Proceedings of the Indian National Science Academy
https://doi.org/10.1007/s43538-025-00403-4

RESEARCH PAPER

Copper Fibonacci, Copper Lucas polynomials and their some special transformations and hyperbolic quaternions

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Received: 5 October 2024 / Accepted: 14 February 2025
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Abstract
In this study, we define the Copper Fibonacci and Copper Lucas polynomials, and some terms of these polynomials are given, including the relationship of the ratio of the biggest to the smallest of two consecutive terms of these polynomials. In addition, we provide important relations among the positive and negative index terms, and the sum of the squares of two consecutive terms is related to these polynomials. We apply Catalan transformations to these polynomials and obtain some of their terms. We also relate the terms of the Hankel transformation of the Catalan Copper Fibonacci polynomials with the classical Fibonacci numbers. Finally, we define the application of the Copper Fibonacci and Copper Lucas polynomials to hyperbolic quaternions.

Keywords Polynomials · Generating function · Binet formula · Fibonacci sequence · Catalan Identity

Mathematics Subject Classification 05A15 · 11B39 · 11B83 · 11C08 · 20G20

Introduction

In mathematics, sequences have an important place in understanding a problem that occurs under certain conditions more clearly. The most famous of these problems is the Rabbit problem in Leonardo Fibonacci's famous book Liber Abaci. Using the outputs of the rabbit problem, the Fibonacci sequence is defined and the relationships between consecutive terms of this sequence are obtained.

The most famous number sequences are the Fibonacci and Lucas sequences. These sequences have attracted the attention of scientists for centuries. Fibonacci sequences have been applied in various fields such as Engineering (Oliveira and Alves 2019), Chemistry (Otto 2022), Biomathematics (Avazzadeh et al. 2023), Phylotaxis (Turner et al. 2023), Cryptology (Aydınyüz and Aşçı 2023), etc. Scientists have done many studies on the generalization of the Fibonacci sequence. Examples of studies carried out are Fermat, k-Chebyshev, k-Jacobsthal-Lucas, Narayana, Pell, Padovan sequences, etc. (see for details in Özkan et al. 2017; Akkuş et al. 2024, 2021; Özkan and Akkus 2023; Hoggatt 1969).

For $n \in \mathbb{N}$, the Fibonacci numbers F_n, Bronze Fibonacci number BF_n, Lucas numbers L_n and Bronze Lucas number BL_n are defined by the recurrence relations,

$$F_{n+2} = F_{n+1} + F_n, BF_{n+2} = 3BF_{n+1} + BF_n,$$

$$L_{n+2} = L_{n+1} + L_n, \text{ and } BL_{n+2} = 3BL_{n+1} + BL_n,$$

with $F_0 = 0, F_1 = 1, BF_0 = 0, BF_1 = 1, L_0 = 2, L_1 = 1$, and $BL_0 = 2, BL_1 = 3$.

For F_n, BF_n, L_n, and BL_n, the Binet formulas are

$$F_n = \frac{k_1^n - k_2^n}{k_1 - k_2}, BF_n = \frac{m_1^n - m_2^n}{m_1 - m_2}, L_n = k_1^n + k_2^n, \text{ and } BL_n = m_1^n + m_2^n.$$

where $k_1 = \frac{1+\sqrt{5}}{2} = 1{,}618033988749894\ldots$, $k_2 = \frac{1-\sqrt{5}}{2}$, $m_1 = \frac{3+\sqrt{13}}{2} = 3.302775638\ldots$, and $m_2 = \frac{3-\sqrt{13}}{2}$ are the roots of the characteristic equations $k^2 - k - 1 = 0$ and

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Published online: 18 March 2025

$m^2 - 3m - 1 = 0$. The k_1, and m_1 numbers here are the golden ratio and bronze ratio, respectively, from the metallic ratios.

Some of the identities between Fibonacci and Lucas sequences are as follows (For more identities see in Akkuş et al. 2021; Koshy 2019).

(i) $5F_n^2 = L_{2n} - 2(-1)^n$, (ii) $F_{n+1}L_n - L_{n+1}F_n = 2(-1)^n$,
(iii) $L_nL_{n+1} - L_{2n+1} = (-1)^n$, (iv) $L_n^2 = L_{2n} + 2(-1)^n$,
(v) $L_n = F_{n+1} + F_{n-1}$, (vi) $F_{3n} = F_{2n}L_n - F_n(-1)^n$,
(vii) $F_{n-1}F_{n+1} - F_{n+2}F_{n-2} = 2(-1)^n$, (viii) $\frac{F_{n+1}}{F_n} - \frac{L_{n+1}}{L_n} = \frac{2(-1)^n}{F_{2n}}$.

The Cassini formulas for Fibonacci and Lucas sequences

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \text{ and } L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n-1}.$$

One of the interesting applications of the Cassini identity is its use in geometric paradoxes (see for details in Koshy 2019, pp. 100–108).

For $n \in \mathbb{N}$, the Copper Fibonacci numbers CF_n and Copper Lucas numbers CL_n are defined by the recurrence relations,

$$CF_{n+2} = 4CF_{n+1} + CF_n \text{ and } CL_{n+2} = 4CL_{n+1} + CL_n$$

with $CF_0 = 0, CF_1 = 1$ and $CL_0 = 2, CL_1 = 4$.

For CF_n and CL_n the Binet formulas are

$$CF_n = \frac{\kappa^n - \theta^n}{\kappa - \theta} \text{ and } CL_n = \kappa^n + \theta^n.$$

where $\kappa = 2 + \sqrt{5} = 4.23606797749979\ldots$, and $\theta = 2 - \sqrt{5}$ are the roots of the characteristic equation $r^2 - 4r - 1 = 0$. The number κ here is the copper ratio from metallic ratios.

In Özkan and Akkuş (2024), Özkan and Akkuş defined the Copper Fibonacci CF_n and Copper Lucas CL_n numbers. In addition, they obtained the following properties of these sequences. For $m, n \in \mathbb{N}$,

(i) $CF_{n+m+1} = CF_{n+1}CF_{m+1} + CF_nCF_m$,
(ii) $CL_{n+m+1} = CL_{m+1}CF_{n+1} + CL_mCF_n$,
(iii) $\theta^{2n} = \frac{CF_{2n}}{2}\left(2\sqrt{5} - \kappa\right)\sqrt{5} - \frac{CL_{2n-1}}{4}$,
(iv) $\kappa^{2n} = \frac{CF_{2n}}{2}\kappa\sqrt{5} - \frac{CL_{2n-1}}{4}$,,
(v) $CF_{n+m}CF_{n-m} - CF_n^2 = (-1)^{n-m+1}CF_m^2$,
(vi) $CL_{n+m}CL_{n-m} - CL_n^2 = 20(-1)^{n-m}CF_m^2$.

With the help of the recurrence relation of the Fibonacci sequence, Fibonacci polynomials were presented and these polynomials had a significant place in algebra and number theory. Fibonacci and Lucas polynomials have applications in different scientific fields like Chemistry, (Avaroğlu et al. 2020), Cryptology (Ali et al. 2021), Medicine (Heydari et al. 2021), etc.

For $n \in \mathbb{N}$, the Fibonacci polynomials $f_n(x)$ and Lucas polynomials $l_n(x)$ are defined by the recurrence relations,

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x) \text{ and } l_{n+2}(x) = xl_{n+1}(x) + l_n(x),$$

with the initial conditions $f_0(x) = 0$, $f_1(x) = 1$ and $l_0(x) = 2$, $l_1(x) = x$.

For $f_n(x)$ and $l_n(x)$, the Binet formulas are

$$f_n(x) = \frac{s_1^n - s_2^n}{s_1 - s_2} \text{ and } l_n(x) = s_1^n + s_2^n.$$

where $s_1 = \frac{x+\sqrt{x^2+4}}{2}$ and $s_2 = \frac{x-\sqrt{x^2+4}}{2}$ are the roots of the characteristic equation $s^2 - xs - 1 = 0$.

In Hoggatt (1973), Hoggatt and Bicknell worked on the roots of Fibonacci polynomials. In addition, they found many properties of these roots. Smajlovic et al. (2024) studied new relations between Chebyshev, Fibonacci, and Lucas polynomials. Also, they obtained new relations on the trigonometric sum formulas. In Kishore and Verma (2023), Kishore and Verma obtained some identities on the sums of finite products of Fibonacci, Chebyshev, and Pell polynomials. Kuloğlu et al. investigated Fibonacci and Lucas polynomials on n-genes and reached very interesting results (Kuloğlu et al. 2023). Florez et al. calculated the identities of the generalized Fibonacci polynomial and related the results b to these polynomials (Flórez et al. 2017). Özkan and Kuloğlu defined Narayana polynomials with the help of the generalization of Fibonacci polynomials and they made many applications of these polynomials (Özkan and Kuloğlu 2021). Özkan and Taştan (2020) introduced the Gauss Fibonacci polynomials and made many applications related to these polynomials.

The quaternions were first used by Hamilton in 1843. In addition, quaternions are used to control rotational movements, especially in Chemistry (Sveier et al. 2019), Eulerian angles (Cariow et al. 2020), mechanics (Danielewski and Sapa 2020), 3D games (Ozgoren 2019), and kinematics (Senjean et al. 2021). In Horadam (1963), Horadam defined Fibonacci quaternions and Complex Fibonacci numbers, and some properties were found. Dağdeviren (2023) worked on the generalization of hyperbolic quaternions. He also found interesting consequences of these generalizations.

