum_{n=2}^{infty}(3\breve{H}BF_{n-1}+\breve{H}BF_{n-2})s^n$$
$$=\breve{H}BF_0+\breve{H}BF_1s+3s\left(-\breve{H}BF_0+f(s)\right)+s^2f(s)$$
$$=\frac{(1-3s)\breve{H}BF_0+\breve{H}BF_1s}{1-3s-s^2}.$$

The proofs of others can also be shown in this way. □

Theorem 15. *For $\breve{H}BF_n$ and $\breve{H}BL_n$ quaternions, the Binet formulas can be obtained with the help of the generating functions.*

Proof. With the help of the roots of the characteristic equation of these quaternions, the roots of the $1-3s-s^2=0$ equation become $\frac{1}{\lambda}$ and $\frac{1}{\psi}$. For $\breve{H}BF_n$ quaternions, we obtain

$$\frac{(1-3s)\breve{H}BF_0+\breve{H}BF_1s}{1-3s-s^2}=\frac{1}{\lambda-\psi}\overline{\lambda}\frac{1}{1-\lambda s}-\frac{1}{\lambda-\psi}\overline{\psi}\frac{1}{1-\psi s}$$
$$=\sum_{n=0}^{infty}\left(\frac{1}{\lambda-\psi}\overline{\lambda}\lambda^n-\frac{1}{\lambda-\psi}\overline{\psi}\psi^n\right)s^n$$
$$=\sum_{n=0}^{infty}\breve{H}BF_ns^n.$$

Similarly, the Binet formula of the $\breve{H}BL_n$ quaternions is found. □

In the next lemma, we give the properties that will be used in the proof of many theorems.

Lemma 1. *We obtain*

$$\textbf{i.}\ \overline{\lambda}+\overline{\psi}=\breve{H}BL_0,\ \textbf{ii.}\ \overline{\lambda\psi}=(0,2\psi,2\psi^2,36+\lambda-\psi),$$

$$\textbf{iii.}\ \overline{\psi\lambda}=(0,2\lambda,2\lambda^2,36+\psi-\lambda),\ \textbf{iv.}\ \overline{\lambda\psi}+\overline{\psi\lambda}=(0,6,22,72),$$

$$\textbf{v.}\ \overline{\lambda\psi}-\overline{\psi\lambda}=(0,1,3,2)\ \ \textbf{vi.}\ \overline{\lambda}^2=2\overline{\lambda}+(-1+\lambda^2+\lambda^4+\lambda^6)i_1,$$

$$\textbf{vii.}\ \overline{\psi}^2=2\overline{\psi}+(-1+\psi^2+\psi^4+\psi^6)i_1,\ \textbf{viii.}\ \overline{\lambda}^2+\overline{\psi}^2=2\breve{H}BL_0+1426i_1.$$

Proof. ii. If the definition is used, we get

$$\overline{\lambda\psi}=(i_1+\lambda i_2+\lambda^2i_3+\lambda^3i_4)(i_1+\psi i_2+\psi^2i_3+\psi^3i_4)=i_1+\lambda i_2+\lambda^2i_3+\lambda^3i_4$$
$$+\psi i_2+\lambda\psi i_1-\lambda^2\psi i_4+\lambda^3\psi i_3+\psi^2i_3+\lambda\psi^2i_4+\lambda^2\psi^2i_1-\lambda^3\psi^2i_2+\psi^3i_4$$
$$-\lambda\psi^3i_3+\lambda^2\psi^3i_2+\lambda^3\psi^3i_1.$$

Thus, we obtain

$$\overline{\lambda\psi} = 2\psi i_2 + 2\psi^2 i_3 + (36 + \lambda - \psi) i_4 = (0, 2\psi, 2\psi^2, 36 + \lambda - \psi\ ).$$

The proofs of others can also be shown in this way. □

In the following theorems, we calculate some identities for $\breve{H}BF_n$ and $\breve{H}BL_n$ quaternions.

