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INVESTIGATION OF HIGHER ORDER HORADAM NUMBERS, THEIR ASSOCIATED TRANSFORMS AND SELF-SIMILARITY

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Summary. This paper gives results for higher-order Horadam numbers, which are akin to higher-order Fibonacci sequence, it weaves intricate connections with the classic Horadam, Pell, Jacobsthal, Mersenne, Balancing numbers. Also, all number sequences used in higher-order Horadam numbers, from Fibonacci to Lucas Mersenne to Fermat numbers, exhibit self-similarity at certain points, and so self-similarity has been exploited in these areas to simplify the analysis of complex systems, improve predictive models, and create efficient designs by exploiting the inherent reproducible and scalable nature of self-similar structures. Methods: We introduce the basic properties and results for Horadam numbers, leading to the Catalan transform of higher order Horadam numbers. Results: We give results for Cassini identity, d'Ocagne identity, Vajda identity, Honsberger identity, Melham's identity. Moreover, this study examines the "Catalan transforms" of these higher-order Horadam numbers, aiming to solidify their comprehension through matrix representations. These powerful tools, along with other equations, act as keys that unlock the heart of these numbers, paving the way for future exploration and deeper insights.

1 INTRODUCTION

By examining the recently existing number sequences, a different perspective has been brought to number sequences by studying Higher-order number sequences, which are defined as a transition to different number sequences.

The first studies with higher order arrays were made by Randic et al. in 1996 [1]. It started out by considering the numbers as 3-step and 4-step, starting with. However, these studies were derived from the numbers previously discussed by Hosoya in 1971, which stated that a Z linear chain constitutes Fibonacci numbers. Where $Z = F_{(n+1)}$ and F_n, nth Fibonacci number. 1_Z, which is used as the generalization of Hosoya numbers, represents the Hosoya Z index and h_Z represents the higher order Hosoya numbers more clearly Hosoya's this work defines a relationship between a generalized Hosoya number (Z) and a function (F) with an increment of 1. Hosoya denotes the base index as 1_Z and represent higher-order values as h_Z. His goal is to develop a structurally related set of descriptors (1_Z, 2_Z, 3_Z,…, h_Z) for improved molecular

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representation.

Traditionally, various topological indices are combined without a unifying structure, leading to limitations. Notably, the Hosoya index itself suffers from degeneracy, where different molecules share the same value. By employing an ordered set of these descriptors instead of a single index, we aim to achieve better discrimination among structures. Furthermore, these descriptors could be applied in multivariable regression analyses, like how connectivity indices (1_x and higher-order m_x) have been used in structure-property studies [2,3,4].

Recent studies have explored "higher-order" sequences inspired by classic number sequences like Fibonacci. These delve deeper into the inherent structures and reveal previously hidden properties. This trend began with Randić et al.'s (1996) [1] exploration of higher-order Fibonacci numbers, followed by Kizilates and Kone's (2003) [5] extension into quaternion algebra. Further explorations include Cook et al.'s (2014) [6] work on "Jacobsthal" sequences and Uysal & Ozkan's (2016) [7,8] foray into quaternion algebra for higher-order versions. Inspired by these, we introduce and investigate "higher-order Mersenne numbers," revealing their connections to classic sequences like Mersenne, Mersenne-Lucas, and Jacobsthal.

However, these studies were discussed based on their relationship with quaternions rather than the general definition and properties of higher order number sequences. Based on all these studies, in this study, we re-examined the Horadam number sequences, which contain many number sequences, with a new definition of higher order by adding a different dimension.

Horadam numbers and properties were discussed by Horadam in 1965 as a generalization of Fibonacci numbers [9] and these numbers have second degree recurrence and are expressed as follows:

$$W_n = kW_{n-1} + mW_{n-2}, W_0 = a, W_1 = b, n \geq 2.$$

Where $a, b, k, m \in \mathbb{R}$.

The Binet formula of Horadam sequence can be calculated using the characteristic equation which is given as

$$x^2 - kx - m = 0.$$

The roots of the equation are

$$\alpha = \frac{k + \sqrt{k^2 + 4m}}{2}, \beta = \frac{k - \sqrt{k^2 + 4m}}{2},$$

$$W_n = \frac{\mathcal{A}\alpha^n - \mathcal{B}\beta^n}{\mathcal{A}\alpha - \mathcal{B}\beta}$$

where $\mathcal{A} = b - a\beta, \mathcal{B} = b - a\alpha$.

The Catalan transform is a sequence transform introduced by Barry [10]. Catalan numbers $C(n)$ are given by

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

Later the formula given above can be written as

$$C_n = \frac{(2n)!}{(n+1)!\,n!}$$

First few Catalan numbers are 1,1,2,5,14,42,132,429, …
Also, one can easily see the following recurrence relation for $C(n)$

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}.$$

It is given that the ordinary generating function $c(x)$ as follows:

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

$$c(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots$$

2 HIGHER ORDER HORADAM NUMBERS

Definition 2.1. The higher order Horadam numbers are defined by

$$W_n^{(s)} = \frac{W_{ns}}{W_s} = \frac{\mathcal{A}\alpha^{ns} - \mathcal{B}\beta^{ns}}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}}$$

where $\mathcal{A} = b - a\beta, \mathcal{B} = b - a\alpha$.

