MATHEMATICA MONTISNIGRI
Vol LXI (2024)

A NEW APPROACH TO OLIVIER SEQUENCES

HAKAN AKKUŞ[1*] AND ENGİN ÖZKAN[2]

[1] Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Erzincan, Türkiye
[2] Department of Mathematics, Faculty of Sciences, Marmara University, İstanbul, Türkiye
*Corresponding author. E-mail: hakan.akkus@ogr.ebyu.edu.tr

DOI: 10.20948/mathmontis-2024-61-2

Summary. In this study, we define the *k*-Olivier, *k*-Olivier-Lucas, and Modified *k*-Olivier sequences and give some terms of these sequences. Then, we obtain the generating functions, summation formulas, etc. Also, we obtain the Binet formulas in three different ways. The first is in the known classical way, the second is with the help of the sequence's generating functions, and the third is with the help of the matrices. In addition, the relations between the terms of the *k*-Olivier, *k*-Olivier-Lucas, Modified *k*-Olivier, third-order Fibonacci-Pell, Adjusted Pell-Padovan, Olivier, Olivier-Lucas, Modified Olivier, third-order Lucas-Pell, Pell-Padovan, and Pell-Perrin sequences are presented. Finally, we associate the terms of these sequences with matrices.

1 INTRODUCTION

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields such as engineering [1], chemistry [2], biomathematics [3], phylotaxis [4], cryptology [5], etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the *k*-Jacobsthal, *k*-Jacobsthal-Lucas, *k*-Oresme, Leonardo, *k*-Fibonacci, *k*-Lucas sequences, etc (see for details in [6-14] and [22-26]).

For $n \in \mathbb{N}$, Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ with } F_0 = 0 \text{ and } F_1 = 1,$$

$$L_{n+2} = L_{n+1} + L_n, \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

For F_n and L_n Binet formulas are given by the following relations, respectively,

$$F_n = \frac{m^n - p^n}{m - p} \text{ and } L_n = m^n + p^n$$

Where $m = \frac{1+\sqrt{5}}{2}$ and $p = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. Here the number m is the known golden ratio.

Olivier numbers, which are related to Fibonacci and Lucas numbers, were first introduced by Yüksel Soykan. The application of Olivier numbers in many areas has attracted the

2020 Mathematics Subject Classification: 05A15, 11B39, 11B83, 05A15.
Key words and Phrases: Olivier sequences; Binet formulas; Generating function; Fibonacci numbers; Summation formulas

attention of scientists. Yüksel Soykan found many interesting properties of Olivier sequences and related these sequences to matrices.

In [15] and [16], for $n \in \mathbb{N}$, Soykan defined the third-order Fibonacci-Pell sequence G_n, third-order Lucas-Pell sequence B_n, Pell-Perrin sequence C_n, Pell-Padovan sequence R_n, and Adjusted Pell-Padovan sequence M_n are defined by the recurrence relations, respectively,

$$G_{n+3} = 2G_{n+1} + G_n, \text{ with } G_0 = 1, G_1 = 0, \text{ and } G_2 = 2,$$
$$B_{n+3} = 2B_{n+1} + B_n, \text{ with } B_0 = 3, B_1 = 0, \text{ and } B_2 = 4,$$
$$C_{n+3} = 2C_{n+1} + C_n, \text{ with } C_0 = 3, C_1 = 0, \text{ and } C_2 = 2,$$
$$R_{n+3} = 2R_{n+1} + R_n, \text{ with } R_0 = 1, R_1 = 1, \text{ and } R_2 = 1,$$
$$M_{n+3} = 2M_{n+1} + M_n, \text{ with } M_0 = 0, M_1 = 1, \text{ and } M_2 = 0.$$

For G_n, B_n, C_n, R_n, and M_n Binet formulas are given by the following relations, respectively,

$$G_n = \frac{1}{\sqrt{5}}m^n - \frac{1}{\sqrt{5}}p^n + r^n,\ B_n = m^n + p^n + r^n,$$
$$C_n = \left(2 - \frac{3}{\sqrt{5}}\right)m^n + \left(2 + \frac{3}{\sqrt{5}}\right)p^n - r^n,\ R_n = \left(1 - \frac{1}{\sqrt{5}}\right)m^n + \left(1 + \frac{1}{\sqrt{5}}\right)p^n - r^n,$$
$$M_n = \frac{m^{n+1}}{(m-p)(m-r)} + \frac{p^{n+1}}{(p-m)(p-r)} + \frac{r^{n+1}}{(r-m)(r-p)}.$$

Hence, $m = \frac{1+\sqrt{5}}{2}$, $p = \frac{1-\sqrt{5}}{2}$, and $r = -1$ are the roots of the characteristic equation $r^3 - 2r - 1 = 0$.

In addition, for $n \in \mathbb{N}$, Olivier numbers O_n, Olivier-Lucas numbers K_n, and Modified Olivier numbers T_n are defined by the recurrence relations, respectively,

$$O_{n+3} = 2O_{n+1} + O_n + 1, \text{ with } O_0 = 0, O_1 = 1, \text{ and } O_2 = 1,$$
$$K_{n+3} = 2K_{n+1} + K_n - 2, \text{ with } K_0 = 4, K_1 = 1, \text{ and } K_2 = 5,$$
$$T_{n+3} = 2T_{n+1} + T_n + 1, \text{ with } T_0 = 1, T_1 = 0, \text{ and } T_2 = 2.$$

For O_n, K_n, and T_n Binet formulas are given by the following relations, respectively,

$$O_n = \frac{2m+1}{(4m+3)(m-1)}m^n + \frac{2p+1}{(4p+3)(p-1)}p^n - \frac{1}{2}r^n - \frac{1}{2},$$
$$K_n = m^n + p^n + r^n + 1,$$
$$T_n = \frac{m-1}{(m-p)(m-r)(m-1)}m^{n+2} + \frac{p-1}{(p-m)(p-r)(p-1)}p^{n+2} + \frac{r-1}{(r-m)(r-p)(r-1)}r^{n+2}.$$

Hence, $m = \frac{1+\sqrt{5}}{2}$, $p = \frac{1-\sqrt{5}}{2}$, and $r = -1$ are the roots of the characteristic equation $r^3 - 2r - 1 = 0$. In [15] and [16], Soykan found many properties about these sequences.

With the help of the recurrence relation of the Fibonacci sequence, k-sequences were introduced, and these sequences have an important place in number theory [17]. In [18], Falcon and Plaza introduced the k-Fibonacci sequence and obtained many properties related to this sequence. In addition, Falcon defined the k-Lucas sequences [19]. Moreover, Falcon

applied the Hankel transform to the k-Fibonacci sequence. In [20], Özkan and Akkuş obtained the terms of Fibonacci sequences by applying different transformations.

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the k-Fibonacci and Olivier sequences. We call these sequences the k-Olivier, k-Olivier-Lucas, and Modified k-Olivier sequences and denote them as $\mathcal{S}_{k,n}$, $\mathcal{G}_{k,n}$ and $\mathcal{V}_{k,n}$, respectively.

We separate the article into three parts.

In chapter 2, we define the k-Olivier, k-Olivier-Lucas, and Modified k-Olivier sequences. Then, we give the characteristic equation, Binet formulas, the summation formulas, generating functions, and some properties for these sequences. In addition, we obtain Binet formulas with the help of generating functions of these sequences.