Theorem 16. ***(Cassini Identity)*** *Let* $n \in \mathbb{N}$. *We obtain*

i. $\breve{H}BF_{n+1}\breve{H}BF_{n-1} - \breve{H}BF_n^2 = (-1)^n(0, 0, 2, 39)$, **ii.** $\breve{H}BL_{n+1}\breve{H}BL_{n-1} - \breve{H}BL_n^2 = (-1)^{n-1}(0, 0, 26, 507)$.

Proof. i. With the Binet formula, we get

$$\breve{H}BF_{n+1}\breve{H}BF_{n-1} - \breve{H}BF_n^2 = \left(\frac{\overline{\lambda}\lambda^{n+1} - \overline{\psi}\psi^{n+1}}{\lambda - \psi}\right)\left(\frac{\overline{\lambda}\lambda^{n-1} - \overline{\psi}\psi^{n-1}}{\lambda - \psi}\right) - \left(\frac{\overline{\lambda}\lambda^{n} - \overline{\psi}\psi^{n}}{\lambda - \psi}\right)\left(\frac{\overline{\lambda}\lambda^{n} - \overline{\psi}\psi^{n}}{\lambda - \psi}\right)$$

$$= \frac{-\overline{\lambda\psi}\lambda^{n+1}\psi^{n-1} - \overline{\psi\lambda}\lambda^{n-1}\psi^{n+1} + \overline{\lambda\psi}\lambda^{n}\psi^{n} + \overline{\psi\lambda}\lambda^{n}\psi^{n}}{(\lambda - \psi)^2} = \frac{\overline{\lambda\psi}\lambda^{n}\psi^{n}\left(-\frac{\lambda}{\psi} + 1\right) + \overline{\psi\lambda}\lambda^{n}\psi^{n}\left(1 - \frac{\psi}{\lambda}\right)}{(\lambda - \psi)^2}$$

$$= \frac{(-1)^n}{\lambda - \psi}\left(\psi\overline{\psi\lambda} - \lambda\overline{\lambda\psi}\right).$$

With the help of Lemma 1., we obtain

$$\breve{H}BF_{n+1}\breve{H}BF_{n-1} - \breve{H}BF_n^2 = (-1)^n(0, 0, 2, 39).$$

ii. With the Binet formula, we have

$$\breve{H}BL_{n+1}\breve{H}BL_{n-1} - \breve{H}BL_n^2 = \left(\overline{\lambda}\lambda^{n+1} + \overline{\psi}\psi^{n+1}\right)\left(\overline{\lambda}\lambda^{n-1} + \overline{\psi}\psi^{n-1}\right) - \left(\overline{\lambda}\lambda^{n} + \overline{\psi}\psi^{n}\right)\left(\overline{\lambda}\lambda^{n} + \overline{\psi}\psi^{n}\right)$$

$$= \overline{\lambda\psi}\lambda^{n+1}\psi^{n-1} + \overline{\psi\lambda}\lambda^{n-1}\psi^{n+1} - \overline{\lambda\psi}\lambda^{n}\psi^{n} - \overline{\psi\lambda}\lambda^{n}\psi^{n}$$

$$= \overline{\lambda\psi}\lambda^{n}\psi^{n}\left(\frac{\lambda}{\psi} - 1\right) + \overline{\psi\lambda}\lambda^{n}\psi^{n}\left(-1 + \frac{\psi}{\lambda}\right) = (-1)^{n-1}(\lambda - \psi)\left(\lambda\overline{\lambda\psi} - \psi\overline{\psi\lambda}\right)$$

Then, with the help of Lemma 1., we obtain

$$\breve{H}BL_{n+1}\breve{H}BL_{n-1} - \breve{H}BL_n^2 = (-1)^{n-1}(0, 0, 26, 507).\ \square$$

Theorem 17. ***(Catalan Identity)*** *Let* $c, n \in \mathbb{N}$. *We obtain*

i. $\breve{H}BF_{n+c}\breve{H}BF_{n-c} - \breve{H}BF_n^2 = (-1)^{n-c}BF_c(0, 2BF_{c-1}, -2BF_{c-2}, -36BF_c - BF_{c+1} - BF_{c-1})$,

ii. $\breve{H}BL_{n+c}\breve{H}BL_{n-c} - \breve{H}BL_n^2 = (-1)^{n-c}BF_c(0, -26BF_{c-1}, 26BF_{c-2}, 468BF_c + 13BF_{c+1} + 13BF_{c-1})$.