Note that for $a = 0, b = 1$ higher order Horadam number is the higher order Fibonacci numbers.

Also, for $a = 0, b = 1, s = 1$ higher order Horadam number is the ordinary Fibonacci numbers.

Some of the sequences obtained for some special values of a, b, k, m are given in the table below:

Name of sequence	$\boldsymbol{W_n^{(s)}(a,b,k,m)}$	**Binet formula**
Higher order Fibonacci	$W_n^{(s)}(0,1,1,1)$	$\left[\left((1+\sqrt{5})/2\right)^{ns} - \left((1-\sqrt{5})/2\right)^{ns}\right]/\left[\left((1+\sqrt{5})/2\right)^{s} - \left((1-\sqrt{5})/2\right)^{s}\right]$
Higher order Lucas	$W_n^{(s)}(2,1,1,1)$	$\left[\left((1+\sqrt{5})/2\right)^{ns} + \left((1-\sqrt{5})/2\right)^{ns}\right]/\left[\left((1+\sqrt{5})/2\right)^{s} + \left((1-\sqrt{5})/2\right)^{s}\right]$
Higher order Pell	$W_n^{(s)}(0,1,2,1)$	$\left[\left(1+\sqrt{2}\right)^{ns} - \left(1-\sqrt{2}\right)^{ns}\right]/\left[\left(1+\sqrt{2}\right)^{s} - \left(1-\sqrt{2}\right)^{s}\right]$
Higher order Jacobsthal	$W_n^{(s)}(0,1,1,2)$	$[2^{ns} - (-1)^{ns}]/[2^{s} - (-1)^{s}]$
Higher order Mersenne	$W_n^{(s)}(0,1,3,-2)$	$[2^{ns} - 1]/[2^{s} - 1]$
Higher order Balancing	$W_n^{(s)}(0,1,6,-1)$	$\left[\left(3+\sqrt{8}\right)^{ns} - \left(3-\sqrt{8}\right)^{ns}\right]/\left[\left(3+\sqrt{8}\right)^{s} - \left(3-\sqrt{8}\right)^{s}\right]$
Higher order Oresme	$W_n^{(s)}(0,1/2,1,-1/4)$	$[ns.2^{s}]/[s.2^{ns}]$
Higher order Fermat	$W_n^{(s)}(2,3,3,-2)$	$[2^{ns} + 1]/[2^{s} + 1]$

Table1. Some higher order number sequences

Additionally, if $s = 1$, higher order sequences turn into the ordinary number sequnces, a few number sequences are discussed in the table 2:

Name of sequence	$\boldsymbol{W_n^{(1)}(a,b,k,m)}$
Fibonacci sequence	$\boldsymbol{W_n^{(1)}}(0,1,1,1)$
Pell sequence	$\boldsymbol{W_n^{(1)}}(0,1,2,1)$
Jacobsthal sequence	$\boldsymbol{W_n^{(1)}}(0,1,1,2)$
Mersenne sequence	$\boldsymbol{W_n^{(1)}}(0,1,3,-2)$
Balancing sequence	$\boldsymbol{W_n^{(1)}}(0,1,6,-1)$

Table2. Some number sequences

Definition 2.2. For negative indices higher order Horadam numbers are defided by

$$W_{-n}^{(s)} = (\alpha\beta)^{-sn}\frac{\mathcal{A}\beta^{ns} - \mathcal{B}\alpha^{ns}}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}}.$$

Definition 2.3. For $n, s \in \mathbb{Z}$, we have

$$W_{n}^{(-s)} = (\alpha\beta)^{(s-1)n}\frac{\mathcal{A}\beta^{ns} - \mathcal{B}\alpha^{ns}}{\mathcal{A}\beta^{s} - \mathcal{B}\alpha^{s}}.$$

Definition 2.4. For $n, s \in \mathbb{Z}$, we have

$$W_{-n}^{(-s)} = (\alpha\beta)^{s}\frac{\mathcal{A}\alpha^{ns} - \mathcal{B}\beta^{ns}}{\mathcal{A}\beta^{s} - \mathcal{B}\alpha^{s}}.$$

Theorem 2.5. For $n, s \in \mathbb{Z}$, the recurrence relation is as follows.

$$W_{n+1}^{(s)} = (\alpha^s + \beta^s)W_{n+1}^{(s)} - (\alpha\beta)^s W_{n-1}^{(s)}.$$

Proof.

$$\begin{aligned}
W_{n+1}^{(s)} &= \frac{\mathcal{A}\alpha^{(n+1)s} - \mathcal{B}\beta^{(n+1)s}}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}} \\
&= \frac{1}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}}(\mathcal{A}\alpha^{ns}\alpha^{s} - \mathcal{B}\beta^{ns}\beta^{s}) \\
&= \frac{1}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}}(\mathcal{A}\alpha^{ns}\alpha^{s} - \mathcal{B}\beta^{ns}\alpha^{s} + \mathcal{B}\beta^{ns}\alpha^{s} - \mathcal{B}\beta^{ns}\beta^{s}) \\
&= \frac{\alpha^s(\mathcal{A}\alpha^{ns} - \mathcal{B}\beta^{ns})}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}} + \frac{\mathcal{B}\beta^{ns}\alpha^{s} - \mathcal{B}\beta^{ns}\beta^{s}}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}} \\
&= \alpha^s W_n^{(s)} + \frac{\mathcal{A}\alpha^{ns}\beta^{s} - \mathcal{A}\alpha^{ns}\beta^{s} + \mathcal{B}\beta^{ns}\alpha^{s} - \mathcal{B}\beta^{ns}\beta^{s}}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}} \\
&= \alpha^s W_n^{(s)} + \frac{\beta^s(\mathcal{A}\alpha^{ns} - \mathcal{B}\beta^{ns})}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}} - \frac{\alpha^s\beta^s(\mathcal{A}\alpha^{ns-s} - \mathcal{B}\beta^{ns-s})}{\mathcal{A}\alpha^{s} - \mathcal{B}\beta^{s}}
\end{aligned}$$