An important feature of the algebra of hyperbolic quaternions is that it is an algebra that has no relation to its elements defined on the set of real numbers.

$$q = xi_1 + yi_2 + zi_3 + ti_4, x, y, z, t \in \mathbb{R}$$

Kösal (2018) examined the q components whose properties are given in Table 1. In addition, he found some features.

In Macfarlane (1902), Macfarlane did a lot of work on hyperbolic quaternions and the properties of these quaternions. The general form for these quaternions is as follows:

$$\hbar = \hbar_1 i_1 + \hbar_2 i_2 + \hbar_3 i_3 + \hbar_4 i_4 = \left(\hbar_1, \hbar_2, \hbar_3, \hbar_4\right).$$

Here, $\hbar_1, \hbar_2, \hbar_3, \hbar_4$ are the terms of the sequence and i_1, i_2, i_3, i_4 are hyperbolic quaternions. Akkuş and Özkan

Table 1 Hyperbolic quaternions units

.	i_1	i_2	i_3	i_4
i_1	i_1	i_2	i_3	i_4
i_2	i_2	i_1	i_4	$-i_3$
i_3	i_3	$-i_4$	i_1	i_2
i_4	i_4	i_3	$-i_2$	i_1

related the generalized k-Leonardo numbers to quaternions and found many of their properties (Akkuş and Özkan 2024a). In Dagdeviren (2023), Dagdeviren, worked on the generalization of binary, complex and hyperbolic quaternions. Studies on hyperbolic quaternions and their applications have been done in the literature (see for details in Özdemir and Ekmekci 2022; Akkuş and Özkan 2024b; Yağmur 2020; Özdemir and Ceyhan 2024).

As can be seen from the studies carried out so far, many generalizations of the Fibonacci and Lucas sequences have been studied so far. In this study, we present a new generalization inspired by the Copper Fibonacci sequence and Fibonacci polynomials. We call these sequences the Copper Fibonacci polynomials and Copper Lucas polynomials and denote them as $CF_n(x)$, and $CL_n(x)$, respectively.

Copper Fibonacci and Copper Lucas polynomials

In this chapter, we define the Copper Fibonacci and Copper Lucas polynomials, and some terms of these polynomials are given. Then there are many properties of these polynomials.

Definition 2.1. For $n \in \mathbb{N}$, the Copper Fibonacci polynomials $CF_n(x)$ and Copper Lucas polynomials $CL_n(x)$ are defined by, respectively,

$$CF_{n+2}(x) = 4xCF_{n+1}(x) + CF_n(x) \tag{2.1}$$

with $CF_0(x) = 0$ and $CF_1(x) = 1$,

$$CL_{n+2}(x) = 4xCL_{n+1}(x) + CL_n(x) \tag{2.2}$$

with $CL_0(x) = 2$ and $CL_1(x) = 4x$.

Note that when $x = 1$, the Copper Fibonacci and Copper Lucas sequences are obtained [see for details in Koshy (2019).

The characteristic equation of the Copper Fibonacci polynomials is

$$c^2 - 4xc - 1 = 0. \tag{2.3}$$

The roots of this equation

$c_1 = 2x + \sqrt{4x^2+1}$ and $c_2 = 2x - \sqrt{4x^2+1}$.

Next, we give the relationships between these roots below:

$c_1 + c_2 = 4x, c_1 - c_2 = 2\sqrt{4x^2+1}, c_1c_2 = -1,$
and $c_1^2 + c_2^2 = 16x^2 + 2.$

Some terms of the $CF_n(x)$ and $CL_n(x)$ polynomials are

$$0,1,4x,16x^2+1,64x^3+8x,256x^4+48x^2 + 1,1024x^5+256x^3+12x,\ldots$$

and

$$2,4x,16x^2+2,64x^3+12x,256x^4+64x^2 + 2,1024x^5+320x^3+20x,\ldots$$

respectively.

Also, the terms of the $CF_n(x)$, and $CL_n(x)$ polynomials can be found with the help of the following relations. These relations are called the closed format of the family of polynomials. Let $n \in \mathbb{N}^+$:

$$CF_n(x) = \sum_{q=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-q}{q} (4x)^{n-1-2q} \text{ and } CL_n(x)$$
$$= \sum_{q=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-q} \binom{n-q}{q} (4x)^{n-2q}.$$

Next, the Binet formulas of the $CF_n(x)$, and $CL_n(x)$ are expressed.

Theorem 2.1. *Let $n \in \mathbb{N}$. We have*

(i) $CF_n(x) = \frac{c_1^n - c_2^n}{c_1 - c_2}$,
(ii) $CL_n(x) = c_1^n + c_2^n$.

Proof. (i) The following general form is used to find equality:

$$CF_n(x) = kc_1^n + mc_2^n$$

k and m values are obtained by solving the equation obtained by giving different values to the above equation. For $n = 0$, $CF_0(x) = 0$ and for $n = 1$, $CF_1(x) = 1$. So, $k = \frac{1}{c_1 - c_2}$ and $m = \frac{-1}{c_1 - c_2}$. Thus, we obtain

$$CF_n(x) = \frac{c_1^n - c_2^n}{c_1 - c_2}.$$

Other proof is shown in a similar way.

□

In the following theorems, we find the relationships among the roots of the characteristic equation of these polynomials and these polynomials.

Theorem 2.2. *Let $n \in \mathbb{N}$. We have*

$(i)\ c_1^{2n} = \frac{CF_{2n}(x)}{2x} c_1\sqrt{4x^2+1} - \frac{CL_{2n-1}(x)}{4x},$

$(ii)\ c_2^{2n} = \frac{CF_{2n}(x)}{4x}\left(2\sqrt{4x^2+1} - c_1\right)\sqrt{5} - \frac{CL_{2n-1}(x)}{4x},$

$(iii)\ c_1^{2n+1} = -\frac{CF_{2n}(x)}{2x}\sqrt{4x^2+1} + c_1\frac{CL_{2n+1}(x)}{4x},$

$(iv)\ c_2^{2n+1} = -\frac{CF_{2n}(x)}{2x}\sqrt{4x^2+1} - \theta\frac{CL_{2n+1}(x)}{4x}$

$(v)\ 2\sqrt{4x^2+1}CF_n(x) + CL_n(x) = 2c_1^n,$

$(vi)\ 2\sqrt{4x^2+1}CF_n(x) - CL_n(x) = -2c_2^n.$

Proof. Binet formulas are used for proofs. We have

(i) $$\begin{aligned}&\frac{CF_{2n}(x)}{2x}c_1\sqrt{4x^2+1} - \frac{CL_{2n-1}(x)}{4x}\\ &= \frac{c_1^{2n}-c_2^{2n}}{2x\left(c_1-c_2\right)}c_1\sqrt{4x^2+1} - \frac{c_1^{2n-1}+c_2^{2n-1}}{4x}\\ &= \frac{c_1^{2n+1}-c_1c_2^{2n}-c_1^{2n-1}-c_2^{2n-1}}{4x}\\ &= \frac{c_1^{2n}\left(c_1-\frac{1}{c_1}\right)+c_2^{2n}\left(-c_1-\frac{1}{c_2}\right)}{4x}\\ &= c_1^{2n}.\end{aligned}$$

Other proofs are shown in a similar way. □

Theorem 2.3. *The relationship between two consecutive terms of $CF_n(x)$ and $CL_n(x)$ polynomials is as follows*:

$$\lim_{n\to\infty}\frac{CF_{n+1}(x)}{CF_n(x)} = c_1 \text{ and } \lim_{n\to\infty}\frac{CL_{n+1}(x)}{CL_n(x)} = c_1.$$

Proof. Binet formulas are used for proofs. We obtain

$$\lim_{n\to\infty}\frac{CF_{n+1}(x)}{CF_n(x)} = \frac{\frac{c_1^{n+1}-c_2^{n+1}}{c_1-c_2}}{\frac{c_1^n-c_2^n}{c_1-c_2}} = \lim_{n\to\infty}\frac{c_1^{n+1}\left(1-\left(\frac{c_2}{c_1}\right)^{n+1}\right)}{c_1^n\left(1-\left(\frac{c_2}{c_1}\right)^n\right)}$$

Thus, we obtain

$$\lim_{n\to\infty}\frac{CF_{n+1}(x)}{CF_n(x)} = c_1.$$

Similarly, we have

$$\lim_{n\to\infty}\frac{CL_{n+1}(x)}{CL_n(x)} = c_1.$$

□

Theorem 2.4. *Let $s = c_1$ or $s = c_2$ and $n, k, m, p \in \mathbb{N}$. We obtain*

$(i)\ s^n = sCF_n(x) + CF_{n-1}(x),$

$(ii)\ 1 + 4xs + s^{2\left(2^{n+1}+1\right)} = s^{2\left(2^n+1\right)}CL_{2^{n+1}}(x).$

$(iii)\ s^{2n} = s^nCL_n(x) - (-1)^n,$

$(iv)\ s^{np} = \frac{s^nCF_{np}(x)}{CF_n(x)} - (-1)^n\frac{CF_{n(p-1)}(x)}{CF_n(x)},$

$(v)\ s^n = s^kCF_{n-k+1}(x) + s^{k-1}CF_{n-k}(x),$

$(vi)\ CF_{n(k-m)}(x) = s^{nm}CF_{nk}(x) + s^{nk}CF_{nm}(x),$

Proof. (i) For $s = c_1$, we have

$$\begin{aligned}sCF_n(x) + CF_{n-1}(x) &= c_1\left(\frac{c_1^n-c_2^n}{c_1-c_2}\right) + \left(\frac{c_1^{n-1}-c_2^{n-1}}{c_1-c_2}\right)\\ &= \frac{c_1^{n-1}\left(c_1^2+1\right) - c_2^{n-1}\left(c_1c_2+1\right)}{c_1-c_2} = c_1^n,\end{aligned}$$

For $s = c_2$, we have

$$\begin{aligned}sCF_n(x) + CF_{n-1}(x) &= c_2\left(\frac{c_1^n-c_2^n}{c_1-c_2}\right) + \left(\frac{c_1^{n-1}-c_2^{n-1}}{c_1-c_2}\right)\\ &= \frac{c_1^{n-1}\left(c_1c_2+1\right) - c_2^n\left(c_2+\frac{1}{c_2}\right)}{c_1-c_2} = c_2^n.\end{aligned}$$

Other proofs are shown in a similar way. □

In the next theorems, we obtain special relations between the $CF_n(x)$, and $CL_n(x)$ polynomials.