Theorem 18. ***(Vajda Identity)*** *Let* $n, a, b \in \mathbb{N}$. *We obtain*

i. $\breve{H}BF_{n+a}\breve{H}BF_{n+b} - \breve{H}BF_n\breve{H}BF_{n+a+b} = (-1)^n BF_a(0, 2BF_{b+1}, 2BF_{b+2}, 36BF_b - BF_{b-1} - BF_{b+1})$,

ii. $\breve{H}BL_{n+a}\breve{H}BL_{n+b} - \breve{H}BL_n\breve{H}BL_{n+a+b} = (-1)^n BF_a(0, 26BF_{b+1}, -26BF_{b+2}, -468BF_b + 13BF_{b-1} + 13BF_{b+1})$.

Theorem 19. ***(D'ocagne Identity)*** *Let* $a, b \in \mathbb{N}$, *and* $a < b$. *We obtain*

i. $\breve{H}BF_b\breve{H}BF_{a+1} - \breve{H}BF_{b+1}\breve{H}BF_a = (-1)^a(0, -2BF_{b-a-1}, 2BF_{b-a-2}, 36BF_{b-a} + BF_{b-a+1} + BF_{b-a-1})$,

ii$\breve{H}BL_b\breve{H}BL_{k,a+1} - \breve{H}BL_{b+1}\breve{H}BL_a = (-1)^a(0, 26BF_{b-a-1}, -26BF_{b-a-2}, -468BF_{b-a} - 13BF_{b-a+1} - 13BF_{b-a-1})$.

The proofs of Theorem 17, 18 and 19 are shown similarly to Theorem 16., using Binet formulas and Lemma 1.

In the following theorems, we examine the relationships between hyperbolic Bronze Fibonacci and Bronze Lucas quaternions.

Theorem 20. *Let* $n \in \mathbb{N}$. *The following equations are true:*

$$\textbf{i. } \breve{H}BF_n = \frac{2}{13}\breve{H}BL_{n+1} - \frac{3}{13}\breve{H}BL_n, \textbf{ ii. } \breve{H}BF_n^* = \frac{2}{13}\breve{H}BL_{n+1}^* - \frac{3}{13}\breve{H}BL_n^*,$$

$$\textbf{iii. } \breve{H}BL_n = 2\breve{H}BF_{n+1} - 3\breve{H}BF_n, \textbf{ iv. } \breve{H}BL_n^* = 2\breve{H}BF_{n+1}^* - 3\breve{H}BF_n^*.$$

Proof. iii. The following relation is used for proofs:

$$\breve{H}BL_n = a\breve{H}BF_{n+1} + b\breve{H}BF_n.$$

For these n values, we obtain

$$\breve{H}BL_0 = a\breve{H}BF_1 + b\breve{H}BF_0,$$

$$\breve{H}BL_1 = a\breve{H}BF_2 + b\breve{H}BF_1.$$

We find

$$a = 2, \text{ and } b = -3.$$

The proofs of others can also be shown in this way. □

Theorem 21. *Let* $a, b \in \mathbb{N}$, *and* $a < b$. *We obtain*

$$\textbf{i. } \breve{H}BF_a\breve{H}BF_b - \breve{H}BF_b\breve{H}BF_a = \frac{1}{\sqrt{13}}(-1)^a BF_{b-a}(0,1,3,2), \textbf{ ii. } \breve{H}BL_a\breve{H}BL_b - \breve{H}BL_b\breve{H}BL_a = (-1)^a BF_{b-a}(0,26,78,-26),$$

$$\textbf{iii. } \breve{H}BF_b\breve{H}BL_a - \breve{H}BF_a\breve{H}BL_b = (-1)^a BF_{b-a}(0,6,22,72), \textbf{ iv. } \breve{H}BL_a\breve{H}BF_{a+1} - \breve{H}BL_{a+1}\breve{H}BF_a = (-1)^a(0,6,22,72),$$