In chapter 3, we examine the relations among the terms of the k-Olivier, k-Olivier-Lucas, Modified k-Olivier, Olivier, Olivier-Lucas, and Modified Olivier sequences. Then, we associate the k-Olivier, k-Olivier-Lucas, and Modified k-Olivier sequences with G_n, B_n, C_n, R_n, and M_n sequences. Finally, we associate the terms of these sequences with matrices.

2 k-OLIVIER, k-OLIVIER-LUCAS, AND MODIFIED k-OLIVIER SEQUENCES

For $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, and Modified k-Olivier $\mathcal{V}_{k,n}$ are defined by, respectively,

$$\mathcal{S}_{k,n+3} = 2\mathcal{S}_{k,n+1} + \mathcal{S}_{k,n} + k, \text{ with } \mathcal{S}_{k,0} = 0, \mathcal{S}_{k,1} = 0, \text{ and } \mathcal{S}_{k,2} = 1,$$

$$\mathcal{G}_{k,n+3} = 2\mathcal{G}_{k,n+1} + \mathcal{G}_{k,n} - 2k, \text{ with } \mathcal{G}_{k,0} = 4, \mathcal{G}_{k,1} = 1 \text{ and } \mathcal{G}_{k,2} = 5,$$

$$\mathcal{V}_{k,n+3} = 2\mathcal{V}_{k,n+1} + \mathcal{V}_{k,n} + k, \text{ with } \mathcal{V}_{k,0} = 1, \mathcal{V}_{k,1} = 0, \text{ and } \mathcal{V}_{k,2} = 2.$$

Also, the four-order recurrence relations of the sequences $\mathcal{S}_{k,n}$, $\mathcal{G}_{k,n}$, and $\mathcal{V}_{k,n}$ are as follows:

$$\mathcal{S}_{k,n+4} = \mathcal{S}_{k,n+3} + 2\mathcal{S}_{k,n+2} - \mathcal{S}_{k,n+1} - \mathcal{S}_{k,n}, \text{ with } \mathcal{S}_{k,0} = 0, \mathcal{S}_{k,1} = 0, \mathcal{S}_{k,2} = 1, \mathcal{S}_{k,3} = k,$$

$$\mathcal{G}_{k,n+4} = \mathcal{G}_{k,n+3} + 2\mathcal{G}_{k,n+2} - \mathcal{G}_{k,n+1} - \mathcal{G}_{k,n}, \text{with } \mathcal{G}_{k,0} = 4, \mathcal{G}_{k,1} = 1, \mathcal{G}_{k,2} = 5, \mathcal{G}_{k,3} = 6 - 2k,$$

$$\mathcal{V}_{k,n+4} = \mathcal{V}_{k,n+3} + 2\mathcal{V}_{k,n+2} - \mathcal{V}_{k,n+1} - \mathcal{V}_{k,n} \text{ with } \mathcal{V}_{k,0} = 0, \mathcal{V}_{k,1} = 0, \mathcal{V}_{k,2} = 1, \mathcal{V}_{k,3} = 1 + k.$$

Then, let's give some information about the equations of these sequences.

The characteristic equation of the k-Olivier, k-Olivier-Lucas, and Modified k-Olivier sequences is

$$t^4 - t^3 - 2t^2 + t + 1 = 0 = (t^2 - t - 1)(t + 1)(t - 1) = 0.$$

The roots of this equation are:

$$m = \frac{1+\sqrt{5}}{2}, p = \frac{1-\sqrt{5}}{2}, r = -1 \text{ and } s = 1.$$

The relationship between these roots is given below:

$$m + p = 1, mp = -1, m + p + r + s = 1, mprs = 1,$$

$$mp + mr + ms + pr + ps + rs = -2, \text{ and } mpr + mps + mrs + prs = -1.$$

The $\mathcal{S}_{k,n}$, $\mathcal{G}_{k,n}$, and $\mathcal{V}_{k,n}$ values for the first six n natural numbers are given below:

- $\mathcal{S}_{k,0} = 0$ $\quad \mathcal{G}_{k,0} = 4$ $\quad \mathcal{V}_{k,0} = 1,$
- $\mathcal{S}_{k,1} = 0$ $\quad \mathcal{G}_{k,1} = 1$ $\quad \mathcal{V}_{k,1} = 0,$
- $\mathcal{S}_{k,2} = 1$ $\quad \mathcal{G}_{k,2} = 5$ $\quad \mathcal{V}_{k,2} = 2,$
- $\mathcal{S}_{k,3} = k$ $\quad \mathcal{G}_{k,3} = 6 - 2k$ $\quad \mathcal{V}_{k,3} = k + 1,$
- $\mathcal{S}_{k,4} = k + 2$ $\quad \mathcal{G}_{k,4} = 11 - 2k$ $\quad \mathcal{V}_{k,4} = k + 4,$
- $\mathcal{S}_{k,5} = 3k + 1$ $\quad \mathcal{G}_{k,5} = 17 - 6k$ $\quad \mathcal{V}_{k,5} = 3k + 4,$

In the following theorem, the Binet formulas of the k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, and Modified k-Olivier $\mathcal{V}_{k,n}$ sequences are expressed.

Theorem 2.1. Let $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. We obtain

i. $\mathcal{S}_{k,n} = \frac{k-p}{(m-p)(m-r)(m-1)} m^n + \frac{k-m}{(p-m)(p-r)(p-1)} p^n + \frac{k-2}{(r-m)(r-p)(r-1)} r^n + \frac{k}{(1-m)(1-p)(1-r)}.$

ii. $\mathcal{G}_{k,n} = \frac{5-2k-p}{(m-p)(m-r)(m-1)} m^n + \frac{5-2k-m}{(p-m)(p-r)(p-1)} p^n - \frac{2k+3}{(r-m)(r-p)(r-1)} r^n - \frac{2k-3}{(1-m)(1-p)(1-r)}.$

iii. $\mathcal{V}_{k,n} = \frac{k+1-p}{(m-p)(m-r)(m-1)} m^n + \frac{k+1-m}{(p-m)(p-r)(p-1)} p^n + \frac{k-2}{(r-m)(r-p)(r-1)} r^n + \frac{k}{(1-m)(1-p)(1-r)}.$

Proof. i. The Binet form of a sequence is as follows

$$\mathcal{S}_{k,n} = xm^n + yp^n + zr^n + ts^n.$$

For these n values, we obtain

$$\mathcal{S}_{k,0} = x + y + z + t,$$
$$\mathcal{S}_{k,1} = xm + yp + zr + ts,$$
$$\mathcal{S}_{k,2} = xm^2 + yp^2 + zr^2 + ts^2,$$
$$\mathcal{S}_{k,3} = xm^3 + yp^3 + zr^3 + ts^3.$$

We find

$x = \frac{k-p}{(m-p)(m-r)(m-1)}$, $y = \frac{k-m}{(p-m)(p-r)(p-1)}$, $z = \frac{k-2}{(r-m)(r-p)(r-1)}$, and $t = \frac{k}{(1-m)(1-p)(1-r)}$.