Theorem 2.5. *Let $n \in \mathbb{N}$. The following equations are satisfied*:

$(i)\ CF_n(x) = \frac{1}{8x^2+2}CL_{n+1}(x) - \frac{x}{4x^2+1}CL_n(x),$

$(ii)\ CL_n(x) = 2CF_{n+1}(x) - 4xCF_n(x)$

Proof. (i) The following relation is used for proofs;

$$CF_n(x) = axCL_{n+1}(x) + bxCL_n(x).$$

For these n values, we obtain;

$$CF_0(x) = axCL_1(x) + bxCL_0(x),$$

$$CF_1(x) = axCL_2(x) + bxCL_1(x).$$

We find

$$a = \frac{1}{8x^2+2} \text{ and } b = -\frac{x}{4x^2+1}.$$

Other proof is shown in a similar way.

□

Theorem 2.6. *Let $m, n \in \mathbb{N}$, and $n > m$. We have*

$(i)\ CF_{n+m+1}(x) = CF_{n+1}(x)CF_{m+1}(x) + CF_n(x)CF_m(x),$

$(ii)\ CL_{n+m+1}(x) = CL_{m+1}(x)CF_{n+1}(x) + CL_m(x)CF_n(x),$

$(iii)\ CF_n(x) = CF_{-n}(x)\Big(\sqrt{4x^2+1}CF_n^2(x) - \frac{1}{2}CL_{2n}(x)\Big),$

$(iv)\ CL_n(x) = \frac{CL_{-n}(x)}{2}\big(CL_n^2(x) - CL_{2n}(x)\big),$

$(v)\ CL_n^2(x) + CL_{n+1}^2(x) = CL_{2n}(x) + CL_{2n+2}(x),$

$(vi)\ CF_n^2(x) + CF_{n+1}^2(x) = CF_{2n+1}(x).$

Proof. Binet formulas are used for proofs. We obtain

$$\begin{aligned}
&\text{(i) } CF_{n+1}(x)CF_{m+1}(x) + CF_n(x)CF_m(x)\\
&= \frac{c_1^{n+1} - c_2^{n+1}}{c_1 - c_2}\frac{c_1^{m+1} - c_2^{m+1}}{c_1 - c_2} + \frac{c_1^n - c_2^n}{c_1 - c_2}\frac{c_1^m - c_2^m}{c_1 - c_2}\\
&= \frac{c_1^{n+m+1}\big(c_1 - c_2\big) - c_2^{n+m+1}\big(c_1 - c_2\big)}{\big(c_1 - c_2\big)^2}\\
&= \frac{c_1^{n+m+1} - c_2^{n+m+1}}{c_1 - c_2}\\
&= CF_{n+m+1}(x).
\end{aligned}$$

(iii) $CF_{-n}(x) = \frac{c_1^{-n} - c_2^{-n}}{c_1 - c_2} = -\frac{c_1^n - c_2^n}{(c_1 - c_2)\kappa^n\theta^n} = -\frac{CF_n(x)}{c_1^n c_2^n}$. So, we get

$$c_1^n c_2^n = -\frac{CF_{-n}(x)}{CF_n(x)}.$$

Then we find

$$\begin{aligned}
CF_n^2(x) &= \frac{c_1^n - c_2^n}{c_1 - c_2}\frac{c_1^n - c_2^n}{c_1 - c_2} = \frac{c_1^{2n} + c_2^{2n} - 2c_1^n c_2^n}{\big(c_1 - c_2\big)\big(c_1 - c_2\big)}\\
&= \frac{1}{2\sqrt{4x^2+1}}CL_{2n} - \frac{1}{\sqrt{4x^2+1}}c_1^n c_2^n.
\end{aligned}$$

Thus, we obtain

$$CF_n(x) = CF_{-n}(x)\Big(\sqrt{4x^2+1}CF_n^2(x) - \frac{1}{2}CL_{2n}(x)\Big).$$

Other proofs are shown in a similar way. □

Theorem 2.7. *Let $k, m, n \in \mathbb{N}$, and $k > m, n$. We obtain*

(i) $CF_{k+m+n} = \frac{1}{4}CL_kCL_mCF_n + \frac{1}{4}CF_kCL_mCL_n + \frac{1}{4}CL_kCF_mCL_n + \big(4x^2+1\big)CF_kCF_mCF_n,$

(ii) $CL_{k+m+n} = \frac{1}{4}CL_kCL_mCL_n + \big(4x^2+1\big)CL_kCF_mCF_n + \big(4x^2+1\big)CF_kCL_mCF_n + \big(4x^2+1\big)CF_kCF_mCL_n.$

The proof of these relations is shown in the same way as the proof of the previous theorem.

In the following theorem, we examine the relationships among $CF_n(x)$, $CL_n(x)$, F_n, and L_n.

Theorem 2.8. *Let $n \in \mathbb{N}$. We have*

$(i)\ CF_n(x) = CF_{-n}(x)\Big(\frac{F_{3n} - F_{2n}(F_{n+1} + F_{n-1})}{F_n}\Big),$

$(ii)\ CL_n(x) = CL_{-n}(x)\big(L_nL_{n+1} - L_{2n+1}\big),$

$(iii)\ CL_n^2(x) - \big(16x^2+4\big)CF_n^2(x) = 4\Big(\frac{F_{2n}F_{n+1}}{2F_n} - \frac{F_{2n}L_{n+1}}{2L_n}\Big).$

Proof. Binet formulas are used for proofs. We obtain

(i) $CF_{-n}(x) = \frac{c_1^{-n} - c_2^{-n}}{c_1 - c_2} = -\frac{c_1^n - c_2^n}{(c_1 - c_2)\kappa^n\theta^n} = -\frac{CF_n(x)}{c_1^n c_2^n}$. Thus, we get

$$\begin{aligned}
CF_n(x) &= -CF_{-n}(x)(-1)^n\\
&= CF_{-n}(x)\frac{F_{3n} - F_{2n}L_n}{F_n}\\
&= CF_{-n}(x)\left(\frac{F_{3n} - F_{2n}(F_{n+1} + F_{n-1})}{F_n}\right).
\end{aligned}$$

(ii) $CL_{-n}(x) = c_1^{-n} + c_2^{-n} = \frac{c_1^n + c_2^n}{c_1^n c_2^n} = \frac{CL_n(x)}{(-1)^n}$. Thus, we have

$$\begin{aligned}
CL_n(x) &= CL_{-n}(x)(-1)^n\\
&= CL_{-n}(x)\big(L_nL_{n+1} - L_{2n+1}\big).
\end{aligned}$$

$$\begin{aligned}
&CL_n^2(x) - \big(16x^2+4\big)CF_n^2(x)\\
&= \big(c_1^n + c_2^n\big)^2 - \big(16x^2+4\big)\left(\frac{c_1^n - c_2^n}{c_1 - c_2}\right)^2\\
&= c_1^{2n} + c_2^{2n} + 2c_1^n c_2^n - c_1^{2n} - c_2^{2n} + 2c_1^n c_2^n\\
\text{(iii)}\ &= 4c_1^n c_2^n\\
&= 4(-1)^n\\
&= 4\left(\frac{F_{2n}F_{n+1}}{2F_n} - \frac{F_{2n}L_{n+1}}{2L_n}\right).
\end{aligned}$$

□

Next, we find the sum formulas of the $CF_n(x)$, and $CL_n(x)$ polynomials. Also, we give binomial formulas of these polynomials.