$$= \alpha^s W_n^{(s)} + \beta^s W_n^{(s)} - (\alpha\beta)^s W_{n-1}^{(s)}$$
$$= (\alpha^s + \beta^s) W_{n+1}^{(s)} - (\alpha\beta)^s W_{n-1}^{(s)}.$$

Theorem 2.6. The generating function of $W_n^{(s)}$ is

$$\sum_{n=0}^{\infty} W_n^{(s)} x^n = \frac{\mathcal{A}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\alpha^s x)} - \frac{\mathcal{B}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\beta^s x)}.$$

Proof. Using Binet formula for higher order Horadam numbers we get

$$\sum_{n=0}^{\infty} W_n^{(s)} x^n = \sum_{n=0}^{\infty} \frac{\mathcal{A}\alpha^{ns} - \mathcal{B}\beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} x^n = \frac{\mathcal{A}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \sum_{n=0}^{\infty} (\alpha^s x)^n - \frac{\mathcal{B}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \sum_{n=0}^{\infty} (\beta^s x)^n$$
$$= \frac{\mathcal{A}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\alpha^s x)} - \frac{\mathcal{B}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\beta^s x)}.$$

This paper carries forward earlier work on self-similarity inherent in Fibonacci number sequences. In this work, mappings are derived from Fibonacci generalizations of Fibonacci mappings, which construct various forms of nonlinear graph with inherent self-similarities. The following graphs are generated by substituting the values of a and b, which are the initial values of higher-order Horadam sequences, into the generating function.