The proofs of the others are shown similarly. □

In the following theorems, we give the generating functions of the k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, and Modified k-Olivier $\mathcal{V}_{k,n}$ sequences. In addition, we obtain Binet formulas of $\mathcal{S}_{k,n}$, $\mathcal{G}_{k,n}$, and $\mathcal{V}_{k,n}$ sequences with the help of generating functions.

Theorem 2.2. The generating functions for k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, and Modified k-Olivier $\mathcal{V}_{k,n}$ sequences are given as follows, respectively,

i. $s(t) = \sum_{n=0}^{\infty} \mathcal{S}_{k,n} t^n = \frac{t^2 + (k-1)t^3}{t^4 + t^3 - 2t^2 - t + 1}$, **ii.** $g(t) = \sum_{n=0}^{\infty} \mathcal{G}_{k,n} t^n = \frac{4 - 3t - 4t^2 - (2k+11)t^3}{t^4 + t^3 - 2t^2 - t + 1}$,

iii. $v(t) = \sum_{n=0}^{\infty} \mathcal{V}_{k,n} t^n = \frac{1 - t + kt^3}{t^4 + t^3 - 2t^2 - t + 1}$.

Proof. i. For the k-Olivier sequence, we have

$$\begin{aligned} s(t) &= \sum_{n=0}^{\infty} \mathcal{S}_{k,n} t^n = t^2 + kt^3 + \sum_{n=4}^{\infty} \mathcal{S}_{k,n} t^n \\ &= t^2 + kt^3 + \sum_{n=4}^{\infty} \mathcal{S}_{k,n-1} t^n + 2\sum_{n=4}^{\infty} \mathcal{S}_{k,n-2} t^n - \sum_{n=4}^{\infty} \mathcal{S}_{k,n-3} t^n - \sum_{n=4}^{\infty} \mathcal{S}_{k,n-4} t^n \\ &= t^2 + kt^3 + t\sum_{n=3}^{\infty} \mathcal{S}_{k,n} t^n + 2t^2 \sum_{n=2}^{\infty} \mathcal{S}_{k,n} t^n - t^3 \sum_{n=1}^{\infty} \mathcal{S}_{k,n} t^n - t^4 \sum_{n=0}^{\infty} \mathcal{S}_{k,n} t^n \\ &= \frac{1+k-t^3}{t^4+t^3-2t^2-t+1}. \end{aligned}$$

The proofs of the others are shown similarly. □

Theorem 2.3. For $\mathcal{S}_{k,n}$, $\mathcal{G}_{k,n}$, and $\mathcal{V}_{k,n}$ the Binet formulas can be obtained with the help of the generating functions.

Proof. With the help of the roots of the characteristic equation of these sequences, the roots of the $t^4 + t^3 - 2t^2 - t + 1 = 0$ equation become $\frac{1}{m}$, $\frac{1}{p}$, $\frac{1}{r}$, and $\frac{1}{s}$. For $\mathcal{S}_{k,n}$, we have

$$\begin{aligned} &\frac{1+k-t^3}{t^4+t^3-2t^2-t+1} \\ &= \frac{k-p}{(m-p)(m-r)(m-1)}\frac{1}{1-mt} + \frac{k-m}{(p-m)(p-r)(p-1)}\frac{1}{1-pt} + \frac{k-2}{(r-m)(r-p)(r-1)}\frac{1}{1-rt} + \frac{k}{(1-m)(1-p)(1-r)}\frac{1}{1-st} \\ &= \sum_{n=0}^{\infty}\left(\frac{k-p}{(m-p)(m-r)(m-1)}m^n + \frac{k-m}{(p-m)(p-r)(p-1)}p^n + \frac{k-2}{(r-m)(r-p)(r-1)}r^n + \frac{k}{(1-m)(1-p)(1-r)}\right)t^n \\ &= \sum_{n=0}^{\infty} \mathcal{S}_{k,n} t^n. \end{aligned}$$

Similarly, the Binet formulas of the sequences $\mathcal{G}_{k,n}$ and $\mathcal{V}_{k,n}$ are found. □

Next, we give special sum formulas of the k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, and Modified k-Olivier $\mathcal{V}_{k,n}$ sequences.

Theorem 2.4. Let $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. We obtain

i. $\sum_{s=0}^{n} \mathcal{S}_{k,s} = \frac{3\mathcal{S}_{k,n}+3\mathcal{S}_{k,n-1}+\mathcal{S}_{k,n-2}-(n-2)k-1}{2}$, **ii.** $\sum_{s=0}^{n} \mathcal{G}_{k,s} = \frac{3\mathcal{G}_{k,n}+3\mathcal{G}_{k,n-1}+\mathcal{G}_{k,n-2}+(n-2)2k-2}{2}$,

iii. $\sum_{s=0}^{n} \mathcal{V}_{k,s} = \frac{3\mathcal{V}_{k,n}+3\mathcal{V}_{k,n-1}+\mathcal{V}_{k,n-2}-(n-2)k-1}{2}$.

Proof. ii. From the definition of the k-Olivier-Lucas sequence, we obtain

$$\begin{aligned} \mathcal{G}_{k,3} &= 2\mathcal{G}_{k,1}+\mathcal{G}_{k,0} - 2k, \\ \mathcal{G}_{k,4} &= 2\mathcal{G}_{k,2} + \mathcal{G}_{k,1} - 2k, \\ &\vdots \\ \mathcal{G}_{k,n} &= 2\mathcal{G}_{k,n-2} + \mathcal{G}_{k,n-3} - 2k. \end{aligned}$$

So, we have

$$\begin{aligned} -10 + \sum_{s=0}^{n} \mathcal{G}_{k,s} &= 2\sum_{s=1}^{n-2} \mathcal{G}_{k,s} + \sum_{s=0}^{n-3} \mathcal{G}_{k,s} - (n-2)2k, \\ -10 + \sum_{s=0}^{n} \mathcal{G}_{k,s} &= 2\left(-\mathcal{G}_{k,n} - \mathcal{G}_{k,n-1} - \mathcal{G}_{k,0} + \sum_{s=0}^{n} \mathcal{G}_{k,s}\right) \\ &\quad + \left(-\mathcal{G}_{k,n} - \mathcal{G}_{k,n-1} - \mathcal{G}_{k,n-2} + \sum_{s=0}^{n} \mathcal{G}_{k,s}\right) - (n-2)2k. \end{aligned}$$

Thus, we obtain

$$\sum_{s=0}^{n} \mathcal{G}_{k,s} = \frac{3\mathcal{G}_{k,n}+3\mathcal{G}_{k,n-1}+\mathcal{G}_{k,n-2}+(n-2)2k-2}{2}.$$

The proofs of the others are shown similarly. □

3 RELATIONS AMONG SPECIAL SEQUENCES

In this chapter, we examine the relationships among the k-Olivier, k-Olivier-Lucas, Modified k-Olivier third-order Fibonacci-Pell, Adjusted Pell-Padovan, Olivier, Olivier-Lucas, Modified Olivier, third-order Lucas-Pell, Pell-Padovan, and Pell-Perrin sequences. In addition, we associate the terms of these sequences with matrices.

In the following theorems, we examine the relations among the k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, Modified k-Olivier $\mathcal{V}_{k,n}$, Olivier O_n, Olivier-Lucas K_n, and Modified Olivier T_n sequences.