Theorem 2.9. Let $m, n, p, r \in \mathbb{N}$. We obtain

(i) $\sum_{m=0}^{n} CF_{rm+p}(x) = \begin{cases} \frac{-CF_{nr+r+p}(x) + (-1)^r CF_{nr+p}(x) + CF_{r-p}(x) + CF_p(x)}{1 + (-1)^r - CL_r(x)}, & \text{if } p < r \\ \frac{-CF_{nr+r+p}(x) + (-1)^r CF_{nr+p}(x) + CF_{p-r}(x) + CF_p(x)}{1 + (-1)^r - CL_r(x)}, & \text{otherwise} \end{cases},$

(ii) $\sum_{m=0}^{n} CL_{rm+p}(x) = \begin{cases} \frac{-CL_{nr+r+p}(x) + (-1)^r CL_{nr+p}(x) - CL_{r-p}(x) + CL_p(x)}{1 + (-1)^r - CL_r(x)}, & \text{if } p < r \\ \frac{(-1)^r CL_{nr+p}(x) - CL_{p-r}(x) + CL_p(x) - CL_{nr+r+p}(x)}{1 + (-1)^r - CL_r(x)}, & \text{otherwise} \end{cases}.$

Proof. (i) Binet formulas are used for proofs. We have

$$\sum_{m=0}^{n} CF_{rm+p}(x) = \sum_{y=0}^{n} \frac{c_1^{rm+p} - c_2^{rm+p}}{c_1 - c_2} = \frac{c_1^p}{c_1 - c_2} \sum_{y=0}^{n} \left(c_1^r\right)^m - \frac{c_2^p}{c_1 - c_2} \sum_{y=0}^{n} \left(c_2^r\right)^m$$
$$= \frac{c_1^p}{c_1 - c_2} \frac{c_1^{nr+r} - 1}{c_1^r - 1} - \frac{c_2^p}{c_1 - c_2} \frac{c_2^{nr+r} - 1}{c_2^r - 1}$$
$$= \begin{cases} \frac{-CF_{nr+r+p}(x)+(-1)^r CF_{nr+p}(x)+CF_{r-p}(x)+CF_p(x)}{1+(-1)^r-CL_r(x)}, & \text{if } p < r \\ \frac{-CF_{nr+r+p}(x)+(-1)^r CF_{nr+p}(x)+CF_{p-r}(x)+CF_p(x)}{1+(-1)^r-CL_r(x)}, & \text{otherwise} \end{cases}.$$

Other proof is shown in a similar way. □

Corollary 2.1. *The special sum formulas of $CF_n(x)$ and $CL_n(x)$ polynomials are as follows*:

(i) $\sum_{m=0}^{n} CF_m(x) = \frac{CF_{n+1}(x)+CF_n(x)-1}{4x}$,

(ii) $\sum_{m=0}^{n} CL_m(x) = \frac{CL_{n+1}(x)+CL_n(x)+4x+2}{4x}$,

(iii) $\sum_{m=0}^{n} CF_{2m}(x) = \frac{CF_{2n+1}(x)-1}{4x}$,

(iv) $\sum_{m=0}^{n} CL_{2m}(x) = \frac{CL_{2n+1}(x)+4x}{4x}$,

(v) $\sum_{m=0}^{n} CF_{2m+1}(x) = \frac{CF_{2n+2}(x)}{4x}$,

(vi) $\sum_{m=0}^{n} CL_{2m+1}(x) = \frac{CL_{2n+2}(x)-2}{4x}$.

Proof. i. In theorem 2.9., if $p = 0$ and $r = 1$, we obtain.

$$\sum_{m=0}^{n} CF_m(x) = \frac{CF_{n+1}(x) + CF_n(x) - 1}{4x}.$$

□

Theorem 2.10. *Let $n \in \mathbb{N}$. We get*

(i) $\sum_{j=0}^{n} \binom{n}{j} (4x)^j CF_j(x) = CF_{2n}(x)$,

(ii) $\sum_{j=0}^{n} \binom{n}{j} (4x)^j CL_j(x) = CL_{2n}(x)$,

(iii) $\sum_{j=0}^{n} \binom{n}{j} (-1)^j CF_{2j}(x) = (-4x)^n CF_n(x)$,

(iv) $\sum_{j=0}^{n} \binom{n}{j} (-1)^j CL_{2j}(x) = (-4x)^n CL_n(x)$.

Proof. (i) With the help of Eq. (2.3), the following relation is obtained:

$c_1^2 = 4xc_1 + 1$ and $c_2^2 = 4xc_2 + 1$.

$$\sum_{j=0}^{n} \binom{n}{j} (4x)^j CF_j(x) = \sum_{j=0}^{n} \binom{n}{j} (4x)^j \left(\frac{c_1^j - c_2^j}{c_1 - c_2}\right)$$
$$= \frac{1}{c_1 - c_2} \sum_{j=0}^{n} \binom{n}{j} (4x)^j c_1^j - \frac{1}{c_1 - c_2} \sum_{j=0}^{n} \binom{n}{j} (4x)^j c_2^j$$
$$= \frac{1}{c_1 - c_2} \left[\left(1 + 4xc_1\right)^n - \left(1 + 4xc_2\right)^n\right]$$
$$= \frac{1}{c_1 - c_2} \left[\left(c_1^2\right)^n - \left(c_2^2\right)^n\right]$$
$$= CF_{2n}(x).$$

Other proofs are shown in a similar way. □

Next, we obtain special generating functions of the $CF_n(x)$, and $CL_n(x)$ polynomials.

Theorem 2.11. *Let $r, n, p \in \mathbb{N}$, and $p > r$, we obtain.*

(i) $\sum_{n=0}^{\infty} CF_{rn+p}(x)t^n = \frac{CF_p(x)-t(-1)^r CF_{p-r}(x)}{1-tCL_r(x)+(-1)^r t^2}$,

(ii) $\sum_{n=0}^{\infty} CL_{rn+p}(x)t^n = \frac{CL_p(x)-t(-1)^r CL_{p-r}(x)}{1-tCL_r(x)+(-1)^r t^2}$,

(iii) $\sum_{n=0}^{\infty} \frac{CF_{rn}(x)}{n!} t^n = \frac{e^{c_1^r t} - e^{c_2^r t}}{c_1 - c_2}$,

(iv) $\sum_{n=0}^{\infty} \frac{CL_{rn}(x)}{n!} t^n = e^{c_1^r t} + e^{c_2^r t}$.

Proof. Binet formulas are used for proofs. We obtain

(i)
$$\sum_{i=0}^{\infty} CF_{rn+p}(x)t^n = \sum_{n=0}^{\infty} \frac{c_1^{rn+p} - c_2^{rn+p}}{c_1 - c_2} t^n$$
$$= \frac{c_1^p}{c_1 - c_2} \sum_{n=0}^{\infty} \left(c_1^r t\right)^n - \frac{c_2^p}{c_1 - c_2} \sum_{n=0}^{\infty} \left(c_2^r t\right)^n$$
$$= \frac{c_1^p}{c_1 - c_2} \frac{1}{1 - c_1^r t} - \frac{c_2^p}{c_1 - c_2} \frac{1}{1 - c_2^r t}$$
$$= \frac{CF_p(x) - t(-1)^r CF_{p-r}(x)}{1 - tCL_r(x) + (-1)^r t^2}.$$

$$\sum_{n=0}^{\infty}\frac{CL_{rn}}{n!}t^n=\sum_{n=0}^{\infty}\frac{c_1^{rn}+c_2^{rn}}{n!}t^n$$

(iv)
$$=\sum_{n=0}^{\infty}\frac{\left(c_1^r t\right)^n}{n!}+\sum_{n=0}^{\infty}\frac{\left(c_2^r t\right)^n}{n!}$$
$$=e^{c_1^r t}+e^{c_2^r t}.$$

Other proofs are shown in a similar way.

□

Corollary 2.2. *The generating functions of $CF_n(x)$ and $CL_n(x)$ polynomials are as follows*:

(*i*) $f(t)=\sum_{n=0}^{\infty}CF_n(x)t^n=\frac{t}{1-4xt-t^2}$,

(*ii*) $l(t)=\sum_{n=0}^{\infty}CL_n(x)t^n=\frac{2-4t}{1-4xt-t^2}$.

Proof. (i) In theorem 2.11., if $p=0$ and $r=1$, we obtain

$$f(t)=\sum_{n=0}^{\infty}CF_n(x)t^n=\frac{t}{1-4xt-t^2}.$$

□

In the following theorems, we calculate the some identities of the $CF_n(x)$, and $CL_n(x)$ polynomials.

Theorem 2.12. (i) (*Cassini Identity*) *Let $n\in\mathbb{N}$. We get*

- $CF_{n+1}(x)CF_{n-1}(x)-CF_n^2(x)=(-1)^n$,
- $CL_{n+1}(x)CL_{n-1}(x)-CL_n^2(x)=4\left(4x^2+1\right)(-1)^{n-1}$.

(*ii*) (*Halton Identity*) *Let $n,i,j\in\mathbb{N}$, and $i,j<n$. We get*

- $$CF_{n+i}(x)CF_{n-i}(x)-CF_{n+j}(x)CF_{n-j}(x)=\frac{1}{4\left(4x^2+1\right)}\left[(-1)^{n-j}CL_{2j}(x)-(-1)^{n-i}CL_{2i}(x)\right],$$
- $$CL_{n+i}(x)CL_{n-i}(x)-CL_{n+j}(x)CL_{n-j}(x)=(-1)^{n-i}CL_{2i}(x)-(-1)^{n-j}CL_{2j}(x).$$

(*iii*) (*D'ocagne Identity*) *Let $n,i\in\mathbb{N}$, and $i<n$. We get*

- $CF_{n+1}(x)CF_i(x)-CF_n(x)CF_{i+1}(x)=(-1)^{i+1}CF_{n-i}(x)$,
- $CL_{n+1}(x)CL_i(x)-CL_n(x)CL_{i+1}(x)=4\left(4x^2+1\right)(-1)^iCF_{n-i}(x)$.