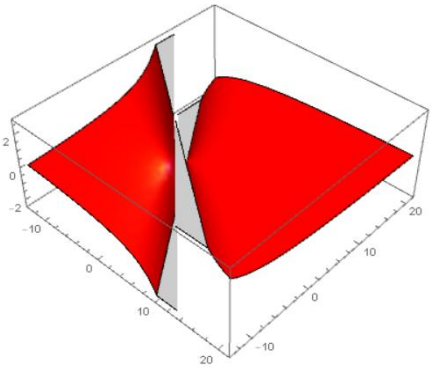


Figure 1. For Fibonacci numbers, self-similarities

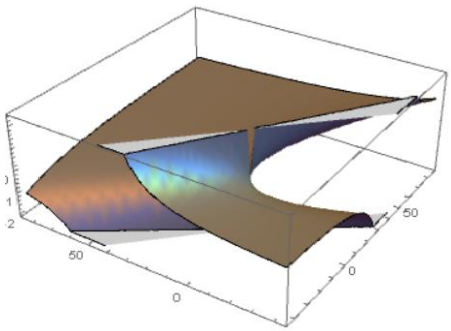


Figure 2. For Jacobsthal numbers, self-similarities

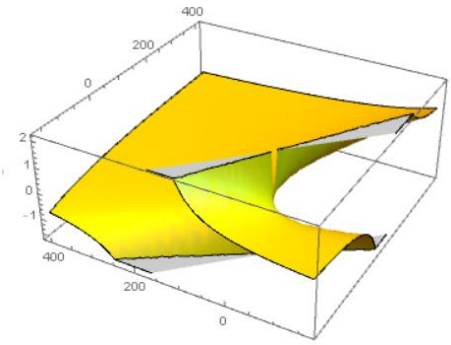


Figure 3. For Pell numbers, self-similarities

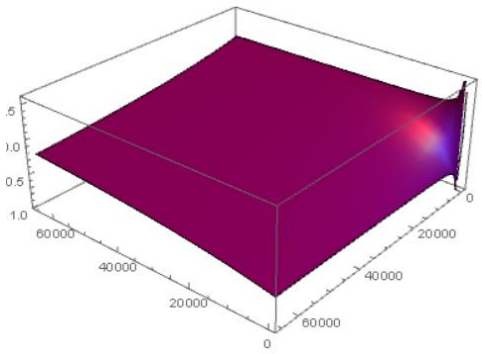


Figure 4. For Mersenne numbers, self-similarities

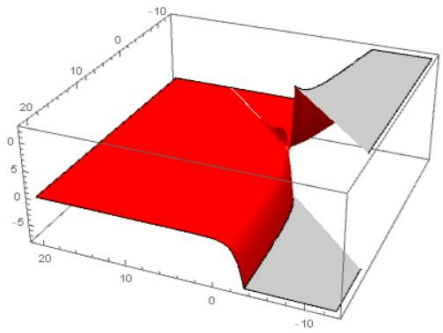


Figure 5. $e^{-a}W_n^{(s)}$, for Fibonacci number, self-similarities

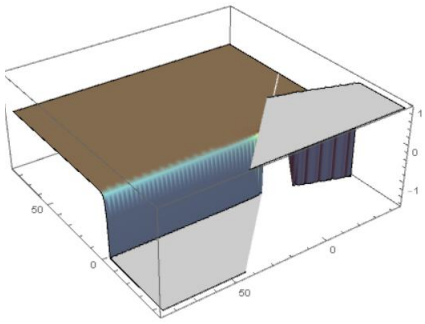


Figure 6. $e^{-a}W_n^{(s)}$, for Jacobsthal number self-similarities

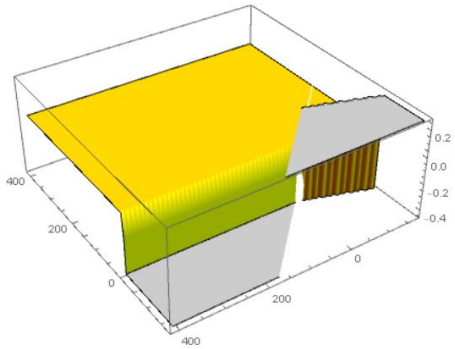


Figure 7. $e^{-a}W_n^{(s)}$, for Pell number, self-similarities

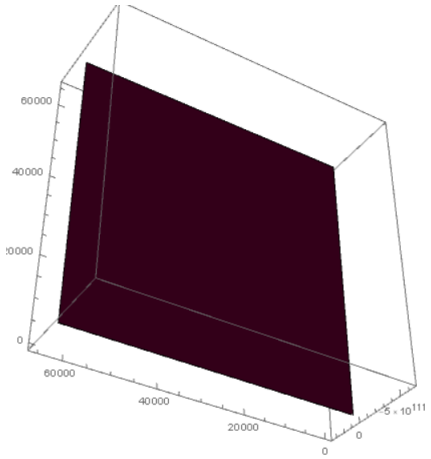


Figure 8. $e^{-a}W_n^{(s)}$, for Mersenne number, self-similarities

These values correspond to terms of Fibonacci, Jacobsthal, Pell, and Mersenne sequences. It can be observed that as we move from Fibonacci to Mersenne, the curvature decreases, resulting in more planar graphs. Moreover, these graphs exhibit clear instances of self-similarity at specific points in Figure 1, 2,3,4.

Also, these graphs are created by multiplying the generating function of exponential functions obtained by substituting a, taken as the initial value, with these sequences. Again, regions of self-similarity at certain points are clearly visible in Figure 5,6,7,8. Self-similarity at these points helps mathematicians understand infinite complexity within finite bounds, allowing properties such as size and scaling to be explored. self-similarity also facilitates forward predictions in many fields, for example; identifying self-similarity in DNA sequences helps understand genetic coding, replication and evolutionary processes.

Techniques such as fractal compression use self-similarity to efficiently encode images and exploit repeating patterns to minimize data redundancy. Self-similarity in financial time series helps model market dynamics and identify recurring patterns across different time scales, aiding risk assessment and strategy development. Fractal analysis of medical images identifies self-similar patterns that can indicate the presence of diseases or abnormalities, aiding

diagnosis and treatment planning. In summary, self-similarity is being exploited in these fields to simplify the analysis of complex systems, improve predictive models, and create efficient designs by taking advantage of the inherent repetitive and scalable nature of self-similar structures.

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Theorem 2.7. The summation formula of $W_n^{(s)}$ is

$$\sum_{k=0}^{n} W_k^{(s)} = \frac{\mathcal{A}(1-\alpha^{sn})}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\alpha^s)} - \frac{\mathcal{B}(1-\beta^{sn})}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\beta^s)}.$$

Proof. From the Binet formula for higher order Horadam numbers

$$\sum_{k=0}^{n} W_k^{(s)} = \sum_{k=0}^{n} \frac{\mathcal{A}\alpha^{ks} - \mathcal{B}\beta^{ks}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} = \frac{\mathcal{A}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\sum_{n=0}^{\infty}\alpha^{ks} - \frac{\mathcal{B}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\sum_{n=0}^{\infty}\beta^{ks}$$

$$= \frac{\mathcal{A}(1-\alpha^{sn})}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\alpha^s)} - \frac{\mathcal{B}(1-\beta^{sn})}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s(1-\beta^s)}.$$

Definition 2.8. The matrix representation of the higher order Lucas and Fibonacci numbers, respectively, $U_n^{(s)}$ *and* $V_n^{(s)}$ is

$$U_n^{(s)} = \begin{bmatrix} W_{n+2}^{(s)} & W_{n+1}^{(s)} \\ W_{n+1}^{(s)} & W_n^{(s)} \end{bmatrix} = \begin{bmatrix} \frac{\alpha^{2s}+\beta^{2s}}{\alpha^s+\beta^s} & 1 \\ 1 & \frac{2}{\alpha^s+\beta^s} \end{bmatrix} \begin{bmatrix} \alpha^s+\beta^s & 1 \\ 1 & 0 \end{bmatrix}^n$$

and

$$V_n^{(s)} = \begin{bmatrix} W_{n+1}^{(s)} & W_n^{(s)} \\ W_n^{(s)} & W_{n-1}^{(s)} \end{bmatrix} = \begin{bmatrix} \alpha^s+\beta^s & 1 \\ 1 & 0 \end{bmatrix}^n$$

where s is odd.

