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(p,q) −Mersenne Numbers and Their Basic Properties

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Abstract

This study focuses on the fundamental properties and algebraic structures of Mersenne numbers. By deriving recurrence relations, Binet formulas, and both ordinary and exponential generating functions, we analyze the behavior of this number sequence in depth. Furthermore, we establish classical identities such as Catalan, Cassini, and d'Ocagne for Mersenne numbers. For these generalized forms, we provide exact formulas and verify structural identities.

Keywords: Mersenne numbers, Generalized Mersenne numbers, Binet formula, Generating function, Special sequences.

1. Introduction

The most well-known of integer sequence is Fibonacci sequence. Many researchers are dedicated to Fibonacci sequence such as Hoggart and Koshy also many researchers investigated the polynomials of integer sequences [7]. Horadam [6] presented a generalization of Fibonacci sequence. Recently, Yazlik and Taskara [13] defined a generalization $\{H_{k,n}\}$ of the special second-order sequences such as Fibonacci, Lucas, k-Fibonacci, k-Lucas, generalized k- Fibonacci and k-Lucas, Horadam, Pell, Jacobsthal, and Jacobsthal-Lucas sequences. Another famous number sequences are Fermat numbers. In 17^{th} century French mathematician Pierre de Fermat presented a new family of integer sequence which are shown

by $F_n = 2^n + 1$, named after he died. Tsang et al. investigated Fermat numbers [14]. Some researcher investigated the connection between Fibonacci numbers and Fermat numbers [1,4]. Later in 17^{th} another French mathematician Marin Mersenne presented Mersenne numbers which satisfies $M_n = 2^n - 1$. Also, many researchers have investigated the properties of Mersenne numbers [2,9,12,5]. Mersenne numbers are important in various branches of mathematics as especially number theory and combinatory. Mersenne numbers M_n by the second order recurrence relation that

$$M_{n+1} = 2M_n + 1 \tag{1}$$

With initial conditions $M_0 = 0\,, M_1 = 1$

Since this recurrence relation is not homogeneous, we substitute n by $n + 1$. Thus, we get the new form of Mersenne numbers as follows

$$M_{n+2} = 2M_{n+1} + 1. \tag{2}$$

Subtracting (1) to (2) we obtain

$M_{n+2} - M_{n+1} = 2M_{n+1} + 1 - (2M_n + 1)$

and

$$M_{n+2} = 3M_{n+1} - 2M_n$$

The characteristic equation of Mersenne sequence as follows,

$r^2 - 3r + 2 = 0$

and the roots are $r_1 = 1,\ r_2 = 2$. One can easily obtain the Binet formula of Mersenne numbers as $2^n - 1$.

Solak and Liu presented the generalized *k*- Horadam sequence formula [10,8], $\{H_{k,n}\}$, formula for any integer $k \geq 1$

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$$

$$H_{k,0} = a, H_{k,1} = b,$$

where $f(k)$ and $g(k)$ are scalar polynomials, $f(k)^2 + g(k)^2 > 0$ and with the initial conditions $H_{k,0} = a, H_{k,1} = b$.

Taking $f(k) = 1, g(k) = 1$ we obtain the Fibonacci sequence as follows,

$$F_{n+2} = F_{n+1} + F_n$$

$$F_0 = 0, F_1 = 1,$$

Suvarbamani and Tatang [11] introduced (p, q) −Fibonacci numbers as follows,

$$F_{n+2}(p, q) = pF_{n+1}(p, q) + qF_n(p, q)$$

$p \geq 1, q \geq 1, n \geq 2$, $F_0(p, q) = 0$, $F_1(p, q) = 1$ and $p^2 + 4q > 0$.

Dişkaya and Menken [3] presented n −dimensional reccurence of (p, q) −Fibonacci numbers Inspired by these studies we encourage to study (p, q) −Mersenne numbers. We present (p, q) −Mersenne numbers in following definition.

2. (p, q) −Mersenne numbers

Definition 1. These numbers are denoted by $\{M_u(p, q)\}$ and defined as follows,

$$M_{n+2}(p, q) = 3pM_{n+1}(p, q) - 2qM_n(p, q) \quad (3)$$

$n \geq 2$, with initial conditions $M_0(p, q) = 0, M_1(p, q) = 1$.

Where p and q are nonzero real numbers such that $9p^2 - 8q > 0$. -

Taking $p = 1, q = 1$ we obtain Mersenne numbers.

$$M_2(p, q) = 3p, M_3(p, q) = 9p^2 - 2q, M_4(p, q) = 27p^3 - 12q, M_5(p, q) = 81p^4 - 54p^2q + 4q^2$$

The characteristic equation of (3) and its roots as follows,

$$\delta^2 - 3p\delta + 2q = 0 \quad (4)$$

$$\alpha(p, q) = \frac{3p - \sqrt{9p^2 - 8q}}{2} \quad , \quad \beta(p, q) = \frac{3p + \sqrt{9p^2 - 8q}}{2}$$

We can easily see that

$$\alpha(p,q).\beta(p,q) = 2q$$

Now we give some important equation of (p,q) −Mersenne numbers including Binet formula, generating function, exponential generating function, Catalan, Cassini, d'Ocagne's Identities and summation formulas.