(*iv*) (*Vajda Identity*) *Let $n,i,j\in\mathbb{N}$, and $i,j<n$. We get*

- $CF_{n+i}(x)CF_{n+j}(x)-CF_n(x)CF_{n+i+j}(x)=(-1)^nCF_i(x)CF_j(x)$,
- $$CL_{n+i}(x)CL_{n+j}(x)-CL_n(x)CL_{n+i+j}(x)=4\left(4x^2+1\right)(-1)^{n+1}CF_i(x)CF_j(x).$$

(*v*) (*Catalan Identity*) *Let $n,i\in\mathbb{N}$, and $i<n$. We get*

- $CF_{n+i}(x)CF_{n-i}(x)-CF_n^2(x)=(-1)^{n-i+1}CF_i^2(x)$,
- $CL_{n+i}(x)CL_{n-i}(x)-CL_n^2(x)=4\left(4x^2+1\right)(-1)^{n-i}CF_i^2(x)$.

(*vi*) (*Gelin-Cesaro Identity*) *Let $n\in\mathbb{N}$. We get*

- $$CF_{n+2}(x)CF_{n+1}(x)CF_{n-1}(x)CF_{n-2}(x)-CF_n^4(x)=\frac{1}{4\left(4x^2+1\right)}\left(\frac{1}{(-1)^{n-3}}CL_{2n+4}(x)+\frac{1}{(-1)^{n-2}}CL_{2n+2}(x)+\frac{1}{(-1)^{2n-3}}CL_6(x)+\frac{1}{(-1)^n}CL_{2n-2}(x)+CL_2(x)+\frac{1}{(-1)^{n+1}}CL_{2n-4}(x)+\frac{1}{(-1)^{n-3}}CL_{2n}(x)-5\right),$$
- $$CL_{n+2}(x)CL_{n+1}(x)CL_{n-1}(x)CL_{n-2}(x)-CL_n^4(x)=\frac{1}{(-1)^{n-2}}CL_{2n+4}(x)+\frac{1}{(-1)^{n-1}}CL_{2n+2}(x)+\frac{1}{(-1)^{2n-3}}CL_6(x)+\frac{1}{(-1)^{n+1}}CL_{2n-2}(x)+\frac{1}{(-1)^{2n-1}}CL_2(x)(x)+\frac{1}{(-1)^{n+2}}CL_{2n-4}(x)-\frac{1}{(-1)^{n-2}}CL_{2n}(x)-5.$$

(*vii*) (*Padilla Identity*) *Let* n $\in\mathbb{N}$. *We get*

- $$CF_{n+2}^3(x)+CF_{n-1}^3(x)-3CF_n(x)CF_{n+1}(x)CF_{n+2}(x)=\frac{1}{4\left(4x^2+1\right)}\left(CF_{3n+6}(x)-3CF_{3n+3}(x)+CF_{3n-3}(x)+3CF_{2n+3}(x)-3CF_{n+2}(x)+3CF_{n+1}(x)\right),$$
- $$CL_{n+2}^3(x)+CL_{n-1}^3(x)-3CL_n(x)CL_{n+1}(x)CL_{n+2}(x)=CL_{3n+6}(x)+3CL_{n+2}(x)+CL_{n-3}(x)+3CL_{n-1}(x)+\left(4\left(4x^2+1\right)\right)\left(-CF_{3n+3}(x)-3CF_{n-1}(x)+3CF_{n+1}(x)+3CF_{2n+3}(x)\right).$$

(*viii*) (*Melham Identity*) *Let* n $\in\mathbb{N}$. *We get*

- $$CF_{n+1}(x)CF_{n+2}(x)CF_{n+6}(x)-CF_n^3(x)=\frac{1}{4\left(4x^2+1\right)}\left(CF_{3n+9}(x)-CF_{3n}(x)-\frac{1}{(-1)^{n+1}}CF_{n+7}(x)-\frac{1}{(-1)^{n+2}}CF_{n+5}(x)+\frac{3}{(-1)^n}CF_n(x)-\frac{1}{(-1)^{n+6}}CF_{n-3}(x)\right),$$

- $CL_{n+1}(x)CL_{n+2}(x)CL_{n+6}(x) - CL_n^3(x)$
$= CL_{3n+9}(x) - CL_{3n}(x) + \frac{1}{(-1)^{n+1}} CL_{k,n+7}(x)$
$+ \frac{1}{(-1)^{n+2}} CL_{k,n+5}(x) - \frac{3}{(-1)^n} CL_n(x)$
$+ \frac{1}{(-1)^{n+6}} CL_{n-3}(x).$

Proof. Binet formulas are used for proofs. We have.

(i)
$$\begin{aligned} & CF_{n+1}(x)CF_{n-1}(x) - CF_n^2(x) \\ &= \frac{c_1^{n+1} - c_2^{n+1}}{c_1 - c_2} \frac{c_1^{n-1} - c_2^{n-1}}{c_1 - c_2} - \frac{c_1^n - c_2^n}{c_1 - c_2} \frac{c_1^n - c_2^n}{c_1 - c_2} \\ &= \frac{c_1^{2n} - c_1^{n+1}c_2^{n-1} - c_2^{n+1}c_1^{n-1} + c_2^{2n}}{(c_1 - c_2)^2} - \frac{c_1^{2n} - 2c_1^n c_2^n + c_2^{2n}}{(c_1 - c_2)^2} \\ &= \frac{\left(c_1 c_2\right)^{n \frac{-c_1}{c_2}}}{(c_1 - c_2)^2} + \frac{\left(c_1 c_2\right)^{n \frac{-c_2}{c_1}}}{(c_1 - c_2)^2} + \frac{2(c_1 c_2)^n}{(c_1 - c_2)^2} \\ & (-1)^n. \end{aligned}$$

(ii)
$$\begin{aligned} & CL_{n+1}(x)CL_{n-1}(x) - CL_n^2(x) \\ &= \left(c_1^{n+1} + c_2^{n+1}\right)\left(c_1^{n-1} + c_2^{n-1}\right) - \left(c_1^n + c_2^n\right)\left(c_1^n + c_2^n\right) \\ &= c_1^{2n} + c_1^{n+1}c_2^{n-1} + c_1^{n-1}c_2^{n+1} + c_2^{2n} - c_1^{2n} - 2c_1^n c_2^n - c_2^{2n} \\ &= \left(c_1 c_2\right)^n \left(-\frac{c_1}{c_2} - \frac{c_2}{c_1} + 2\right) \\ &= 4\left(4x^2 + 1\right)(-1)^{n-1} \end{aligned}$$

Other proofs are shown in a similar way. □

Applications

This section includes two applications of the Copper Fibonacci and Copper Lucas polynomials. The first of these applications is the Catalan transformation of these polynomials, and the second is the hyperbolic quaternions of these polynomials.

Catalan transformation

In this section, we apply the Catalan transformation to these polynomials and find their first few terms. In addition, we obtain some properties of the Catalan transformation of these polynomials. Moreover, the Hankel transform is applied to the Catalan transformation of these polynomials. Moreover, very interesting results are obtained with the terms of the Hankel transformation.

In Barry (2005), for $n \in \mathbb{N}^+$, the Catalan numbers and the generating function of the Catalan numbers are

$$C_n = \frac{C(2n, n)}{n+1} \text{ and } c(t) = \frac{1 - \sqrt{1 - 4t}}{2t}.$$

Using the definition of the Catalan transformation, the Catalan transformation of the Copper Fibonacci and Copper Lucas polynomials are defined as follows:

$$CCF_n(x) = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} CF_i(x), n \in \mathbb{N}^+ \text{ with } CCF_0(x) = 0$$

and

$$CCL_n(x) = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} CL_i(x), n \in \mathbb{N}^+ \text{ with } CCL_0(x) = 0.$$

Next, the first five values of $CCF_n(x)$, and $CCL_n(x)$ are given below:

$$\begin{aligned} CCF_0(x) &= 0, CCF_1(x) = 1, CCF_2(x) \\ &= 4x + 1, CCF_3(x) = 16x^2 + 8x + 3, \end{aligned}$$

$$\begin{aligned} CCF_4(x) &= 64x^3 + 48x^2 + 28x + 8, CCF_5(x) \\ &= 256x^4 + 256x^3 + 192x^2 + 88x + 24, \end{aligned}$$

and

$$CCL_0(x) = 0, CCL_1(x) = 4x, CCL_2(x) = 16x^2 + 4x + 2,$$

$$\begin{aligned} CCL_3(x) &= 64x^3 + 32x^2 + 20x + 4, CCL_4(x) \\ &= 256x^4 + 192x^3 + 144x^2 + 56x + 12, \end{aligned}$$

$$CCL_5(x) = 1024x^5 + 1024x^4 + 896x^3 + 480x^2 + 184x + 36.$$

We can write $CCF_n(x)$ and $CCL_n(x)$ as the product of $nx1$ type $CF_n(x)$ and $CL_n(x)$'s of the lower triangular Catalan matrix C:

$$\begin{bmatrix} CCF_1(x) \\ CCF_2(x) \\ CCF_3(x) \\ CCF_4(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \dots \\ 1 & 1 & & & \dots \\ 2 & 2 & 1 & & \dots \\ 5 & 5 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} CF_1(x) \\ CF_2(x) \\ CF_3(x) \\ CF_4(x) \\ \vdots \end{bmatrix} \text{ and } \begin{bmatrix} CCL_1(x) \\ CCL_2(x) \\ CCL_3(x) \\ CCL_4(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \dots \\ 1 & 1 & & & \dots \\ 2 & 2 & 1 & & \dots \\ 5 & 5 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} CL_1(x) \\ CL_2(x) \\ CL_3(x) \\ CL_4(x) \\ \vdots \end{bmatrix}.$$

So,

$$\begin{bmatrix} 1 \\ 4x+1 \\ 16x^2+8x+3 \\ 64x^3+48x^2+28x+8 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \dots \\ 1 & 1 & & & \dots \\ 2 & 2 & 1 & & \dots \\ 5 & 5 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 4x \\ 16x^2+1 \\ 64x^3+8x \\ \vdots \end{bmatrix} \text{ and}$$

$$H_n = det \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \\ v_2 & v_3 & \vdots & v_n & \vdots \\ v_3 & \vdots & v_n & \vdots & \vdots \\ \vdots & v_n & \vdots & \vdots & \vdots \\ v_n & \cdots & \cdots & \cdots & v_{2n-1} \end{bmatrix}.$$

Here $\{v_1, v_2, v_3, \dots\}$ are the elements of the sequence.

$$\begin{bmatrix} 4x \\ 16x^2+4x+2 \\ 64x^3+32x^2+20x+4 \\ 256x^4+192x^3+144x^2+56x+12 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \dots \\ 1 & 1 & & & \dots \\ 2 & 2 & 1 & & \dots \\ 5 & 5 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 4x \\ 16x^2+2 \\ 64x^3+12x \\ 256x^4+64x^2+2 \\ \vdots \end{bmatrix}.$$

Next, we obtain the generating functions of the Catalan transformation of these polynomials.