Theorem 3.1. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

i. $\mathcal{S}_{k,n} = \frac{-8k^2+25k-19}{8k^3-36k^2+38k}\mathcal{G}_{k,n+3} + \frac{6k^2-33k+38}{8k^3-36k^2+38k}\mathcal{G}_{k,n+2} + \frac{4k^2-7k}{8k^3-36k^2+38k}\mathcal{G}_{k,n+1}$
$+ \frac{-4k^3+16k^2-4k-19}{8k^3-36k^2+38k}\mathcal{G}_{k,n},$

ii. $\mathcal{S}_{k,n} = -\frac{k+1}{k^2+k-1}\mathcal{V}_{k,n+3} + \frac{k+2}{k^2+k-1}\mathcal{V}_{k,n+2} + \frac{k+1}{k^2+k-1}\mathcal{V}_{k,n+1} + \frac{k^2-3}{k^2+k-1}\mathcal{V}_{k,n},$

iii. $\mathcal{S}_{k,n} = (k-2)O_{n+3} + (-2k+3)O_{n+2} + (-k+3)O_{n+1} + (3k-4)O_n,$

iv. $\mathcal{S}_{k,n} = \frac{11k-13}{10}K_{n+3} + \frac{-13k+24}{10}K_{n+2} + \frac{-11k+8}{10}K_{n+1} + \frac{8k-19}{10}K_n,$

v. $\mathcal{S}_{k,n} = (2k-4)T_{n+3} + (-4k+7)T_{n+2} + (-k+3)T_{n+1} + (4k-6)T_n.$

Proof. i. The following relation is used for proofs:

$$\mathcal{S}_{k,n} = a\,x\,\mathcal{G}_{k,n+3} + b\,x\,\mathcal{G}_{k,n+2} + c\,x\,\mathcal{G}_{k,n+1} + d\,x\,\mathcal{G}_{k,n}.$$

For these n values, we obtain

$$\mathcal{S}_{k,0} = a\,x\,\mathcal{G}_{k,3} + b\,x\,\mathcal{G}_{k,2} + c\,x\,\mathcal{G}_{k,1} + d\,x\,\mathcal{G}_{k,0},$$
$$\mathcal{S}_{k,1} = a\,x\,\mathcal{G}_{k,4} + b\,x\,\mathcal{G}_{k,3} + c\,x\,\mathcal{G}_{k,2} + d\,x\,\mathcal{G}_{k,1},$$
$$\mathcal{S}_{k,2} = a\,x\,\mathcal{G}_{k,5} + b\,x\,\mathcal{G}_{k,4} + c\,x\,\mathcal{G}_{k,3} + d\,x\,\mathcal{G}_{k,2},$$
$$\mathcal{S}_{k,3} = a\,x\,\mathcal{G}_{k,6} + b\,x\,\mathcal{G}_{k,5} + c\,x\,\mathcal{G}_{k,4} + d\,x\,\mathcal{G}_{k,3}.$$

We find

$$a = \frac{-8k^2+25k-19}{8k^3-36k^2+38k},\ b = \frac{6k^2-33k+38}{8k^3-36k^2+38k},\ c = \frac{4k^2-7k}{8k^3-36k^2+38k} \text{ and } d = \frac{-4k^3+16k^2-4k-19}{8k^3-36k^2+38k}.$$

The proofs of the others are shown similarly. □

Theorem 3.2. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

i. $\mathcal{G}_{k,n}=\frac{4k^2-9k+4}{k^3-3k^2+k+2}\mathcal{S}_{k,n+3}+\frac{-3k^2+8}{k^3-3k^2+k+2}\mathcal{S}_{k,n+2}+\frac{-2k+3}{k^2-k-1}\mathcal{S}_{k,n+1}+\frac{-2k^3+7k^2-10}{k^3-3k^2+k+2}\mathcal{S}_{k,n}$,

ii. $\mathcal{G}_{k,n}=\frac{6k^2-5k-4}{k^3-3k^2-3k+2}\mathcal{V}_{k,n+3}+\frac{-5k^2-7k+14}{k^3-3k^2-3k+2}\mathcal{V}_{k,n+2}-\frac{4k+1}{k^2+k-1}\mathcal{V}_{k,n+1}+\frac{-2k^3+5k^2+11k-16}{k^3-3k^2-3k+2}\mathcal{V}_{k,n}$,

iii. $\mathcal{G}_{k,n}=(-2k+7)O_{n+3}+(4k-10)O_{n+2}+(2k-7)O_{n+1}+(-6k+10)O_n$,

iv. $\mathcal{G}_{k,n}=\frac{-11k+11}{5}K_{n+3}+\frac{13k-13}{5}K_{n+2}+\frac{11k-11}{5}K_{n+1}+\frac{-8k+13}{5}K_n$,

v. $\mathcal{G}_{k,n}=(-4k+7)T_{n+3}+(8k-10)T_{n+2}+(2k-7)T_{n+1}+(-8k+10)T_n$.

Proof. v. If Binet formulas are used for proofs, we get

$$
\begin{aligned}
&(-4k+7)T_{n+3}+(8k-10)T_{n+2}+(2k-7)T_{n+1}+(-8k+10)T_n\\
&=(-4k+7)\left(\frac{m-1}{(m-p)(m-r)(m-1)}m^{n+5}+\frac{p-1}{(p-m)(p-r)(p-1)}p^{n+5}+\frac{r-1}{(r-m)(r-p)(r-1)}r^{n+5}\right)\\
&+(8k-10)\left(\frac{m-1}{(m-p)(m-r)(m-1)}m^{n+4}+\frac{p-1}{(p-m)(p-r)(p-1)}p^{n+4}+\frac{r-1}{(r-m)(r-p)(r-1)}r^{n+4}\right)\\
&+(2k-7)\frac{m-1}{(m-p)(m-r)(m-1)}m^{n+3}+\frac{p-1}{(p-m)(p-r)(p-1)}p^{n+3}+\frac{r-1}{(r-m)(r-p)(r-1)}r^{n+3}\\
&+(-8k+10)\frac{m-1}{(m-p)(m-r)(m-1)}m^{n+2}+\frac{p-1}{(p-m)(p-r)(p-1)}p^{n+2}+\frac{r-1}{(r-m)(r-p)(r-1)}r^{n+2}\\
&=\frac{5-2k-p}{(m-p)(m-r)(m-1)}m^n+\frac{5-2k-m}{(p-m)(p-r)(p-1)}p^n-\frac{2k+3}{(r-m)(r-p)(r-1)}r^n-\frac{2k-3}{(1-m)(1-p)(1-r)}\\
&=\mathcal{G}_{k,n}.
\end{aligned}
$$

The proofs of the others are shown similarly. □

Theorem 3.3. Let $k\in\mathbb{R}^+$, and $n\in\mathbb{N}$. The following equations are true:

i. $\mathcal{V}_{k,n}=\frac{k}{k^2-k-1}\mathcal{S}_{k,n+3}-\frac{k+1}{k^2-k-1}\mathcal{S}_{k,n+2}-\frac{k}{k^2-k-1}\mathcal{S}_{k,n+1}+\frac{k^2}{k^2-k-1}\mathcal{S}_{k,n}$,

ii. $\mathcal{V}_{k,n}=\frac{-12k^2+35k-19}{8k^3-36k^2+38k}\mathcal{G}_{k,n+3}+\frac{10k^2-45k+38}{8k^3-36k^2+38k}\mathcal{G}_{k,n+2}+\frac{8k-17}{8k^2-36k+38}\mathcal{G}_{k,n+1}$
$+\frac{-4k^3+12k^2+8k-19}{8k^3-36k^2+38k}\mathcal{G}_{k,n}$,

iii. $\mathcal{V}_{k,n}=kO_{n+3}-2kO_{n+2}+(-k+1)O_{n+1}+(3k-1)O_n$,

iv. $\mathcal{V}_{k,n}=\frac{11k-7}{10}K_{n+3}+\frac{-13k+16}{10}K_{n+2}+\frac{-11k+2}{10}K_{n+1}+\frac{8k-11}{10}K_n$,

v. $\mathcal{V}_{k,n}=(2k-2)T_{n+3}+(-4k+4)T_{n+2}+(-k+1)T_{n+1}+(4k-3)T_n$.