Theorem 2. Binet formula of (p,q) −Mersenne numbers holds the following relation,

$$M_n(p,q) = \frac{\alpha^n(p,q)-\beta^n(p,q)}{\sqrt{9p^2-8q}} \tag{5}$$

where

$$\alpha(p,q) = \frac{3p-\sqrt{9p^2-8q}}{2} \quad , \quad \beta(p,q) = \frac{3p+\sqrt{9p^2-8q}}{2}$$

Proof.

$$M_n(p,q) = c_1\alpha^n(p,q) + c_2\beta^n(p,q)$$

For $n = 0$ we have

$$M_0(p,q) = c_1 + c_2 = 0$$

For $n = 1$ we have

$$M_1(p,q) = c_1\alpha(p,q) + c_2\beta(p,q) = 1,$$

If this system of equations is solved, then the coefficients , c_1, c_2 are found as

$$c_1 = \frac{1}{\sqrt{9p^2-8q}} \text{ , } c_2 = -\frac{1}{\sqrt{9p^2-8q}}$$

So, we can find Binet formula as follows,

$$M_n(p,q) = \frac{\alpha^n(p,q)-\beta^n(p,q)}{\sqrt{9p^2-8q}}$$

where $\alpha(p,q) = \frac{3p-\sqrt{9p^2-8q}}{2}$, $\beta(p,q) = \frac{3p+\sqrt{9p^2-8q}}{2}$

Theorem 3. The generating function of $\{M_u(p,q)\}$ as follows,

$$GM_u(p,q)(t)(t) = \frac{t}{1-3pt+2qt^2}$$

Proof.

$$GM_u(p,q)(t) = M_0(p,q) + M_1(p,q)t + M_2(p,q)t^2 + \cdots + M_m(p,q)t^m + \cdots$$

$$-3ptGM_u(p,q)(t) = -3pM_0(p,q)t - 3pM_1(p,q)t^2 - 3pM_2(p,q)t^3 + \cdots + -3pM_m(p,q)t^{m+1} + \cdots$$

$$2qt^2GM_u(p,q)(t) = 2qM_0(p,q)t^2 + 2qM_1(p,q)t^3 + 2qM_2(p,q)t^4 + \cdots + 2qM_m(p,q)t^{m+2} + \cdots$$

Summation the both side of equations,

$$\begin{aligned}(1-3pt+2qt^2)\,GM_u(p,q)(t) &= M_0(p,q) + (M_1(p,q) - 3pBM_0)t + (M_2(p,q) - 3pM_1(p,q) + 2qBM_0)t^2 + \cdots \\ &+ (M_m(p,q) - 3pM_{m-1}(p,q) + 2qM_{m-2}(p,q)) + \cdots\end{aligned}$$

Using (7) and initial conditions we get

$$(1-3pt+2qt^2)M_u(p,q) = M_0(p,q) + (M_1(p,q) - 3pM_0(p,q))t$$

$$GM_u(p,q)(t) = \frac{M_0(p,q) + (M_1(p,q) - 3pM_0(p,q))t}{1-3pt+2qt^2}$$

$$GM_u(p,q)(t) = \frac{t}{1-3pt+2qt^2}$$

Theorem 4. Exponential generating function of $\{M_u(p,q)\}$ as follows,

$$EM_u(p,q)(t) = c_1e^{\alpha(p,q)t} + c_2e^{\beta(p,q)t}$$

where

$$c_1 = \frac{1}{\sqrt{9p^2-8q}}, c_2 = -\frac{1}{\sqrt{9p^2-8q}}, \alpha(p,q) = \frac{3p-\sqrt{9p^2-8q}}{2}, \beta(p,q) = \frac{3p+\sqrt{9p^2-8q}}{2}$$

Proof. We know that

$$e^t = \sum_{u=0}^{\infty} \frac{t^u}{u!} \tag{6}$$

To find the exponential generating function of $\{M_u(p,q)\}$, we use (5)-(6)

$EM_u(p,q)(t) = \sum_{u=0}^{\infty} M_u(p,q)\frac{t^u}{u!}$

$$EM_u(p,q)(t) = \sum_{u=0}^{\infty} (c_1\alpha^u(p,q) + c_2\beta^u(p,q))\frac{t^u}{u!}$$

$$EM_u(p,q)(t) = \sum_{u=0}^{\infty} c_1\alpha^u(p,q)\frac{t^u}{u!} + \sum_{u=0}^{\infty} c_2\beta^u(p,q)\frac{t^u}{u!}$$

we get

$$EM_u(p,q)(t) = c_1 e^{\alpha(p,q)t} + c_2 e^{\beta(p,q)t}$$

where

$$c_1 = \frac{1}{\sqrt{9p^2-8q}}, c_2 = -\frac{1}{\sqrt{9p^2-8q}},$$

$$\alpha(p,q) = \frac{3p-\sqrt{9p^2-8q}}{2}, \beta(p,q) = \frac{3p+\sqrt{9p^2-8q}}{2} \tag{7}$$

