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New Recurrence Relations for Tribonacci Sequence and Generalized Tribonacci Sequence

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Abstract

Tribonacci numbers are placed on the corners of polygons clockwise with each corner receiving a term. Then, it is argued whether there is a relationship among the numbers in the same corners. It is shown to be able to find m^{th} term corresponding to the corner A_k in an n-gon by a relation:

$$T_{(m-1)n+k} = T_{(m-2)n+k}C_n + T_{(m-3)n+k}S_n + T_{(m-4)n+k}$$

where, T_n is a Tribonaccisequence and C_n is a Tribonacci- Lucas sequence. The relation is proved by using a binet formula obtained for S_n. It is also argued whether the relation corrects all Generalized Tribonacci sequences by changing the beginning terms. Moreover, it is shown that m^{th} term corresponding to the corner A_k in an n-gon can be found by using the following relation:

$$V_{(m-1)n+k} = V_{(m-2)n+k}C_n + V_{(m-3)n+k}S_n + V_{(m-4)n+k}$$

Keywords: *Binet formula, Fibonacci Sequences, n-gon, Tribonacci Numbers, Tribonacci-Lucas Numbers*

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INTRODUCTION

The Tribonacci sequence, a natural generalization of the Fibonacci sequence, has been studied extensively by many authors [1–3, 11]. Tribonacci numbers were first studied by Feinberg in 1963 [4]. In this study, the author approached the Tribonacci numbers coined the term "Silver Ratio" to name the two consecutive Tribonacci numbers. Since the promotion of Generalized Fibonacci sequence, the approach to solving the Tribonacci sequence has been analog to solving Fibonacci sequence [4, 5].

TanackovIlija introduced a detailed procedure for determining the formula for calculation Tribonacci sequence numbers with arbitrary initial numbers $T_{a,b,c}(n)$. Tan and Wen considered the factor properties of the Tribonacci sequence and then gave the singular factorization and the Lyndon factorization. Tana and Wenb [6] studied the powers of the factors and the overlap of the factors. Özkan, Aydın and Dikici [7], proved two theorems concerning wall number of the Tribonacci sequences. Özkan gave 3-step Fibonacci sequences by the three generating elements of a group of exponent *p* (*p* is prime) and nilpotency class *n* [8].

In this study, we place the terms of the Tribonacci sequence on the corners of polygons clockwise with each corner receiving a term with each cycle. By doing this, we attempt to observe whether there is a link among the numbers on each corner. After observing that the result is positive, we attempt to provide a proof for the link. We have obtained a new link by implementing the method that was used for the Tribonacci sequence and use it for the Generalized Tribonacci sequence. Finally, we attempt to prove whether this new link works.

MATERIALS AND METHODS

Definition: The Tribonacci sequence is defined by

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$T_n = T_{n-1} + T_{n-2} + T_{n-3}, n \geq 3$
with $T_0 = 0, T_1 = 0$ and $T_2 = 1$.

A few terms of this sequence can be given as $0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, ...$

Multitudinous number sequences can be originated by changing the original terms of Tribonacci sequence. One of these originated sequences is the Tribonacci-Lucas sequence.

Definition: The Tribonacci-Lucas sequence is defined by
$C_n = C_{n-1} + C_{n-2} + C_{n-3}, n \geq 3$
with $C_0 = 3, C_1 = 1$ and $C_2 = 3$.

The first few terms of this sequence can be given as $3, 1, 3, 7, 11, ...$

The characteristic equation for both the Tribonacci and the Tribonacci-Lucas sequences is $x^3 - x^2 - x - 1 = 0$. This characteristic equation has three roots: one real root and two complex roots. These roots are as follows:

$$\alpha = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right)$$

$$\beta = \frac{1}{6}\left(2 + (-1 - i\sqrt{3})\sqrt[3]{19 - 3\sqrt{33}} + i(\sqrt{3} + i)\sqrt[3]{19 + 3\sqrt{33}}\right)$$

and

$$\theta = \frac{1}{6}\left(2 + i(\sqrt{3} + i)\sqrt[3]{19 - 3\sqrt{33}} + (-1 - i\sqrt{3})\sqrt[3]{19 + 3\sqrt{33}}\right)$$

The Binnet formulas for Tribonacci and Tribonacci-Lucas numbers are as respectively follows [9]:

$$T_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \theta)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \theta)} + \frac{\theta^n}{(\theta - \alpha)(\theta - \beta)}$$

and
$C_n = \alpha^n + \beta^n + \theta^n$.

RESULTS AND DISCUSSION

New Recurrence Relations for the Tribonacci Sequences

We place the Tribonacci numbers on the corners of a triangle clockwise as shown in the Figure 1. We can find the relation among the numbers on any corner of the triangle by multiplying the three terms, except for the first three terms, respectively by 7, -5, and 1 then, adding them. Indeed, if we are to apply the numbers in corner A at Figure 1 then we have $274.7 + 44.(-5) + 7.1 = 1705$.