Proposition 2.9. For $n \geq 0$, we have

$$\det\left(U_n^{(s)}\right) = (-1)^n$$

and

$$\det\left(V_n^{(s)}\right) = (-1)^n \left(\frac{\alpha^s-\beta^s}{\alpha^s+\beta^s}\right)^2.$$

Proof. We can do the proof by induction method on n such that;
For $n = 0$, we have

$$U_0^{(s)} = \begin{bmatrix} W_2^{(s)} & W_1^{(s)} \\ W_1^{(s)} & W_0^{(s)} \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha^{2s}+\beta^{2s}}{\alpha^s+\beta^s} & 1 \\ 1 & \dfrac{2}{\alpha^s+\beta^s} \end{bmatrix} \begin{bmatrix} \alpha^s+\beta^s & 1 \\ 1 & 0 \end{bmatrix}^0 = \left(\frac{\alpha^s-\beta^s}{\alpha^s+\beta^s}\right)^2$$

For $n = k$, assume that the equation holds. That is;

$$U_k^{(s)} = \begin{bmatrix} W_{k+2}^{(s)} & W_{k+1}^{(s)} \\ W_{k+1}^{(s)} & W_k^{(s)} \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha^{2s}+\beta^{2s}}{\alpha^s+\beta^s} & 1 \\ 1 & \dfrac{2}{\alpha^s+\beta^s} \end{bmatrix} \begin{bmatrix} \alpha^s+\beta^s & 1 \\ 1 & 0 \end{bmatrix}^k = (-1)^k \left(\frac{\alpha^s-\beta^s}{\alpha^s+\beta^s}\right)^2$$

Then $n = k+1$, we have

$$U_{k+1}^{(s)} = \begin{bmatrix} W_{k+3}^{(s)} & W_{k+2}^{(s)} \\ W_{k+2}^{(s)} & W_{k+1}^{(s)} \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha^{2s}+\beta^{2s}}{\alpha^s+\beta^s} & 1 \\ 1 & \dfrac{2}{\alpha^s+\beta^s} \end{bmatrix} \begin{bmatrix} \alpha^s+\beta^s & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} \alpha^s+\beta^s & 1 \\ 1 & 0 \end{bmatrix}$$

$$= -1(-1)^k \left(\frac{\alpha^s-\beta^s}{\alpha^s+\beta^s}\right)^2 = (-1)^{k+1} \left(\frac{\alpha^s-\beta^s}{\alpha^s+\beta^s}\right)^2$$

as desired.

Theorem 2.10. Let $n \to \infty, \alpha > \beta, n, s \in \mathbb{Z}$. Then the ratio of the bigger to the smallest of the two consecutive terms of the higher order Horadam numbers are as follows:

$$\lim_{n\to\infty} \frac{W_{n+1}^{(s)}}{W_n^{(s)}} = \alpha^s.$$

Proof. The Binet Formula of the higher order Horadam numbers, we get

$$\lim_{n\to\infty} \frac{W_{n+1}^{(s)}}{W_n^{(s)}} = \frac{\mathcal{A}\alpha^{(n+1)s} - \mathcal{B}\beta^{(n+1)s}}{\mathcal{A}\alpha^{ns} - \mathcal{B}\beta n^{ns}} = \frac{\alpha^{(n+1)s}\left(\mathcal{A} - \mathcal{B}\left(\frac{\beta}{\alpha}\right)^{(n+1)s}\right)}{\alpha^{ns}\left(\mathcal{A} - \mathcal{B}\left(\frac{\beta}{\alpha}\right)^{ns}\right)} = \alpha^s.$$

3 CATALAN TRANSFORMATION OF HIGHER ORDER HORADAM NUMBERS

We define the Catalan transform of higher order Horadam numbers $\left(CW_n^{(s)}\right)$ as

$$CW_n^{(s)} = \sum_1^1 \frac{i}{2n-i}\binom{2n-i}{n-i} W_i^{(s)}.$$

First few terms of $CW_n^{(s)}$ as follow:

$$CW_1^{(s)} = \sum_1^1 \frac{i}{2-i}\binom{2-i}{1-i} W_i^{(s)} = 1$$

$$CW_2^{(s)} = \sum_1^2 \frac{i}{4-i}\binom{4-i}{2-i} W_i^{(s)} = \frac{\mathcal{A}(\alpha^{2s}+\alpha^s) - \mathcal{B}(\beta^{2s}+\beta^s)}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}$$

$$CW_3^{(s)} = \sum_1^3 \frac{i}{6-i}\binom{6-i}{3-i} W_i^{(s)} = \frac{\mathcal{A}(\alpha^{3s}+2\alpha^{2s}+2\alpha^s) - \mathcal{B}(\beta^{3s}+2\beta^{2s}+2\beta^s)}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}$$

$$CW_4^{(s)} = \sum_1^4 \frac{i}{8-i}\binom{8-i}{4-i} W_i^{(s)} = \frac{\mathcal{A}(\alpha^{4s}+3\alpha^{3s}+5\alpha^{2s}+5\alpha^s) - \mathcal{B}(\beta^{4s}+3\beta^{3s}+5\beta^{2s}+5\beta^s)}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}$$

$$CW_5^{(s)} = \sum_1^4 \frac{i}{10-i}\binom{10-i}{5-i} W_i^{(s)} = \frac{\mathcal{A}(\alpha^{5s}+4\alpha^{4s}+9\alpha^{3s}+14\alpha^{2s}+14\alpha^s) - \mathcal{B}(\beta^{5s}+4\beta^{4s}+9\beta^{3s}+14\beta^{2s}+14\beta^s)}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}$$

obtained.