Proof. With the Binet formula, we obtain

$$\textbf{i. } \breve{H}BF_a\breve{H}BF_b - \breve{H}BF_b\breve{H}BF_a = \left(\frac{\overline{\lambda}\lambda^a - \overline{\psi}\psi^a}{\lambda-\psi}\right)\left(\frac{\overline{\lambda}\lambda^b - \overline{\psi}\psi^b}{\lambda-\psi}\right) - \left(\frac{\overline{\lambda}\lambda^b - \overline{\psi}\psi^b}{\lambda-\psi}\right)\left(\frac{\overline{\lambda}\lambda^a - \overline{\psi}\psi^a}{\lambda-\psi}\right)$$

$$= \frac{-\lambda^a\psi^b\overline{\lambda\psi} - \lambda^b\psi^a\overline{\psi\lambda} + \lambda^b\psi^a\overline{\lambda\psi} + \lambda^a\psi^b\overline{\psi\lambda}}{(\lambda-\psi)(\lambda-\psi)} = \frac{\lambda^a\psi^a(\lambda^{b-a} - \psi^{b-a})(\overline{\lambda\psi} - \overline{\psi\lambda})}{(\lambda-\psi)(\lambda-\psi)}.$$

With the help of Lemma 1., we have

$$\breve{H}BF_a\breve{H}BF_b - \breve{H}BF_b\breve{H}BF_a = \frac{1}{\sqrt{13}}(-1)^a BF_{b-a}(0,1,3,2).$$

The proofs of others can also be shown in this way. □

In the following theorems, we associate the terms of the Bronze Fibonacci and Bronze Lucas sequences with their hyperbolic quaternion values.

Theorem 22. *Let* $a, b \in \mathbb{N}$, *and* $a < b$. *We obtain*

$$\textbf{i. } BL_a\breve{H}BF_b = \breve{H}BF_{a+b} + (-1)^a\breve{H}BF_{b-a}, \textbf{ ii. } BF_a\breve{H}BL_b = \breve{H}BF_{a+b} - (-1)^a\breve{H}BF_{b-a},$$

$$\textbf{iii. } BL_a\breve{H}BL_b = \breve{H}BF_{a+b} + (-1)^a\breve{H}BL_{b-a}, \textbf{ iv. } 2(-1)^a\breve{H}BF_0 = \breve{H}BF_aBL_a - \breve{H}BL_aBF_a.$$

Proof. i. If Binet formulas are used, we get

$$BL_a\breve{H}BF_b = (\lambda^a + \psi^a)\left(\frac{\overline{\lambda}\lambda^b - \overline{\psi}\psi^b}{\lambda-\psi}\right) = \frac{\overline{\lambda}\lambda^{a+b} - \overline{\psi}\psi^{a+b} + \overline{\lambda}\lambda^b\psi^a - \overline{\psi}\psi^b\lambda^a}{\lambda-\psi}$$

$$= \frac{\overline{\lambda}\lambda^{a+b} - \overline{\psi}\psi^{a+b}}{\lambda-\psi} + \frac{\lambda^a\psi^a(\overline{\lambda}\lambda^{b-a} - \overline{\psi}\psi^{b-a})}{\lambda-\psi} = BL_a\breve{H}BF_b.$$

The proofs of the others may be found similarly. □

Theorem 23. *Let* $a, b \in \mathbb{N}$, *and* $a < b$. *We obtain*

i. $\breve{H}BF_{a+2b} = \frac{BF_{2b}}{3}\breve{H}BL_{a+1} - \frac{BL_{2b-1}}{3}\breve{H}BF_a$, **ii.** $\breve{H}BL_{a+2b} = \sqrt{13}\frac{BF_{2b}}{3}\breve{H}BF_{a+1} - \frac{BL_{2b-1}}{3}\breve{H}BL_a$,

iii. $\breve{H}BF_{a+2b} = \frac{BF_{2b}}{3}\breve{H}BF_{a+2} - \frac{BL_{2b-2}}{3}\breve{H}BF_a$, **iv.** $\breve{H}BL_{a+2b} = \frac{BF_{2b}}{3}\breve{H}BL_{a+2} - \frac{BF_{2b-2}}{3}\breve{H}BL_a$.