Theorem 3.1.1. *We get*

$(i)\ F(t) = \frac{1-\sqrt{1-4t}}{1+2t-4x+(1-4t)\sqrt{1-4t}}$,

$(ii)\ L(t) = \frac{-2-4\sqrt{1-4t}}{1+2t-4x+(1-4t)\sqrt{1-4t}}$.

Proof. (i)
The following equations are used for proof:

$$F(t) = CCF_n(t) = CCF_n(t * c(t)).$$

Thus, we obtain

$$F(t) = \frac{1-\sqrt{1-4t}}{1+2t-4x+(1-4t)\sqrt{1-4t}}.$$

Other proof is shown in a similar way. □

In Layman (2001), for $n \in \mathbb{N}^+$, the Hankel transform H_n of the terms of this sequence was defined as follows:

Next, we apply Hankel's work to the Catalan Copper Lucas $CCL_n(x)$ polynomials and Copper Fibonacci $CCF_n(x)$ polynomials, respectively. We obtain

$$HCCL_1(x) = H_1 = det\left[CCL_1(x)\right] = det[4x] = 4x = CL_1(x),$$

$$HCCL_2(x) = H_2 = det\begin{bmatrix} CCL_1(x) & CCL_2(x) \\ CCL_2(x) & CCL_3(x) \end{bmatrix}$$

$$= det\begin{bmatrix} 4x & 16x^2+4x+2 \\ 16x^2+4x+2 & 64x^3+32x^2+20x+4 \end{bmatrix} = -4 = -2CL_1(x),$$

$$HCCL_3(x) = H_3 = det\begin{bmatrix} CCL_1(x) & CCL_2(x) & CCL_3(x) \\ CCL_2(x) & CCL_3(x) & CCL_4(x) \\ CCL_3(x) & CCL_4(x) & CCL_5(x) \end{bmatrix}$$

$$= -64x^3 - 64x^2 - 32x - 16$$

$$= -CL_3(x) - 4CL_2(x) - 5CL_1(x) - 2CL_0(x),$$

and

$$HCCF_1(x) = det\left[CCF_1(x)\right] = det[1] = 1,$$

$$HCCF_2(x) = det\begin{bmatrix} CCF_1(x) & CCF_2(x) \\ CCF_2(x) & CCF_3(x) \end{bmatrix}$$

$$= det\begin{bmatrix} 1 & 4x+1 \\ 4x+1 & 16x^2+8x+3 \end{bmatrix} = 2,$$

$$HCCF_3(x) = det\begin{bmatrix} CCF_1(x) & CCF_2(x) & CCF_3(x) \\ CCF_2(x) & CCF_3(x) & CCF_4(x) \\ CCF_3(x) & CCF_4(x) & CCF_5(x) \end{bmatrix}$$

$$= det\begin{bmatrix} 1 & 4x+1 & 16x^2+8x+3 \\ 4x+1 & 16x^2+8x+3 & 64x^3+48x^2+28x+8 \\ 16x^2+8x+3 & 64x^3+48x^2+28x+8 & 256x^4+256x^3+192x^2+102x+15 \end{bmatrix}$$

$$= 5,$$

$$HCCF_4(x) = det\begin{bmatrix} CCF_1(x) & CCF_2(x) & CCF_3(x) & CCF_4(x) \\ CCF_2(x) & CCF_3(x) & CCF_4(x) & CCF_5(x) \\ CCF_3(x) & CCF_4(x) & CCF_5(x) & CCF_6(x) \\ CCF_4(x) & CCF_5(x) & CCF_6(x) & CCF_7(x) \end{bmatrix} = 13.$$

In the next theorem, we obtain important feature.

Theorem 3.1.2. *For $n \in \mathbb{N}^+$, When the Hankel transformation is applied to the Catalan transformation of Copper Fibonacci polynomials, the relation that relates the results obtained to the classical Fibonacci sequence is as follows*:

$$HCCF_n(x) = F_{2n-1}.$$

Proof. $HCCF_n(x) \neq 0$ and let us write it as $HCCF_n(x) = detK_n detM_n$. Here, the properties of K_n and M_n matrices are as follows.

K_n is the *nxn* type lower triangular matrix, the first column $\{CCF_1(x), CCF_2(x), CCF_3(x), \dots\}$, and the matrix with the principal diagonal $\{1, 1, 1, \dots\}$. M_n the *nxn* type upper triangular matrix, the first row $\{CCF_1(x), CCF_2(x), CCF_3(x), \dots\}$, and is the matrix with the principal diagonal $\{1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \frac{89}{34}, \dots\}$. Thus, we obtain

$$HCCF_n(x) = det\begin{bmatrix} CCF_1(x) & 0 & 0 & 0 & 0 \\ CCF_2(x) & 1 & 0 & 0 & 0 \\ CCF_3(x) & \dots & 1 & 0 & 0 \\ CCF_4(x) & \dots & \cdots & 1 & 0 \\ \dots & \dots & \dots & \dots & 1 \end{bmatrix} x det\begin{bmatrix} CCF_1(x) & CCF_2(x) & CCF_3(x) & CCF_4(x) & \dots \\ 0 & 2 & \ddots & \vdots & \dots \\ 0 & 0 & \frac{5}{2} & \dots & \dots \\ 0 & 0 & 0 & \frac{13}{5} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Thus, we obtain

$$\begin{aligned} HCCF_n(x) = detK_n detM_n = detM_n &= \{m_{11}m_{22}m_{33}m_{44}\dots m_{nn}\} \\ &= \{1, 2, 5, 13, 34, 89, \dots\} \\ &= F_{2n-1}. \end{aligned}$$

□

Hyperbolic Copper Fibonacci and Copper Lucas quaternions

In this section, we introduce the application of these polynomials hyperbolic to quaternions. For these quaternions, we define hyperbolic Copper Fibonacci and Copper Lucas quaternions. Moreover, we obtain many features like Binet formulas. At the end of this section, the terms of these polynomials are related with to their hyperbolic quaternion values.

Definition 3.2.1. For $n \in \mathbb{N}$, the hyperbolic Copper Fibonacci $\overset{\vee}{Q}CF_n(x)$ and hyperbolic Copper Lucas $\overset{\vee}{Q}CL_n(x)$ quaternions are defined by,

$$\begin{aligned} \overset{\vee}{Q}CF_n(x) &= CF_n(x)i_1 + CF_{n+1}(x)i_2 + CF_{n+2}(x)i_3 + CF_{n+3}(x) \\ &= \big(CF_n(x), CF_{n+1}(x), CF_{n+2}(x), CF_{n+3}(x)\big) \end{aligned}$$

and

$$\begin{aligned} \overset{\vee}{Q}CL_n(x) &= CL_n(x)i_1 + CL_{n+1}(x)i_2 + CL_{n+2}(x)i_3 + CL_{n+3}(x)i_4 \\ &= \big(CL_n(x), CL_{n+1}(x), CL_{n+2}(x), CL_{n+3}(x)\big) \end{aligned}$$

where $CF_n(x)$ is n^{th} Copper Fibonacci polynomials, $CL_n(x)$ is n^{th} Copper Lucas polynomials, and i_1, i_2, i_3 and i_4 are the quaternion units in Table 1.