Here, the coefficients of the terms in the coefficients of $\mathcal{A}, \mathcal{B}$ are obtained as $\{1,2,5,14,42,132,\dots\}$ which are Catalan numbers and are numbered A000108 in OEIS [11].

The Catalan transform of higher order Horadam numbers can be written as matrix product as follow:

$$\begin{bmatrix} CW_1^{(s)} \\ CW_2^{(s)} \\ CW_3^{(s)} \\ CW_4^{(s)} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 5 & 5 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \cdot \begin{bmatrix} W_1^{(s)} \\ W_2^{(s)} \\ W_3^{(s)} \\ W_4^{(s)} \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \frac{\mathcal{A}(\alpha^{2s}+\alpha^s) - \mathcal{B}(\beta^{2s}+\beta^s)}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \\ \frac{\mathcal{A}(\alpha^{3s}+2\alpha^{2s}+2\alpha^s) - \mathcal{B}(\beta^{3s}+2\beta^{2s}+2\beta^s)}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \\ \frac{\mathcal{A}(\alpha^{4s}+3\alpha^{3s}+5\alpha^{2s}+5\alpha^s) - \mathcal{B}(\beta^{4s}+3\beta^{3s}+5\beta^{2s}+5\beta^s)}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & & \dots \\ 1 & 1 & & & & \dots \\ 2 & 2 & 1 & & & \dots \\ 5 & 5 & 3 & 1 & & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{\mathcal{A}(\alpha^{2s}) - \mathcal{B}(\beta^{2s})}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \\ \frac{\mathcal{A}(\alpha^{3s}) - \mathcal{B}(\beta^{3s})}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \\ \frac{\mathcal{A}(\alpha^{4s}) - \mathcal{B}(\beta^{4s})}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s} \\ \vdots \end{bmatrix}$$

4 SOME IDENTITIES OF THE HIGHER ORDER HORODAM NUMBERS

In this chapter, we have calculated some fundamental identities for number sequences.

Theorem 4.1. (Cassini Identity) For $n \geq 1$, the equation given below is satisfied.

$$W_{n-1}^{(s)} W_{n+1}^{(s)} - \left(W_n^{(s)}\right)^2 = -m^{ns-s} \mathcal{AB} \left(\frac{\alpha^s - \beta^s}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)^2.$$

Proof. We will use Binet's formula to prove it.

$$W_{n-1}^{(s)} W_{n+1}^{(s)} - \left(W_n^{(s)}\right)^2 =$$

$$\left(\frac{\alpha^{(n-1)s} - \beta^{(n-1)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{(n+1)s} - \beta^{(n+1)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right) - \left(\frac{\alpha^{ns} - \beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{ns} - \beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)$$

$$= \frac{2\mathcal{AB}\alpha^{ns}\beta^{ns} - \mathcal{AB}\alpha^{ns-s}\beta^{ns-s} - \mathcal{AB}\alpha^{ns+s}\beta^{ns+s}}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{AB}\alpha^{ns}\beta^{ns}(2 - \alpha^{-s}\beta^s - \alpha^s\beta^{-s})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{AB}\alpha^{ns}\beta^{ns}\left(2 - \left(\frac{\beta}{\alpha}\right)^s - \left(\frac{\alpha}{\beta}\right)^s\right)}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$= \frac{-\mathcal{AB}(\alpha\beta)^{ns-s}(\alpha^s - \beta^s)^2}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$ from $\alpha\beta = m$, we have

$$= -m^{ns-s} \mathcal{AB} \left(\frac{\alpha^s - \beta^s}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)^2.$$

Thus, the proof is completed.

Theorem 4.2. (Catalan Identity) For $n \geq t$, the equation given below is satisfied.

$$W_{n-t}^{(s)} W_{n+t}^{(s)} - \left(W_n^{(s)}\right)^2 = -m^{ns-ts} \mathcal{AB} \left(\frac{\alpha^{ts} - \beta^{ts}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)^2.$$

Proof. We will use Binet's formula to prove it.

$$W_{n-t}^{(s)}W_{n+t}^{(s)} - \left(W_n^{(s)}\right)^2 =$$

$$\left(\frac{\alpha^{(n-t)s} - \beta^{(n-t)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{(n+t)s} - \beta^{(n+t)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right) - \left(\frac{\alpha^{ns} - \beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{ns} - \beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)$$

$$= \frac{2\mathcal{AB}\alpha^{ns}\beta^{ns} - \mathcal{AB}\alpha^{ns-ts}\beta^{ns-ts} - \mathcal{AB}\alpha^{ns+st}\beta^{ns-ts}}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{AB}\alpha^{ns}\beta^{ns}(2 - \alpha^{-ts}\beta^{ts} - \alpha^{ts}\beta^{-ts})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{AB}\alpha^{ns}\beta^{ns}\left(2 - \left(\frac{\beta}{\alpha}\right)^{ts} - \left(\frac{\alpha}{\beta}\right)^{ts}\right)}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{-\mathcal{AB}(\alpha\beta)^{ns-ts}(\alpha^{ts}-\beta^{ts})^2}{(\mathcal{A}\alpha^s-\mathcal{B}\beta^s)^2} \quad \text{from } \alpha\beta = m, \text{ we have}$$

$$= -m^{ns-ts}\mathcal{AB}\left(\frac{\alpha^{ts} - \beta^{ts}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)^2.$$

Thus, the proof is completed.