Proof. iv. With the Binet formula, we have

$$\frac{BF_{2b}}{3}\breve{H}BL_{a+2} - \frac{BF_{2b-2}}{3}\breve{H}BL_a = \frac{1}{3}\frac{\lambda^{2b}-\psi^{2b}}{\lambda-\psi}\left(\overline{\lambda}\lambda^{a+2}+\overline{\psi}\psi^{a+2}\right) - \frac{1}{3}\frac{\lambda^{2b-2}-\psi^{2b-2}}{\lambda-\psi}\left(\overline{\lambda}\lambda^{a}+\overline{\psi}\psi^{a}\right)$$

$$= \frac{\overline{\lambda}\lambda^{a+2b}\left(\lambda^2-\frac{1}{\lambda^2}\right) - \overline{\psi}\psi^{2a+b}\left(\psi^2-\frac{1}{\psi^2}\right)}{3(\lambda-\psi)}$$

Since, $\psi^2 = \frac{1}{\lambda^2}$, $\lambda^2 = \frac{1}{\psi^2}$, and $\lambda + \psi = 3$ we obtain

$$\breve{H}BL_{a+2b} = \frac{BF_{2b}}{3}\breve{H}BL_{a+2} - \frac{BF_{2b-2}}{3}\breve{H}BL_a.$$

The proofs of others can also be shown in this way. □

Theorem 24. *Let* $a, b \in \mathbb{N}$, *and* $a < b$. *We obtain*

i. $\breve{H}BF_{2a+b+1} = \frac{BL_{2a+1}}{3}\breve{H}BF_{b+1} - \frac{BF_{2a}}{3}\breve{H}BL_b$, **ii.** $\breve{H}BL_{2a+b+1} = \frac{BL_{2a+1}}{3}\breve{H}BL_{b+1} - \frac{13}{3}BF_{2a}\breve{H}BF_b$,

iii. $\breve{H}BF_{2a+b+1} = \frac{BL_{2a+1}}{36}\breve{H}BF_{b+3} - \frac{BF_{2a-2}}{36}\breve{H}BL_b$, **iv.** $\breve{H}BL_{2a+b+1} = \frac{BL_{2a+1}}{36}\breve{H}BL_{b+3} - \frac{13}{36}BF_{2a-2}\breve{H}BF_b$.

Proof. i. If Binet formulas are used, we get

$$BL_{2a+1}\breve{H}BF_{b+1} - BF_{2a}\breve{H}BL_b$$

$$= (\lambda^{2a+1}+\psi^{2a+1})\left(\frac{\overline{\lambda}\lambda^{b+1}-\overline{\psi}\psi^{b+1}}{\lambda-\psi}\right) - \left(\frac{\lambda^{2a}-\psi^{2a}}{\lambda-\psi}\right)\left(\overline{\lambda}\lambda^{b}+\overline{\psi}\psi^{b}\right)$$

$$= \frac{1}{\lambda-\psi}\left(\overline{\lambda}\lambda^{2a+b+1}\left(\lambda-\frac{1}{\lambda}\right) - \overline{\psi}\psi^{2a+b+1}\left(\psi-\frac{1}{\psi}\right)\right)$$

Since, $\lambda + \psi = 3$ and $\lambda\psi = -1$ we obtain

$$\breve{H}BF_{2a+b+1} = \frac{BL_{2a+1}}{3}\breve{H}BF_{b+1} - \frac{BF_{2a}}{3}\breve{H}BL_b.$$

The proofs of others can also be shown in this way. □

Theorem 25. *Let* $a, b, c, d \in \mathbb{N}$ *and* $c < a, b$. *We obtain*

i. $\breve{H}BF_{a+b} = BF_a\breve{H}BF_{b+1} + BF_{a-1}\breve{H}BF_b$, **ii.** $\breve{H}BF_{2a+b} = BL_a\breve{H}BF_{a+b} - (-1)^a\breve{H}BF_b$,

iii. $\breve{H}BF_{ac+d} = \frac{BF_{ac}}{BF_a}\breve{H}BF_{a+d} - (-1)^a\frac{BF_{ac-a}}{BF_a}\breve{H}BF_d$, **iv.** $(-1)^{ac}BF_{ab-ac}\breve{H}BF_d = \breve{H}FB_{ac+d}BF_{ab} - \breve{H}BF_{ab+d}BF_{ac}$,

v. $\breve{H}BF_{a+b}\breve{H}BL_{a+c} - \breve{H}BF_{a+c}\breve{H}BL_{a+b} = (-1)^{a+b+1}(0{,}6, 22{,}72)$.