Proof. The proofs are shown similarly to Theorem 3.2. □

Theorem 3.4. Let $k\in\mathbb{R}^+$, and $n\in\mathbb{N}$. The following equations are true:

i.

- $O_n=\frac{-k^2+2k-1}{k^4-3k^3+k^2+2k}\mathcal{S}_{k,n+3}+\frac{k^2-2k+1}{k^3-3k^2+k+2}\mathcal{S}_{k,n+2}-\frac{1}{k^3-k^2-k}\mathcal{S}_{k,n+1}+\frac{-k+1}{k^4-3k^3+k^2+2k}\mathcal{S}_{k,n}$,
- $O_n=\frac{-10k+19}{8k^3-36k^2+38k}\mathcal{G}_{k,n+3}+\frac{-4k^2+16k-19}{k^3+32k^2+128k}\mathcal{G}_{k,n+2}+\frac{10k-19}{k^3+32k^2+128k}\mathcal{G}_{k,n+1}$

$$+\frac{1}{4k^2-18k+19}\mathcal{G}_{k,n},$$

- $O_n=\frac{-2k^2+2k-1}{k^4-k^3-3k^2+2k}\mathcal{V}_{k,n+3}+\frac{k^2+1}{k^3-k^2-3k+2}\mathcal{V}_{k,n+2}+\frac{k-1}{k^3+k^2-k}\mathcal{V}_{k,n+1}+\frac{-3k+1}{k^4-k^3-3k^2+2k}\mathcal{V}_{k,n}.$

ii.

- $K_n=\frac{4k^3-9k^2+2k+2}{k^4-3k^3+k^2+2k}\mathcal{S}_{k,n+3}+\frac{-3k^2+2k+6}{k^3-3k^2+k+2}\mathcal{S}_{k,n+2}+\frac{-4k^2+3k+2}{k^3-k^2-k}\mathcal{S}_{k,n+1}+\frac{k^3+2k^2-6k-2}{k^4-3k^3+k^2+2k}\mathcal{S}_{k,n},$
- $K_n=\frac{-8k^2+27k-19}{4k^3-18k^2+19k}\mathcal{G}_{k,n+3}+\frac{6k^2-25k+19}{4k^3-18k^2+19k}\mathcal{G}_{k,n+2}+\frac{8k^2-27k+19}{4k^3-18k^2+19k}\mathcal{G}_{k,n+1}+\frac{-2k+7}{4k^2-18k+19}\mathcal{G}_{k,n},$
- $K_n=\frac{4k^3-3k^2-6k+2}{k^4-k^3-3k^2+2k}\mathcal{V}_{k,n+3}+\frac{-3k^2-3k+8}{k^3-k^2-3k+2}\mathcal{V}_{k,n+2}+\frac{-4k^2-3k+2}{k^3+k^2-k}\mathcal{V}_{k,n+1}+\frac{k^3+3k^2-4k-2}{k^4-k^3-3k^2+2k}\mathcal{V}_{k,n}.$

iii.

- $T_n=\frac{k^3-2k^2+k-1}{k^4-3k^3+k^2+2k}\mathcal{S}_{k,n+3}+\frac{-k^2+3}{k^3-3k^2+k+2}\mathcal{S}_{k,n+2}-\frac{1}{k^3-k^2-k}\mathcal{S}_{k,n+1}+\frac{k^3-k^2-2k+1}{k^4-3k^3+k^2+2k}\mathcal{S}_{k,n},$
- $T_n=\frac{-2k+4}{4k^2-18k+19}\mathcal{G}_{k,n+3}+\frac{4k^2-20k+19}{8k^3-36k^2+38k}\mathcal{G}_{k,n+2}+\frac{10k-19}{8k^3-36k^2+38k}\mathcal{G}_{k,n+1}+\frac{-4k^2+20k-19}{8k^3-36k^2+38k}\mathcal{G}_{k,n},$
- $T_n=\frac{k^3-k^2+k-1}{k^4-k^3-3k^2+2k}\mathcal{V}_{k,n+3}+\frac{-k^2-2k+3}{k^3-k^2-3k+2}\mathcal{V}_{k,n+2}+\frac{k-1}{k^3+k^2-k}\mathcal{V}_{k,n+1}+\frac{k^3+k^2-4k+1}{k^4-k^3-3k^2+2k}\mathcal{V}_{k,n}.$

Proof. The proofs are shown similarly to Theorem 3.1. □

In the following theorem, we examine the relationships between the k-Olivier, k-Olivier-Lucas, and Modified k-Olivier sequences and special numbers sequences.

Theorem 3.5. Let $k\in\mathbb{R}^+$, and $n\in\mathbb{N}$. The relationships of the k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, and Modified k-Olivier $\mathcal{V}_{k,n}$ sequences with the third-order Fibonacci-Pell G_n, Adjusted Pell-Padovan M_n, third-order Lucas-Pell B_n, Pell-Padovan R_n, and Pell-Perrin C_n sequences. are as follows:

i.

- $\mathcal{S}_{k,n}=\frac{3k-4}{2}G_{n+2}+\frac{-k+2}{2}G_{n+1}+\frac{-5k+8}{2}G_n-\frac{k}{2},$
- $\mathcal{S}_{k,n}=\frac{7k-12}{22}C_{n+2}+\frac{-5k+18}{22}C_{n+1}+\frac{-k+8}{22}C_n-\frac{k}{2},$
- $\mathcal{S}_{k,n}=\frac{1}{2}R_{n+1}+\frac{k-1}{2}R_n-\frac{k}{2},$
- $\mathcal{S}_{k,n}=\frac{-13k+24}{10}B_{n+2}+\frac{11k-18}{10}B_{n+1}+\frac{19k-32}{10}B_n-\frac{k}{2},$
- $\mathcal{S}_{k,n}=\frac{-k+2}{2}M_{n+2}+\frac{k}{2}M_{n+1}+\frac{3k-4}{2}M_n-\frac{k}{2},$
- $G_n=\frac{1}{k^3-3k^2+k+2}((2-k^2)\mathcal{S}_{k,n+2}+(2k^2-3k)\mathcal{S}_{k,n+1}+(k^2-k)\mathcal{S}_{k,n}+k^3-2k^2+k),$
- $C_n=\frac{1}{k^3-3k^2+k+2}((-3k^2+4k+6)\mathcal{S}_{k,n+2}+(2k^2-5k)\mathcal{S}_{k,n+1}+(7k^2-11k-8)\mathcal{S}_{k,n}+3k^3-6k^2-k),$
- $R_n=\frac{1}{k^3-3k^2+k+2}(2\mathcal{S}_{k,n+2}+(-2k+2)\mathcal{S}_{k,n+1}+(2k^2-4k-2)\mathcal{S}_{k,n}+k^3-3k^2+k),$
- $B_n=\frac{1}{k^3-3k^2+k+2}((-3k^2+2k+6)\mathcal{S}_{k,n+2}+(4k^2-7k)\mathcal{S}_{k,n+1}+(5k^2-7k-4)\mathcal{S}_{k,n}+3k^3-6k^2+k),$
- $M_n=\frac{1}{k^3-3k^2+k+2}((k^2-k)\mathcal{S}_{k,n+2}+(-k^2+2)\mathcal{S}_{k,n+1}-k\mathcal{S}_{k,n}+k-k^2).$