Note that, if we take $t = 1$ then we obtain the Cassini identity. Thus, the proof is completed.

Theorem 4.3. (d' Ocagne Identity) For $n, k \in \mathbb{Z}$, the equation given below is satisfied.

$$W_k^{(s)}W_{n+1}^{(s)} - W_n^{(s)}W_{k+1}^{(s)} = \frac{\mathcal{AB}(\alpha^s - \beta^s)(\alpha^{ks}\beta^{ns} - \alpha^{ns}\beta^{ks})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

Proof. We will use Binet's formula to prove it.

$$W_m^{(s)}W_{n+1}^{(s)} - W_n^{(s)}W_{k+1}^{(s)} =$$

$$\left(\frac{\alpha^{ks} - \beta^{ks}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{(n+1)s} - \beta^{(n+1)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right) - \left(\frac{\alpha^{ns} - \beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{(k+1)s} - \beta^{(k+1)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)$$

$$= \frac{\mathcal{AB}\alpha^{ns}\beta^{ks+s} + \mathcal{AB}\alpha^{ks+s}\beta^{ns} - \mathcal{AB}\alpha^{ks}\beta^{ns+s} - \mathcal{AB}\alpha^{ns+s}\beta^{ks}}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{AB}\alpha^{ns}\beta^{ks}(\beta^s - \alpha^s) + \mathcal{AB}\alpha^{ks}\beta^{ns}(\alpha^s - \beta^s)}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{AB}(\alpha^s - \beta^s)(\alpha^{ks}\beta^{ns} - \alpha^{ns}\beta^{ks})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}.$$

Theorem 4.4. (Vajda Identity) For $n, k, r \in \mathbb{Z}$, the equation given below is satisfied.

$$W_{n+k}^{(s)}W_{n+r}^{(s)} - W_n^{(s)}W_{n+k+r}^{(s)} = \frac{m^{ns}\mathcal{AB}(\alpha^{rs} - \beta^{rs})(\alpha^{ks} - \beta^{ks})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}.$$

Proof. We will use Binet's formula to prove it.

$$W_{n+k}^{(s)}W_{n+r}^{(s)} - W_n^{(s)}W_{n+k+r}^{(s)} =$$

$$\left(\frac{\alpha^{(n+k)s} - \beta^{(n+k)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{(n+r)s} - \beta^{(n+r)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right) - \left(\frac{\alpha^{ns} - \beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{(n+k+r)s} - \beta^{(n+k+r)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)$$

$$= \frac{-\mathcal{AB}\alpha^{ns+ks}\beta^{ns+rs} - \mathcal{AB}\alpha^{ns+rs}\beta^{ns+ks} + \mathcal{AB}\alpha^{ns}\beta^{ns+ks+rs} + \mathcal{AB}\alpha^{ns+ks+rs}\beta^{ns}}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{AB}\alpha^{ns+ks}\beta^{ns}(\alpha^{rs} - \beta^{rs}) - \mathcal{AB}\alpha^{ns}\beta^{ns+ks}(\alpha^{rs} - \beta^{rs})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$= \frac{(\alpha\beta)^{ns}\mathcal{AB}(\alpha^{rs}-\beta^{rs})(\alpha^{ks}-\beta^{ks})}{(\mathcal{A}\alpha^s-\mathcal{B}\beta^s)^2}$ from $\alpha\beta = m$, we have

$$= \frac{m^{ns}\mathcal{AB}(\alpha^{rs} - \beta^{rs})(\alpha^{ks} - \beta^{ks})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}.$$

Thus, the proof is completed.

Theorem 4.5. (Honsberger Identity) For $m, n \in \mathbb{Z}$, the equation given below is satisfied.

$$W_{m-1}^{(s)}W_n^{(s)} + W_m^{(s)}W_{n-1}^{(s)} =$$
$$\frac{\mathcal{A}^2\alpha^{ms+ns-s}(1+\alpha^{2s}) + \mathcal{B}^2\beta^{ms+ns-s}(1+\beta^{2s}) - \mathcal{AB}(1+m^s)(\alpha^{ms-s}\beta^{ns} + \alpha^{ns}\beta^{ms-s})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}.$$

Proof. We will use Binet's formula to prove it.