Proof. i. If Binet formulas are used, we get

$$BF_a\breve{H}BF_{b+1} + BF_{a-1}\breve{H}BF_b = \left(\frac{\lambda^a-\psi^a}{\lambda-\psi}\right)\left(\frac{\overline{\lambda}\lambda^{b+1}-\overline{\psi}\psi^{b+1}}{\lambda-\psi}\right) + \left(\frac{\lambda^{a-1}-\psi^{a-1}}{\lambda-\psi}\right)\left(\frac{\overline{\lambda}\lambda^{b}-\overline{\psi}\psi^{b}}{\lambda-\psi}\right)$$

$$= \frac{1}{(\lambda-\psi)(\lambda-\psi)}\left(\overline{\lambda}\lambda^{a+b}\left(\lambda+\frac{1}{\lambda}\right) + \overline{\psi}\psi^{a+b}\left(\psi+\frac{1}{\psi}\right)\right).$$

Since, $\frac{1}{\lambda} = -\psi$ and $\frac{1}{\psi} = -\lambda$ we obtain

$$\breve{H}BF_{a+b} = BF_a\breve{H}BF_{b+1} + BF_{a-1}\breve{H}BF_b.$$

The proofs of others can also be shown in this way. □

Theorem 26. *Let* $a, b \in \mathbb{N}$, *and* $a < b$. *We obtain*

i. $2\breve{H}BF_{a+b} = BF_a\breve{H}BL_b + BL_a\breve{H}BF_b$, **ii.** $2(-1)^a\breve{H}BL_{b-a} = \breve{H}BL_bBL_a - 13\breve{H}BF_bBF_a$,

iii. $2\breve{H}BL_{a+b} = BL_a\breve{H}BL_b + 13BF_a\breve{H}BF_b$, **iv.** $2(-1)^a\breve{H}BF_{b-a} = \breve{H}BF_bBL_a - \breve{H}BL_bBF_a$.

Theorem 27. *Let* $a \in \mathbb{N}$. *We obtain*

i. $13\breve{H}BF_a^2 - \breve{H}BL_a^2 = 4(-1)^{a+1}(0, 3, 11, 36)$,

ii. $13\breve{H}BF_a^2 + \breve{H}BL_a^2 = 4\breve{H}BL_{2a} + (-2BL_{2a} + 2BL_{2a+2} + 2BL_{2a+4} + 2BL_{2a+6})i_1$.

The proofs of Theorem 26 and 27 are shown similarly to Theorem 25, using Binet formulas and Lemma 1.

4. Conclusions

In this study, we determined some properties of Bronze Fibonacci and Bronze Lucas sequences. Then we found many relations between the roots of the characteristic equation of these sequences and these sequences. Also, we obtained interesting relations between the terms of these sequences. We established relations between the positive and negative index terms of these sequences and tried to relate the sums of squares of two consecutive terms to these sequences. In addition, we presented the application of generalized Bronze Fibonacci sequences to hyperbolic quaternions. For these hyperbolic quaternions, we gave general summation formulas, general generating functions, exponential generating functions, etc. Moreover, we obtained Binet formulas in different ways. Furthermore, we calculated special identities of these hyperbolic quaternions such as the Cassini identity, and Vajda identity. We found relations between hyperbolic Bronze Fibonacci and Bronze Lucas quaternions. We produced some special results with the conjugates of these hyperbolic quaternions. Finally, we related the terms of the generalized Bronze Fibonacci sequences to the hyperbolic quaternion values. In the future, we can spread the new approach to hyperbolic Bronze Fibonacci and Bronze Lucas octonions and sedenions.

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