ii.

- $\mathcal{G}_{k,n}=(-3k+7)G_{n+2}+(k-3)G_{n+1}+(5k-10)G_n+k,$

- $\mathcal{G}_{k,n} = -\frac{7k+5}{11}C_{n+2} + \frac{5k+13}{11}C_{n+1} + \frac{k+18}{11}C_n + k,$
- $\mathcal{G}_{k,n} = -\frac{3}{2}R_{n+2} + 2R_{n+1} + \frac{-2k+7}{2}R_n + k,$
- $\mathcal{G}_{k,n} = \frac{13k-13}{5}B_{n+2} + \frac{-11k+11}{5}B_{n+1} + \frac{-19k+24}{5}B_n + k,$
- $\mathcal{G}_{k,n} = (k-3)M_{n+2} + (-k+4)M_{n+1} + (-3k+7)M_n + k,$
- $G_n = \frac{1}{4k^3-18k^2+19k}((2k^2-13k+19)\mathcal{G}_{k,n+2} + (-4k^2+17k-19)\mathcal{G}_{k,n+1} + (-2k^2+12k-19)\mathcal{G}_{k,n} + 4k^3 - 16k^2 + 19k),$
- $C_n = \frac{1}{4k^3-18k^2+19k}((6k^2-11k-19)\mathcal{G}_{k,n+2} + (-4k^2+3k+19)\mathcal{G}_{k,n+1} + (-14k^2+32k+19)\mathcal{G}_{k,n} + 12k^3 - 24k^2 - 19k),$
- $R_n = \frac{1}{4k^3-18k^2+19k}((6k-19)\mathcal{G}_{k,n+2} + (-8k+19)\mathcal{G}_{k,n+1} + (-4k^2+4k+19)\mathcal{G}_{k,n} + 4k^3 - 2k^2 - 19k),$
- $B_n = \frac{1}{4k^3-18k^2+19k}((6k^2-25k+19)\mathcal{G}_{k,n+2} + (-8k^2+27k-19)\mathcal{G}_{k,n+1} + (-10k^2+34k-19)\mathcal{G}_{k,n} + 12k^3 - 36k^2 + 19k),$
- $M_n = \frac{1}{4k^3-18k^2+19k}((-2k^2+12k-19)\mathcal{G}_{k,n+2} + (2k^2-13k+19)\mathcal{G}_{k,n+1} + (-7k+19)\mathcal{G}_{k,n} + 8k^2 - 19k).$

iii.

- $\mathcal{V}_{k,n} = \frac{3k}{2}G_{n+2} - \frac{k}{2}G_{n+1} + \frac{-5k+2}{2}G_n - \frac{k}{2},$
- $\mathcal{V}_{k,n} = \frac{7k-16}{22}C_{n+2} + \frac{-5k+24}{22}C_{n+1} + \frac{-k+18}{22}C_n - \frac{k}{2},$
- $\mathcal{V}_{k,n} = -\frac{1}{2}R_{n+2} + R_{n+1} + \frac{k+1}{2}R_n - \frac{k}{2},$
- $\mathcal{V}_{k,n} = \frac{-13k+16}{10}B_{n+2} + \frac{11k-12}{10}B_{n+1} + \frac{19k-18}{10}B_n - \frac{k}{2},$
- $\mathcal{V}_{k,n} = -\frac{k}{2}M_{n+2} + \frac{k+2}{2}M_{n+1} + \frac{3k}{2}M_n - \frac{k}{2},$
- $G_n = \frac{1}{k^3-k^2-3k+2}(-(k^2+3k)\mathcal{V}_{k,n+2} + (2k^2+k)\mathcal{V}_{k,n+1} + (k^2+2k+2)\mathcal{V}_{k,n} + k^3 + k),$
- $C_n = \frac{1}{k^3-k^2-3k+2}((-3k^2+3k+16)\mathcal{V}_{k,n+2} + (2k^2-5k-8)\mathcal{V}_{k,n+1} + (7k^2-6k-26)\mathcal{V}_{k,n} + 3k^3 - 4k^2 - 9k),$
- $R_n = \frac{1}{k^3-k^2-3k+2}((2k+6)\mathcal{V}_{k,n+2} - (4k+2)\mathcal{V}_{k,n+1} + (2k^2-4k-10)\mathcal{V}_{k,n} + k^3 + k^2 - 3k),$
- $B_n = \frac{1}{k^3-k^2-3k+2}((-3k^2-3k+8)\mathcal{V}_{k,n+2} + (4k^2-k-4)\mathcal{V}_{k,n+1} + (5k^2-10)\mathcal{V}_{k,n} + 3k^3 - 2k^2 - 3k),$
- $M_n = \frac{1}{k^3-k^2-3k+2}((k^2+2k+2)\mathcal{V}_{k,n+2} - (k^2+3k)\mathcal{V}_{k,n+1} - (3k+4)\mathcal{V}_{k,n} - 2k^2 - k).$

Proof. i. If Binet formulas are used for proofs, we obtain

$$\frac{3k-4}{2}G_{n+2} + \frac{-k+2}{2}G_{n+1} + \frac{-5k+8}{2}G_n - \frac{k}{2} = \frac{3k-4}{2}\left(\frac{1}{\sqrt{5}}m^{n+2} - \frac{1}{\sqrt{5}}p^{n+2} + r^{n+2}\right)$$
$$+\frac{-k+2}{2}\left(\frac{1}{\sqrt{5}}m^{n+1} - \frac{1}{\sqrt{5}}p^{n+1} + r^{n+1}\right) + \frac{-5k+8}{2}\left(\frac{1}{\sqrt{5}}m^{n} - \frac{1}{\sqrt{5}}p^{n} + r^{n}\right) - \frac{k}{2}$$
$$= \frac{k-p}{(m-p)(m-r)(m-1)}m^n + \frac{k-m}{(p-m)(p-r)(p-1)}p^n + \frac{k-2}{(r-m)(r-p)(r-1)}r^n + \frac{k}{(1-m)(1-p)(1-r)}$$

$= \mathcal{S}_{k,n}$.