$$W_{m-1}^{(s)}W_n^{(s)} + W_m^{(s)}W_{n-1}^{(s)} =$$

$$\left(\frac{\alpha^{(m-1)s} - \beta^{(m-1)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{ns} - \beta^{ns}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right) + \left(\frac{\alpha^{ms} - \beta^{ms}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)\left(\frac{\alpha^{(n+1)s} - \beta^{(n+1)s}}{\mathcal{A}\alpha^s - \mathcal{B}\beta^s}\right)$$

$$= \frac{\mathcal{A}^2\alpha^{ms+ns-s} - \mathcal{AB}\alpha^{ms-s}\beta^{ns} - \mathcal{AB}\alpha^{ns}\beta^{ms-s} + \mathcal{B}^2\beta^{ms+ns-s}}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$+ \frac{\mathcal{A}^2\alpha^{ms+ns+s} - \mathcal{AB}\alpha^{ms}\beta^{ns+s} - \mathcal{AB}\alpha^{ns+s}\beta^{ms} + \mathcal{B}^2\beta^{ms+ns+s}}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$= \frac{\mathcal{A}^2\alpha^{ms+ns-s}(1+\alpha^{2s}) + \mathcal{B}^2\beta^{ms+ns-s}(1+\beta^{2s}) - \mathcal{AB}(1+m^s)(\alpha^{ms-s}\beta^{ns} + \alpha^{ns}\beta^{ms-s})}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}.$$

Thus, the proof is completed.

Theorem 4.5. (Melham's Identity) For $n \in \mathbb{Z}$, the equation given below is satisfied.

$$W_{n+1}^{(s)}W_{n+2}^{(s)}W_{n+6}^{(s)} - {W_{n+3}^{(s)}}^3 = \frac{\mathcal{A}^2\mathcal{B}\alpha^{2ns+3s}\beta^{ns+s}(3\alpha^{3s}\beta^{2s} - \alpha^{5s} - \beta^{5s} - \alpha^{4s}\beta^s)}{(\mathcal{A}\alpha^s - \mathcal{B}\beta^s)^2}$$

$$+\frac{\mathcal{B}^2\mathcal{A}\alpha^{ns+s}\beta^{2ns+3s}(\alpha^{5s}+\beta^{5s}-3\alpha^{2s}\beta^{3s}+\alpha^{s}\beta^{4s})}{(\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s})^{\mathbf{2}}}.$$

Proof. We will use Binet's formula to prove it.

$$W_{n+1}^{(s)}W_{n+2}^{(s)}W_{n+6}^{(s)}-{W_{n+3}^{(s)}}^{3}=$$

$$\left(\frac{\alpha^{(n+1)s}-\beta^{(n+1)s}}{\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s}}\right)\left(\frac{\alpha^{ns}-\beta^{ns}}{\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s}}\right)\left(\frac{\alpha^{(n+6)s}-\beta^{(n+6)s}}{\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s}}\right)$$

$$-\left(\frac{\alpha^{(n+3)s}-\beta^{(n+3)s}}{\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s}}\right)\left(\frac{\alpha^{(n+3)s}-\beta^{(n+3)s}}{\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s}}\right)\left(\frac{\alpha^{(n+3)s}-\beta^{(n+3)s}}{\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s}}\right)$$

$$=\frac{1}{(\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s})^{\mathbf{2}}}[\mathcal{A}^2\mathcal{B}(\alpha^{2ns+6s}\beta^{ns+3s}+\alpha^{2ns+6s}\beta^{ns+3s}-\alpha^{2ns+3s}\beta^{ns+6s}$$

$$-\alpha^{2ns+37s}\beta^{ns+2s}+\alpha^{2ns+6s}\beta^{ns+3s}-\alpha^{2ns+8s}\beta^{ns+s})$$

$$+\mathcal{B}^2\mathcal{A}(\alpha^{ns+s}\beta^{2ns+8s}+\alpha^{ns+2s}\beta^{2ns+7s}+\alpha^{ns+6s}\beta^{2ns+3s}-\alpha^{ns+3s}\beta^{2ns+6s}$$

$$-\alpha^{ns+3s}\beta^{2ns+6s}-\alpha^{ns+3s}\beta^{2ns+6s})$$

$$=\frac{\mathcal{A}^2\mathcal{B}\alpha^{2ns+3s}\beta^{ns+s}(3\alpha^{3s}\beta^{2s}-\alpha^{5s}-\beta^{5s}-\alpha^{4s}\beta^{s})}{(\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s})^{\mathbf{2}}}$$

$$+\frac{\mathcal{B}^2\mathcal{A}\alpha^{ns+s}\beta^{2ns+3s}(\alpha^{5s}+\beta^{5s}-3\alpha^{2s}\beta^{3s}+\alpha^{s}\beta^{4s})}{(\mathcal{A}\alpha^{s}-\mathcal{B}\beta^{s})^{\mathbf{2}}}.$$

Thus, the proof is completed.

5 CONCLUSIONS

We unveiled a new realm of numbers – the "higher-order Horadam numbers" – intimately linked to their classic counterparts. By uncovering their secrets: binet's formula, hidden identities, and diverse representations.

But our journey didn't stop there. We unearthed yet another hidden sequence, born from transforming these horadam numbers. Unraveling its intricate recurrence relation and unique properties, we provided a detailed map to this unexplored territory. Additionally, in this study, catalan transforms of higher order Horadam numbers were also examined and tried to be concretized with matrix representations.

Finally, we presented powerful tools – matrices and equations – that unlock the heart of these higher-order horadam numbers, paving the way for further exploration and deeper understanding.

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DATA AVAILABILITY STATEMENT

No data are associated in this manuscript.

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