The first two values of $\overset{\vee}{Q}CF_n(x)$ and $\overset{\vee}{Q}CL_n(x)$ are as follows, respectively,

- $\overset{\vee}{H}CF_0(x) = i_2 + 4xi_3 + \big(16x^2 + 1\big)i_4,$
- $\overset{\vee}{H}CF_1(x) = i_1 + 4xi_2 + \big(16x^2 + 1\big)i_3 + \big(64x^3 + 8x\big)i_4,$
- $\overset{\vee}{H}CL_0(x) = 2i_1 + 4xi_2 + \big(16x^2 + 2\big)i_3 + \big(64x^3 + 12x\big)i_4,$
- $\overset{\vee}{H}CL_1(x) = 4xi_1 + \big(16x^2 + 2\big)i_2 + \big(64x^3 + 12x\big)i_3 + \big(256x^4 + 64x^2 + 2\big)i_4,$

Definition 3.2.2. For $n \in \mathbb{N}$, the norms of the hyperbolic Copper Fibonacci $\|\overset{\vee}{Q}CF_n(x)\|$ and hyperbolic Copper Lucas $\|\overset{\vee}{Q}CL_n(x)\|$ quaternions are defined by,

$$\left\|\overset{\vee}{Q}CF_n(x)\right\| = \sqrt{CF_n^2(x) + CF_{n+1}^2(x) + CF_{n+2}^2(x) + CF_{n+3}^2(x)}$$

and

$$\left\|\overset{\vee}{Q}CL_n(x)\right\| = \sqrt{CL_n^2(x) + CL_{n+1}^2(x) + CL_{n+2}^2(x) + CL_{n+3}^2(x)}.$$

Definition 3.2.3. For $n \in \mathbb{N}$, the conjugate of hyperbolic Copper Fibonacci $\overset{\vee}{Q}CF_n^*(x)$ and Copper Lucas $\overset{\vee}{Q}CL_n^*(x)$ quaternions are defined by,

$$\begin{aligned} \overset{\vee}{Q}CF_n^*(x) &= CF_n(x)i_1 - CF_{n+1}(x)i_2 - CF_{n+2}(x)i_3 - CF_{n+3}(x)i_4 \\ &= \big(CF_n(x), -CF_{n+1}(x), -CF_{n+2}(x), -CF_{n+3}(x)\big), \end{aligned}$$

and

$$\overset{\vee}{Q}CL_n^*(x) = CL_n(x)i_1 - CL_{n+1}(x)i_2 - CL_{n+2}(x)i_3 - CL_{n+3}(x)i_4 = \big(CL_n(x), -CL_{n+1}(x), -CL_{n+2}(x), -CL_{n+3}(x)\big).$$

Next, we find the relations among the $\overset{\vee}{Q}CF_n$, $\overset{\vee}{Q}CF_n^*$, $\overset{\vee}{Q}CL_n$, and $\overset{\vee}{Q}CL_n^*$.

Theorem 3.2.1. *Let $n \in \mathbb{N}$. We obtain*

$(i)\ \overset{\vee}{Q}CF_{n+2}(x) = 4x\overset{\vee}{Q}CF_{n+1}(x) + \overset{\vee}{Q}CF_n(x),$

$(ii)\ \overset{\vee}{Q}CL_{n+2}(x) = 4x\overset{\vee}{Q}CL_{n+1}(x) + \overset{\vee}{Q}CL_n(x),$

$(iii)\ \overset{\vee}{Q}CF_{n+2}^*(x) = 4x\overset{\vee}{Q}CF_{n+1}^*(x) + \overset{\vee}{Q}CF_n^*(x),$

$(iv)\ \overset{\vee}{Q}CL_{n+2}^*(x) = 4x\overset{\vee}{Q}CL_{n+1}^*(x) + \overset{\vee}{Q}CL_n^*(x).$

Proof. (i) If the definition is used, we get

$$4x\overset{\vee}{Q}CF_{n+1}(x) + \overset{\vee}{Q}CF_n(x) = 4x\big(CF_{n+1}(x)i_1 + CF_{n+2}(x)i_2 + CF_{n+3}(x)i_3 + CF_{n+4}(x)i_4\big) + \big(CF_n(x)i_1 + CF_{n+1}(x)i_2 + CF_{n+2}(x)i_3 + CF_{n+3}(x)i_4\big) = \big(4xCF_{n+1}(x) + CF_n(x)\big)i_1 + \big(4xCF_{n+2}(x) + CF_{n+1}(x)\big)i_2 + \big(4xCF_{n+3}(x) + CF_{n+2}(x)\big)i_3 + \big(4xCF_{n+4}(x) + CF_{n+3}(x)\big)i_4.$$

Since, $CF_{n+2}(x) = 4xCF_{n+1}(x) + CF_n(x)$. Thus, we obtain

$$\overset{\vee}{Q}CF_{n+2}(x) = 4x\overset{\vee}{Q}CF_{n+1}(x) + \overset{\vee}{Q}CF_n(x).$$

Other proofs are shown in a similar way. □

Theorem 3.2.2. *We obtain*

$(i)\ \overset{\vee}{Q}CF_n(x) + \overset{\vee}{Q}CF_n^*(x) = 2CF_n(x)i_1,$

$(ii)\ \overset{\vee}{Q}CF_n^2(x) = 2CF_n(x)\overset{\vee}{Q}CF_n(x) + \left(\overset{\vee}{Q}CF_n(x)^2 - 2CF_n^2(x)\right)i_1,$

$(iii)\ \overset{\vee}{Q}CL_n(x) + \overset{\vee}{Q}CL_n^*(x) = 2CL_n(x)i_1$

$(iv)\ \overset{\vee}{Q}CL_n^2(x) = 2CL_n(x)\overset{\vee}{Q}CL_n(x) + \left(\overset{\vee}{Q}CL_n(x)^2 - 2CL_n^2(x)\right)i_1.$

Proof. (ii) If the definition and properties of quaternions are used, we obtain

Thus, we obtain

$$\overset{\vee}{Q}CF_n^2 = 2CF_n\overset{\vee}{Q}CF_n + \left(\overset{\vee}{Q}CF_n^2 - 2CF_n^2\right)i_1.$$

Other proofs are shown in a similar way. □

Theorem 3.2.3. *We obtain*

$(i)\ i_1\overset{\vee}{Q}CF_n(x) - i_2\overset{\vee}{Q}CF_{n+1}(x) - i_3\overset{\vee}{Q}CF_{n+2}(x) - i_4\overset{\vee}{Q}CF_{n+3}(x) = 4x\big(CF_{n+5}(x) - CF_{n+1}(x)\big)i_1$

$(ii)\ i_1\overset{\vee}{Q}CL_n(x) - i_2\overset{\vee}{Q}CL_{n+1}(x) - i_3\overset{\vee}{Q}CL_{n+2}(x) - i_4\overset{\vee}{Q}CL_{n+3}(x) = \big(16x^2CL_{n+4}(x) + (4x-1)CL_{n+2}(x) + CL_{n+1}(x) + CL_n(x)\big)i_1.$

Proof. (i) If the definition and properties of quaternions are used, we get

$$\begin{aligned} &i_1\overset{\vee}{Q}CF_n(x) - i_2\overset{\vee}{Q}CF_{n+1}(x) - i_3\overset{\vee}{Q}CF_{n+2}(x) - i_4\overset{\vee}{Q}CF_{n+3}(x)\\ &= \big(CF_n(x) - CF_{n+2}(x) - CF_{n+4}(x) + CF_{n+6}(x)\big)i_1\\ &+ \big(CF_{n+1}(x) - CF_{n+1}(x) - CF_{n+5}(x) + CF_{n+5}(x)\big)i_2\\ &+ \big(CF_{n+2}(x) + CF_{n+4}(x) - CF_{n+2}(x) - CF_{n+4}(x)\big)i_3\\ &+ \big(CF_{n+3}(x) - CF_{n+3}(x) + CF_{n+3}(x) - CF_{n+3}(x)\big)i_4\\ &= \big(CF_n(x) - CF_{n+2}(x) - CF_{n+4}(x) + CF_{n+6}(x)\big)i_1\\ &= 4x\big(CF_{n+5}(x) - CF_{n+1}(x)\big)i_1. \end{aligned}$$

Other proof is shown in a similar way. □

Next, we express the Binet formulas of the $\overset{\vee}{Q}CF_n$, $\overset{\vee}{Q}CF_n^*$, $\overset{\vee}{Q}CL_n$, and $\overset{\vee}{Q}CL_n^*$ quaternions.

Theorem 3.2.4. *Let $n \in \mathbb{N}$. We have*

$$\begin{aligned} \overset{\vee}{Q}CF_n^2 &= \big(CF_ni_1 + CF_{n+1}i_2 + CF_{n+2}i_3 + CF_{n+3}i_4\big)\big(CF_ni_1 + CF_{n+1}i_2 + CF_{n+2}i_3 + CF_{n+3}i_4\big)\\ &= \big(CF_n^2 + CF_{n+1}^2 + CF_{n+2}^2 + CF_{n+3}^2\big)i_1 + \big(2CF_nCF_{n+1}\big)i_2 + \big(2CF_nCF_{n+2}\big)i_3 + \big(2CF_nCF_{n+3}\big)i_4\\ &= 2CF_n\big(CF_ni_1 + CF_{n+1}i_2 + CF_{n+2}i_3 + CF_{n+3}i_4\big) + \big(-CF_n^2 + CF_{n+1}^2 + CF_{n+2}^2 + CF_{n+3}^2\big)i_1 \end{aligned}$$

$(i)\ \overset{\vee}{Q}\,CF_n(x) = \frac{\overline{c_1}c_1^n - \overline{c_2}c_2^n}{c_1 - c_2}$,

$(ii)\ \overset{\vee}{Q}\,CL_n(x) = \overline{c_1}c_1^n + \overline{c_2}c_2^n$,

$(iii)\ \overset{\vee}{Q}\,CF_n^*(x) = \frac{\overline{c_1}^*c_1^n - \overline{c_2}^*c_2^n}{c_1 - c_2}$,

$(iv)\ \overset{\vee}{Q}\,CL_n^*(x) = \overline{c_1}^*c_1^n + \overline{c_2}^*c_2^n$, *where*

$\overline{c_1} = i_1 + c_1 i_2 + c_1^2 i_3 + c_1^3 i_4 = (1, c_1, c_1^2, c_1^3)$, $\overline{c_1}^* = (1, -c_1, -c_1^2, -c_1^3)$,

$\overline{c_2} = i_1 + c_2 i_2 + c_2^2 i_3 + c_2^3 i_4 = \left(1, c_2, c_2^2, c_2^3\right)$ and
$\overline{c_2}^* = \left(1, -c_2, -c_2^2, -c_2^3\right)$.