Theorem 5. Catalan identity of $\{M_u(p,q)\}$ as follows,

For $u \geq v$,

$$M_{u-v}(p,q)M_{u+v}(p,q) - M_u(p,q)^2 = \frac{-(2q)^{u-v}(\beta^v(p,q) - \alpha^v(p,q))^2}{9p^2-8q}$$

Proof. Using (5)

$$M_{u-v}(p,q)M_{u+v}(p,q) - M_u(p,q)^2$$
$$= \frac{\alpha^{u-v}(p,q) - \beta^{u-v}(p,q)}{\alpha(p,q) - \beta(p,q)} \cdot \frac{\alpha^{u+v}(p,q) - \beta^{u+v}(p,q)}{\alpha(p,q) - \beta(p,q)}$$
$$- \frac{\alpha^{u}(p,q) - \beta^{u}(p,q)}{\alpha(p,q) - \beta(p,q)} \cdot \frac{\alpha^{u}(p,q) - \beta^{u}(p,q)}{\alpha(p,q) - \beta(p,q)} = \frac{-(2q)^{u-v}(\beta^{v}(p,q) - \alpha^{v}(p,q))^2}{9p^2 - 8q}$$

As we desired.

Corollary 6. Taking $v = 1$ in Catalan Identity we get Cassini identity as it follows,

$$M_{u-1}(p,q)M_{u+1}(p,q) - {M_u}^2(p,q) = -(2q)^{u-1}$$

Theorem 7. D'ocagne Identity of $\{M_u(p,q)\}$ as follows,

$$M_u(p,q)M_{v+1}(p,q) - M_{u+1}(p,q)M_v(p,q) = \frac{(2q)^v(\alpha^{u-v}(p,q) - \beta^{u-v}(p,q))}{\sqrt{9p^2 - 8q}}$$

Proof.

$$M_u(p,q)M_{v+1}(p,q) - M_{u+1}(p,q)M_v(p,q)$$
$$= \frac{\alpha^{u}(p,q) - \beta^{u}(p,q)}{\alpha(p,q) - \beta(p,q)} \cdot \frac{\alpha^{v+1}(p,q) - \beta^{v+1}(p,q)}{\alpha(p,q) - \beta(p,q)}$$
$$- \frac{\alpha^{u+1}(p,q) - \beta^{u+1}(p,q)}{\alpha(p,q) - \beta(p,q)} \cdot \frac{\alpha^{v}(p,q) - \beta^{v}(p,q)}{\alpha(p,q) - \beta(p,q)}$$

$$= \frac{\alpha^{u}(p,q)\beta^{v}(p,q)(\alpha(p,q)-\beta(p,q)) - \beta^{u}(p,q)\alpha^{v}(p,q)(\beta(p,q)-\alpha(p,q))}{(\alpha(p,q)-\beta(p,q))^2}$$

$$= \frac{(\alpha(p,q) - \beta(p,q))(\alpha^{u}(p,q)\beta^{v}(p,q) - \beta^{u}(p,q)\alpha^{v}(p,q))}{(\alpha(p,q) - \beta(p,q))^2}$$

$$= \frac{\alpha^{u}(p,q)\beta^{v}(p,q) - \beta^{u}(p,q)\alpha^{v}(p,q)}{\sqrt{9p^2 - 8q}}$$

For any integer $k \geq 1$, the generalized *k*- Horadam sequence $\{H_{k,n}\}$ summation formula [14] is given by

(1) $If\ f(k) + g(k) \neq 1$, then

$$\sum_{k=0}^{n} H_k = \frac{H_{k,n+1}+g(k)H_{k,n}+f(k).H_0 -H_0 -H_1}{f(k)+g(k)-1} \quad (8)$$

For $f(k)+g(k)=1$

$$\sum_{k=0}^{n} H_k = \frac{g(k)H_n+n[g(k).H_0+H_1] -H_0}{g(k)+1} \quad (9)$$

Theorem 8. Summation formula of $\{M_u(p,q)\}$ as follows,

(1) $3p-2q \neq 1$

$$\sum_{k=0}^{n} M_k(p,q) = \frac{M_{n+1}(p,q) - 2qM_n(p,q) - 1}{3p-2q-1}$$

(2) $3p-2q=1$

$$\sum_{k=0}^{n} M_k(p,q) = -\frac{-2qM_n(p,q)+n}{2q+1}$$

Proof. Taking $H_n = M_n(p,q), f(k) = 3p, g(k) = -2q$ in (8) and (9), we easily get the summation formula of $(p,q)-$ Mersenne numbers.

3. Conclusion

In this work, we explored Mersenne numbers from a structural and functional perspective. We derived recurrence relations, the Binet formula, and generating functions—both ordinary and exponential—laying the foundation for further analytical developments. We established classical identities such as Catalan, Cassini, and d'Ocagne's for the Mersenne sequence. These results demonstrate the richness of the Mersenne number framework and its potential for application in abstract algebra, number theory, and mathematical physics.

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