The proofs of the others are shown similarly. □

In the following theorems, we associate the terms of the k-Olivier $\mathcal{S}_{k,n}$, k-Olivier-Lucas $\mathcal{G}_{k,n}$, and Modified k-Olivier $\mathcal{V}_{k,n}$ sequences with matrices.

Theorem 3.6. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

i. For the k-Olivier sequence,

$$\textbf{1.}\begin{bmatrix}\mathcal{S}_{k,n+3}\\ \mathcal{S}_{k,n+2}\\ \mathcal{S}_{k,n+1}\\ \mathcal{S}_{k,n}\end{bmatrix}=\begin{bmatrix}1&2&-1&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}^n\begin{bmatrix}k\\1\\0\\0\end{bmatrix},\qquad \textbf{2.}\begin{bmatrix}\mathcal{S}_{k,n}\\ \mathcal{S}_{k,n+1}\\ \mathcal{S}_{k,n+2}\\ \mathcal{S}_{k,n+3}\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&2&-1&1\end{bmatrix}^n\begin{bmatrix}0\\0\\1\\k\end{bmatrix},$$

ii. For the k-Olivier-Lucas sequence,

$$\textbf{1.}\begin{bmatrix}\mathcal{G}_{k,n+3}\\ \mathcal{G}_{k,n+2}\\ \mathcal{G}_{k,n+1}\\ \mathcal{G}_{k,n}\end{bmatrix}=\begin{bmatrix}1&2&-1&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}^n\begin{bmatrix}6-2k\\5\\1\\4\end{bmatrix},\ \textbf{2.}\begin{bmatrix}\mathcal{G}_{k,n}\\ \mathcal{G}_{k,n+1}\\ \mathcal{G}_{k,n+2}\\ \mathcal{G}_{k,n+3}\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&2&-1&1\end{bmatrix}^n\begin{bmatrix}4\\1\\5\\6-2k\end{bmatrix},$$

iii. For the Modified k-Olivier sequence,

$$\textbf{1.}\begin{bmatrix}\mathcal{V}_{k,n+3}\\ \mathcal{V}_{k,n+2}\\ \mathcal{V}_{k,n+1}\\ \mathcal{V}_{k,n}\end{bmatrix}=\begin{bmatrix}1&2&-1&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}^n\begin{bmatrix}k+1\\2\\0\\1\end{bmatrix},\ \textbf{2.}\begin{bmatrix}\mathcal{V}_{k,n}\\ \mathcal{V}_{k,n+1}\\ \mathcal{V}_{k,n+2}\\ \mathcal{V}_{k,n+3}\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&2&-1&1\end{bmatrix}^n\begin{bmatrix}1\\0\\2\\k+1\end{bmatrix},$$

Proof. The proofs are shown by the induction method using definitions of sequences and properties of multiplication in matrices. □

Theorem 3.7. (Simpson Formulas) Let $k \in \mathbb{R}^+$ and $t \in \mathbb{N}$. The following equations are satisfied.

$$\textbf{i.}\ det\begin{bmatrix}\mathcal{S}_{k,t+3}&\mathcal{S}_{k,t+2}&\mathcal{S}_{k,t+1}&\mathcal{S}_{k,t}\\ \mathcal{S}_{k,t+2}&\mathcal{S}_{k,t+1}&\mathcal{S}_{k,t}&\mathcal{S}_{k,t-1}\\ \mathcal{S}_{k,t+1}&\mathcal{S}_{k,t}&\mathcal{S}_{k,t-1}&\mathcal{S}_{k,t-2}\\ \mathcal{S}_{k,t}&\mathcal{S}_{k,t-1}&\mathcal{S}_{k,t-2}&\mathcal{S}_{k,t-3}\end{bmatrix}=k(k-2)(k^2-k-1).$$

$$\textbf{ii.}\ det\begin{bmatrix}\mathcal{G}_{k,t+3}&\mathcal{G}_{k,t+2}&\mathcal{G}_{k,t+1}&\mathcal{G}_{k,t}\\ \mathcal{G}_{k,t+2}&\mathcal{G}_{k,t+1}&\mathcal{G}_{k,t}&\mathcal{G}_{k,t-1}\\ \mathcal{G}_{k,t+1}&\mathcal{G}_{k,t}&\mathcal{G}_{k,t-1}&\mathcal{G}_{k,t-2}\\ \mathcal{G}_{k,t}&\mathcal{G}_{k,t-1}&\mathcal{G}_{k,t-2}&\mathcal{G}_{k,t-3}\end{bmatrix}=4k^2(4k^2-18k+19).$$

$$\textbf{iii.}\ det\begin{bmatrix}\mathcal{V}_{k,t+3}&\mathcal{V}_{k,t+2}&\mathcal{V}_{k,t+1}&\mathcal{V}_{k,t}\\ \mathcal{V}_{k,t+2}&\mathcal{V}_{k,t+1}&\mathcal{V}_{k,t}&\mathcal{V}_{k,t-1}\\ \mathcal{V}_{k,t+1}&\mathcal{V}_{k,t}&\mathcal{V}_{k,t-1}&\mathcal{V}_{k,t-2}\\ \mathcal{V}_{k,t}&\mathcal{V}_{k,t-1}&\mathcal{V}_{k,t-2}&\mathcal{V}_{k,t-3}\end{bmatrix}=k(k-2)(k^2+k-1).$$

Proof. The proofs are shown by the induction method using definitions and determinant properties. □

Theorem 3.8. For $\mathcal{S}_{k,n}$, $\mathcal{G}_{k,n}$, and $\mathcal{V}_{k,n}$ sequences, the Binet formulas can be obtained with the help of the matrices.

Proof. The following relation is used for proof (see for details Corollary 3.1 in [21]).

$$t_n = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} r_{c+1-j} \det(\Lambda_j).$$

Thus,

$$\mathcal{S}_{k,n} = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} \mathcal{S}_{k,c+1-j} \det(\Lambda_j).$$

Let $c = i = 4$,

$$\Lambda = \begin{bmatrix} m^3 & m^2 & m & 1 \\ p^3 & p^2 & p & 1 \\ r^3 & r^2 & r & 1 \\ s^3 & s^2 & s & 1 \end{bmatrix}, \Lambda_1 = \begin{bmatrix} m^{n-1} & m^2 & m & 1 \\ p^{n-1} & p^2 & p & 1 \\ r^{n-1} & r^2 & r & 1 \\ s^{n-1} & s^2 & s & 1 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} m^3 & m^{n-1} & m & 1 \\ p^3 & p^{n-1} & p & 1 \\ r^3 & r^{n-1} & r & 1 \\ s^3 & s^{n-1} & s & 1 \end{bmatrix},$$

$\Lambda_3 = \begin{bmatrix} m^3 & m^2 & m^{n-1} & 1 \\ p^3 & p^2 & p^{n-1} & 1 \\ r^3 & r^2 & r^{n-1} & 1 \\ s^3 & s^2 & s^{n-1} & 1 \end{bmatrix}$, and $\Lambda_4 = \begin{bmatrix} m^3 & m^2 & m & m^{n-1} \\ p^3 & p^2 & p & p^{n-1} \\ r^3 & r^2 & r & r^{n-1} \\ s^3 & s^2 & s & s^{n-1} \end{bmatrix}$. So, we obtain

$$\mathcal{S}_{k,n} = \frac{1}{\det(\Lambda)} \sum_{j=1}^{4} \mathcal{S}_{k,5-j} \det(\Lambda_j)$$

$$= \frac{1}{\det(\Lambda)} (\mathcal{S}_{k,4} \det(\Lambda_1) + \mathcal{S}_{k,3} \det(\Lambda_2) + \mathcal{S}_{k,2} \det(\Lambda_3) + \mathcal{S}_{k,1} \det(\Lambda_4)).$$