Proof. (i) The following general form is used to find equality:

$$\overset{\vee}{Q}\,CF_n(x) = Tc_1^n + Sc_2^n.$$

T and S values are obtained by solving the equation obtained by giving different values to the above equation.

$$\overset{\vee}{Q}\,CF_0(x) = i_2 + 4xi_3 + \left(16x^2 + 1\right)i_4 = T + S$$

and

$$\overset{\vee}{Q}\,CF_1(x) = i_1 + 4xi_2 + \left(16x^2 + 1\right)i_3 + \left(64x^3 + 8x\right)i_4 = Tc_1 + Sc_2.$$

Thus, we have

$$x = \frac{\overset{\vee}{Q}\,CF_1(x) - c_2\,\overset{\vee}{Q}\,CF_0(x)}{c_1 - c_2} = \frac{i_1 + c_1 i_2 + c_1^2 i_3 + c_1^3 i_4}{c_1 - c_2} = \frac{\overline{c_1}}{c_1 - c_2}$$

$$y = \frac{\overset{\vee}{Q}\,CF_1(x) - c_1\,\overset{\vee}{Q}\,CF_0(x)}{-c_1 + c_2} = \frac{i_1 + c_2 i_2 + c_2^2 i_3 + c_2^3 i_4}{-\left(c_1 - c_2\right)} = \frac{-\overline{c_2}}{c_1 - c_2}.$$

So, we obtain

$$\overset{\vee}{Q}\,CF_n(x) = \frac{\overline{c_1}c_1^n - \overline{c_2}c_2^n}{c_1 - c_2}.$$

Other proofs are shown in a similar way.

□

Next theorems, are the Copper Fibonacci and Copper Lucas polynomials with hyperbolic quaternion values.

Theorem 3.2.5. *Let $a, b \in \mathbb{N}$, and $a < b$. We get*

$(i)\ \overset{\vee}{Q}\,CF_{a+2b}(x) = \frac{CF_{2b}(x)}{4x}\,\overset{\vee}{Q}\,CL_{a+1}(x) - \frac{CL_{2b-1}(x)}{4x}\,\overset{\vee}{Q}\,CF_a(x)$,

$(ii)\ \overset{\vee}{Q}\,CL_{a+2b}(x) = \left(2\sqrt{4x^2+1}\right)\frac{CF_{2b}(x)}{4x}\,\overset{\vee}{Q}\,CF_{a+1}(x) - \frac{CL_{2b-1}(x)}{4x}\,\overset{\vee}{Q}\,CL_a(x)$,

$(iii)\ \overset{\vee}{Q}\,CF_{a+2b}(x) = \frac{CF_{2b}(x)}{4x}\,\overset{\vee}{Q}\,CF_{a+2}(x) - \frac{CL_{2b-2}(x)}{4x}\,\overset{\vee}{Q}\,CF_a(x)$,

$(iv)\ \overset{\vee}{Q}\,CL_{a+2b}(x) = \frac{CF_{2b}(x)}{4x}\,\overset{\vee}{Q}\,CL_{a+2}(x) - \frac{CF_{2b-2}(x)}{4x}\,\overset{\vee}{Q}\,CL_a(x)$.

Proof. Binet formulas are used for proofs. We get.

$$\text{(iv)} = \frac{CF_{2b}(x)}{4}\,\overset{\vee}{Q}\,CL_{a+2}(x) - \frac{CF_{2b-2}(x)}{4}\,\overset{\vee}{Q}\,CL_a(x)$$
$$= \frac{1}{4x}\frac{c_1^{2b} - c_2^{2b}}{c_1 - c_2}\left(\overline{c_1}c_1^{a+2} + \overline{c_2}c_2^{a+2}\right) - \frac{1}{4x}\frac{c_1^{2b-2} - c_2^{2b-2}}{c_1 - c_2}\left(\overline{c_1}c_1^{a} + \overline{c_2}c_2^{a}\right)$$
$$= \frac{\overline{c_1}c_1^{a+2b}\left(c_1^2 - \frac{1}{c_1^2}\right) - \overline{c_2}c_2^{a+2b}\left(c_2^2 - \frac{1}{c_2^2}\right)}{4x\left(c_1 - c_2\right)}.$$

Since, $c_2^2 = \frac{1}{c_1^2}$, $c_1^2 = \frac{1}{c_2^2}$, and $c_1 + c_2 = 4x$. Thus, we obtain

$$\overset{\vee}{Q}\,CL_{a+2b}(x) = \frac{CF_{2b}(x)}{4x}\,\overset{\vee}{Q}\,CL_{a+2}(x) - \frac{CF_{2b-2}(x)}{4x}\,\overset{\vee}{Q}\,CL_a(x).$$

Other proofs are shown in a similar way. □

Theorem 3.2.6. *Let $a, b, c, d \in \mathbb{N}$ and $c < a, b$. We have.*

$(i)\ \overset{\vee}{Q}\,CF_{a+b}(x) = CF_a(x)\,\overset{\vee}{Q}\,CF_{b+1}(x) + CF_{a-1}(x)\,\overset{\vee}{Q}\,CF_b(x)$,

$(ii)\ \overset{\vee}{Q}\,CF_{2a+b}(x) = CL_a(x)\,\overset{\vee}{Q}\,CF_{a+b}(x) - (-1)^a\,\overset{\vee}{Q}\,CF_b(x)$,

$(iii)\ \overset{\vee}{Q}\,CF_{ac+d}(x) = \frac{CF_{ac}(x)}{CF_a(x)}\,\overset{\vee}{Q}\,CF_{a+d}(x) - (-1)^a\frac{CF_{ac-a}(x)}{CF_a(x)}\,\overset{\vee}{Q}\,CF_d(x)$,

$(iv)\ (-1)^{ac}CF_{ab-ac}(x)\,\overset{\vee}{Q}\,CF_d(x) = \overset{\vee}{Q}\,CF_{ac+d}(x)CF_{ab}(x) - \overset{\vee}{Q}\,CF_{ab+d}(x)CF_{ac}(x)$,

Proof. Binet formulas are used for proofs. We obtain

$$\text{(i)} = CF_a\,\overset{\vee}{Q}\,CF_{b+1} + CF_{a-1}\,\overset{\vee}{Q}\,CF_b$$
$$= \left(\frac{c_1^a - c_2^a}{c_1 - c_2}\right)\left(\frac{\overline{c_1}c_1^{b+1} - \overline{c_2}c_2^{b+1}}{c_1 - c_2}\right) + \left(\frac{c_1^{a-1} - c_2^{a-1}}{c_1 - c_2}\right)\left(\frac{\overline{c_1}c_1^{b} - \overline{c_2}c_2^{b}}{c_1 - c_2}\right)$$
$$= \frac{1}{\left(c_1 - c_2\right)\left(c_1 - c_2\right)}\left[\overline{c_1}c_1^{a+b}\left(c_1 + \frac{1}{c_1}\right) + \overline{c_2}c_2^{a+b}\left(c_2 + \frac{1}{c_2}\right)\right].$$

Since, $\frac{1}{c_1} = -c_2$ and $\frac{1}{c_2} = -c_1$. Thus, we obtain

$$\overset{\vee}{Q}\,CF_{a+b}(x) = CF_a(x)\,\overset{\vee}{Q}\,CF_{b+1}(x) + CF_{a-1}(x)\,\overset{\vee}{Q}\,CF_b(x).$$

Other proofs are shown in a similar way. □

Conclusions

In this study, we defined the Copper Fibonacci and Copper Lucas polynomials and gave some terms for them. Then we presented some properties like the special summation formulas, Binet formulas, etc. Also, we found special relations between these polynomials and the roots of the characteristic equations of these polynomials. We obtained the relationship between the ratio of the biggest to the smallest of two consecutive terms of these polynomials. In addition, special relations between the positive and negative index terms of these polynomials are given, and it is seen that the sum of the squares of two consecutive terms is related to these polynomials. We calculated the important identities of these polynomials.

Moreover, we find the relationships among the Fibonacci and Lucas sequences of the polynomials. We applied Catalan transformations to the polynomials and obtained some of their terms. We found some features of the Catalan transformations like generating functions. We applied the Hankel transform to the Catalan transform to the polynomials. Furthermore, with an interesting feature, we related the terms of the Hankel transformation of the Catalan copper Fibonacci polynomials to the classical Fibonacci numbers. Finally, we defined the application of the Copper Fibonacci and Copper Lucas polynomials to hyperbolic quaternions. For these hyperbolic quaternions, we obtained many properties like Binet formulas. We found the relationships between these quaternions. We related Copper Fibonacci and Copper Lucas polynomials with hyperbolic quaternion values.

Acknowledgements The authors would like to thank the referees who carefully read the article and made valuable comments.

Funding This study received no support.

Data availability The data supporting the findings of this study are available from the corresponding authors [Hakan Akkuş and Engin Özkan] upon reasonable request.

Declarations

Conflict of interest The authors have no conflict of interest.

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