We do the same operations for a quadrangle. To reach a term in a quadrangle, we multiply the previous three terms, except for the first three terms, by 11, 5, 1, respectively and add them. If we are to apply the terms in corner A at Figure 2 then we get
$274.11 + 24.5 + 2.1 = 3136$.

If we also apply these operations for other polygons, a line, and a single point, and present the results then we obtain ordered triples in Table 1. In Table 1, we write the numbers that we use for each polygon as ordered triples.

If we take the first components of ordered triples in Table 1, {1, 3, 7, 11, 21, 39, 75, ... }, one can recognize that they are the terms of the Tribonacci-Lucas sequence. If we examine Table 1 carefully, it is corresponded that $C_3 = 7$ for 3-gon, $C_4 = 11$ for 4-gon and similarly C_n for n-gon.

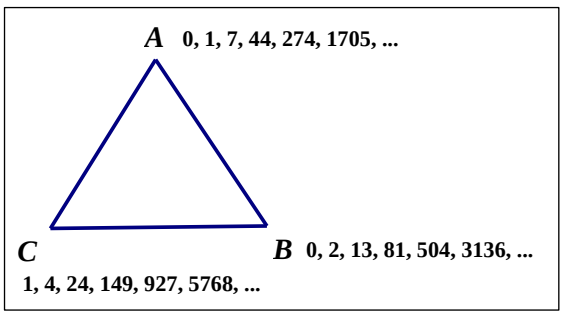


Fig. 1: *The Tribonacci Terms on the Corners of a Triangle Clockwise.*

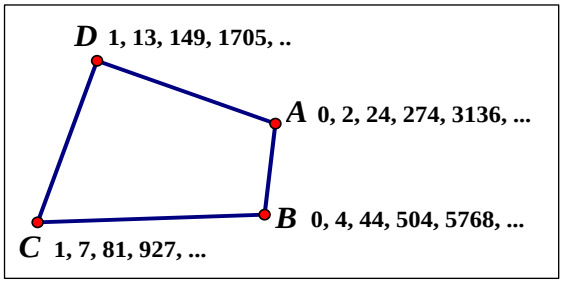


Fig. 2: *The Tribonacci Terms on the Corners of a Quadrilateral Clockwise.*

Table 1: *Ordered Triples for the Polygon.*

Point	Line	Triangle	Quadrilateral	Pentagon	Hexagon	Heptagon
(1, 1, 1)	(3, 1, 1)	(7, -5, 1)	(11, 5, 1)	(21, 1, 1)	(39, -11, 1)	(75, 15, 1)

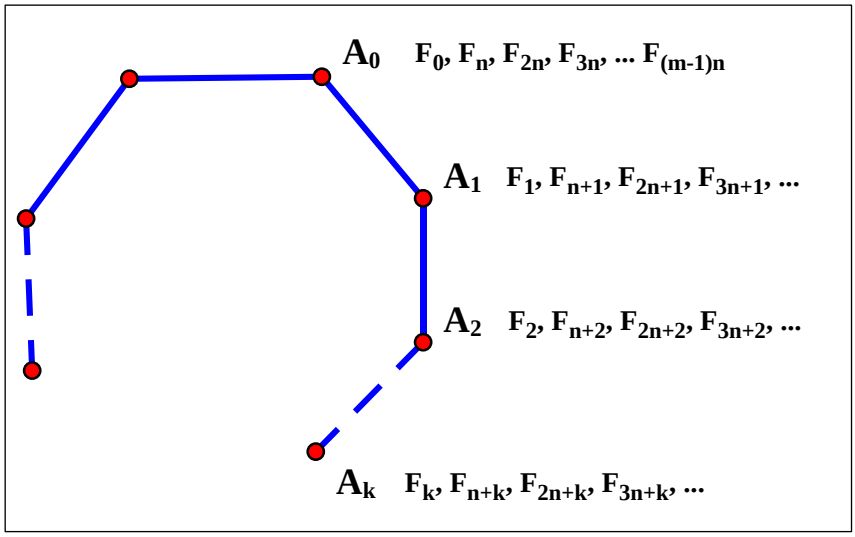


Fig. 3: *The Tribonacci Terms on the Corners of a n-gon Clockwise.*

The sequence formed by the second components are {1, 1, -5, 5, 1, -11, 15, -3, ... }. If we name the terms of the sequence as S_n, we notice that there is a recurrence relation such that $S_n = -S_{n-1} - S_{n-2} + S_{n-3}$ among the terms of the sequence [10].

Like the Tribonacci-Lucas sequence, the term used in an n-gon in this sequence is also S_n. Examining the third components, we can conclude that it is always 1 for all polygons. Next, we apply our relation to an *n*-gon. To do this application, we name the corners of an *n*-gon as $A_0, A_1, A_2, A_3, \dots, A_k$ in Figure 3.

Thus, the m^{th} term that corresponds to the corner A_0 in an n-gon can be found with the relation
$T_{(m-1)n} = T_{(m-2)n}C_n + T_{(m-3)n}S_n + T_{(m-4)n}$
where the m^{th} term in the corner A_0 is $T_{(m-1)\,n}$.