Similarly, the Binet formula of the $\mathcal{G}_{k,n}$, and $\mathcal{V}_{k,n}$ sequences are found. □

4 CONCLUSIONS

In this study, we defined the k-Olivier, k-Olivier-Lucas, and Modified k-Olivier sequences. Then, we found the features of these sequences. In addition, we obtained the relationships between the terms of these sequences. Moreover, we examined k-Olivier, k-Olivier-Lucas, and Modified k-Olivier sequences with third-order Fibonacci-Pell, Adjusted Pell-Padovan, Olivier, Olivier-Lucas, third-order Lucas-Pell, Pell-Padovan, and Pell-Perrin sequences. Finally, we associated the terms of these sequences with matrices. If this study is examined, such features can be found in other sequences such as Fermat, and Mersenne sequences.

Acknowledgements: The authors would like to thank the referees who carefully read the article and made valuable comments.

REFERENCES

[1] R. R. de Oliveira and F. R. V. Alves, "An investigation of the bivariate complex fibonacci polynomials supported in didactic engineering: an application of Theory of Didactics Situations (TSD)", *Acta Scientiae*, **21**(3), 170-195, (2019).

[2] H. H. Otto, "Fibonacci Stoichiometry and Superb Performance of Nb16W5O55 and Related Super-Battery Materials", *JAMP*, **10**(6), 1936-1950, (2022). DOI: 10.4236/jamp.2022.106133

[3] Z. Avazzadeh, H. Hassani, P. Agarwal, S. Mehrabi, M. Javad Ebadi, and M. K. Hosseini Asl,

"Optimal study on fractional fascioliasis disease model based on generalized Fibonacci polynomials", *Math. Methods Appl. Sci.*, **46**(8), 9332-9350, (2023). DOI:10.1002/mma.9057
[4] H. A. Turner, M. Humpage, H. Kerp and A. J. Hetherington, "Leaves and sporangia developed in rare non-Fibonacci spirals in early leafy plants", *Science*, **380**(6650), 1188-1192, (2023). DOI: 10.1126/science.adg4014
[5] S. Aydınyüz and M. Aşcı, "The Moore-Penrose Inverse of the Rectangular Fibonacci Matrix and Applications to the Cryptology". *AADM*, **40**(2), 195-211, (2023). DOI: 10.17654/0974165823066
[6] H. Akkuş and E. Özkan, "Generalization of the k-Leonardo Sequence and their Hyperbolic Quaternions". *Math. Montis.*, **60**, 14-31 (2024). DOI: 10.20948/mathmontis-2024-60-2
[7] H. Akkuş, R. N. Üregen and E. Özkan, "A New Approach to $k-$Jacobsthal Lucas Sequences", *SAUJS*, **25**(4), 969-973, (2021). DOI: 10.16984/saufenbilder.842489
[8] S. Falcon, "Catalan Transform of the k-Fibonacci sequence", *Commun. Korean Math. Soc.*, **28**(4), 827-832, (2011). DOI: https://doi.org/10.4134/CKMS.2013.28.4.827
[9] E. Özkan, and T. Alp, "Bigaussian Pell and Pell-Lucas Polynomials", *Math. Montis.*, **53**, 17-25 (2022). DOI: 10.20948/mathmontis-2022-53-3
[10] Y. Soykan, "Generalized Leonardo numbers", *JPRM*, **18**(4), 58-84, (2021).
[11] E. Özkan, and H. Akkus, "On k-Chebsyhev Sequence", *WSEAS Trans. Math.*, **22**, 503-507, (2023). DOI: 10.37394/23206.2023.22.56
[12] N.Yılmaz, "The generalized bivariate fibonacci and lucas matrix polynomials", *Math. Montis.*, **53**, 33-44, (2022). DOI: 10.20948/mathmontis-2022-53-5
[13] H. Akkuş, Ö. Deveci, E. Özkan and A. G. Shannon, "Discatenated and lacunary recurrences", *NNTDM*, **30**(1), 8-19, (2024). DOI: 10.7546/nntdm.2024.30.1.8-19
[14] E. Özkan, and H. Akkuş, "A New Approach to k-Oresme and k-Oresme-Lucas Sequences", *Symmetry*, **16**, 1-12 (2024). DOI: 10.3390/sym16111407
[15] Y. Soykan, "Generalized Olivier Numbers", *Asian Res. J. Math.*, **19**(1), 1-22, (2022).
[16] Y. Soykan, "Properties of Generalized (r, s, t, u)-Numbers", *Earthline J. Math. Sci.*, **5**(2), 297-327 (2021). DOI:10.34198/ejms.5221.297327
[17] J. R. Griggs, P. Hanlon, A. M. Odlyzko, and M. S. Waterman, "On the number of alignments of k sequences", *Graphs Combin.*, **6**(2), 133-146, (1990). DOI: 10.1007/BF01787724
[18] S. Falcon, and A. Plaza, "On the Fibonacci k-numbers", *Chaos, Solitons & Fractals*, **32**(5), 1615–1624, (2007). DOI: https://doi.org/10.1016/j.chaos.2006.09.022
[19] S. Falcon, "On the k-Lucas numbers", *Int. J. Contemp. Math. Sci.*, **6**(21), 1039-1050 (2011).
[20] E. Özkan, and H. Akkuş, "Copper ratio obtained by generalizing the Fibonacci sequence", *AIP Advances*, **14**(7), 1-11, (2024). DOI: 10.1063/5.0207147
[21] E. Kiliç, and P. Stanica, "A matrix approach for general higher order linear recurrences", *B. Malays. Math. Sci. So.*, **34**(1), 51-67, (2011).
[22] A. G. Shannon, H. Akkuş, Y. Aküzüm, Ö. Deveci, and E. Özkan, "A partial recurrence Fibonacci link", *NNTDM*, **30**(3), 530-537, (2024). DOI: 10.7546/nntdm.2024.30.3.530-537
[23] P. M. Catarino, and A. Borges, "On leonardo numbers", *Acta Math. Univ. Comen.*, **89**(1), 75-86, (2019).
[24] E. Özkan, and H. Akkuş, "Generalized Bronze Leonardo sequence", *NNTDM*, **30**(4), 811-824, (2024). DOI: 10.7546/nntdm.2024.30.4.811-824
[25] H. Akkuş, and E. Özkan, E., "Hyperbolic (s,t)-Fibonacci and (s,t)-Lucas Quaternions", *Proof*, **4**, 97-105, (2024). DOI: 10.37394/232020.2024.4.9
[26] F. Yilmaz, and D. Bozkurt, "The generalized order-k Jacobsthal numbers", *Int. J. Contemp. Math. Sci.*, **4**(34), 1685-1694, (2009).

Received November 19, 2024