Lemma: For the S_n sequence, the Binet formula is $S_n = -\gamma^n - \delta^n - \varphi^n$ where γ, δ, φ are roots of third-degree equation of $x^3 + x^2 + x - 1 = 0$.

Proof: To form the Binet formula for the sequence, we are required to form the characteristic equation for the sequence.
Make the necessary calculations in the equation $S_n = -S_{n-1} - S_{n-2} + S_{n-3}$ by replacing S_n with x^n.
$x^n = -x^{n-1} - x^{n-2} + x^{n-3}$, if each term here is abbreviated with the term x^{n-3};
$x^3 = -x^2 - x + 1$ the third-degree equation of $x^3 + x^2 + x - 1 = 0$ is obtained.

The roots of the third-degree equation are

$$\gamma = -\frac{1}{3} + \frac{1}{3}\left(\sqrt[3]{17 + 3\sqrt{33}} - \frac{2}{\sqrt[3]{\frac{17+3\sqrt{33}}{27}}}\right)$$

$$\beta = -\frac{1}{3} + \frac{1 + i\sqrt{3}}{3\sqrt[3]{17 + 3\sqrt{33}}} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{17 + 3\sqrt{33}}$$

and

$$\varphi = -\frac{1}{3} + \frac{1 - i\sqrt{3}}{3\sqrt[3]{17 + 3\sqrt{33}}} - \frac{1}{6}(1 + i\sqrt{3})\sqrt[3]{17 + 3\sqrt{33}}$$

Having a third-degree equation, the Binet formula for a sequence is in the format of $S_n = a\gamma^n + b\delta^n + c\varphi^n$. Therefore, we write the starting values of $S_0 = -3, S_1 = 1, S_2 = 1$ in the formula.
$S_n = a\gamma^n + b\delta^n + c\varphi^n$
$a + b + c = -3$
$a\gamma + b\delta + c\varphi = 1$
$a\gamma^2 + b\delta^2 + c\varphi^2 = 1$

If we solve the equation system, we find;

$$a = \frac{1-\varphi-\delta-3\delta\varphi}{(\gamma-\delta)(\gamma-\varphi)} \quad (1)$$

$$b = \frac{1-\varphi-\gamma-3\delta\varphi}{(\delta-\gamma)(\delta-\varphi)} \quad (2)$$

$$c = \frac{-1+\gamma+\delta+3\delta\gamma}{(\varphi-\gamma)(\varphi-\delta)} \quad (3)$$

Being aware that for the third-degree equation of $x^3 + x^2 + x - 1 = 0$
$\gamma + \delta + \varphi = -1$ and $.\,\delta.\varphi = 1$; from here
If we put the equations $\delta + \varphi = -1 - \gamma$ and $\delta.\varphi = \frac{1}{\gamma}$ and place them in (1), we achieve

$$a = \frac{2\gamma + \gamma^2 - 3}{2\gamma^3 + \gamma^2 + 1}$$

To continue with our proof, then, we place the γ root in the equation $x^3 + x^2 + x - 1 = 0$.
$\gamma^3 + \gamma^2 + \gamma - 1 = 0$, we obtain $\gamma^3 = -\gamma^2 - \gamma + 1$.
We write this equation once again in (1).

$$a = \frac{\gamma^2 + 2\gamma - 3}{2(-\gamma^2 - \gamma + 1) + \gamma^2 + 1}$$

$$a = \frac{\gamma^2 + 2\gamma - 3}{-\gamma^2 - 2\gamma + 3}$$

We reach the result of $a = -1$.
If we implement the same calculations for (2) and (3),
we find $b = -1$ and $c = -1$.
As a result, we have achieved the Binet formula of S_n as $S_n = -\gamma^n - \delta^n - \varphi^n$.
Theorem: If we take the m^{th} term coinciding with corner A_k in an n- gon, $n \geq 3$, we have
$T_{(m-1)n+k} = T_{(m-2)n+k}C_n + T_{(m-3)n+k}S_n + T_{(m-4)n+k}$.

***Proof*:** Our relation is $T_{(m-1)n} = T_{(m-2)n}C_n + T_{(m-3)n}S_n + T_{(m-4)n}$. We shall use the right side of the equation to find the left side.
When we write the Binet like formulae in our relation:

$$T_{(m-2)n}C_n + T_{(m-3)n}S_n + T_{(m-4)n} = \left(\frac{\alpha^{(m-2)n}}{A} + \frac{\beta^{(m-2)n}}{B} + \frac{\theta^{(m-2)n}}{C}\right).(\alpha^n + \beta^n + \theta^n)$$
$$+\left(\frac{\alpha^{(m-3)n}}{A} + \frac{\beta^{(m-3)n}}{B} + \frac{\theta^{(m-3)n}}{C}\right).(-\gamma^n - \delta^n - \varphi^n) + \left(\frac{\alpha^{(m-4)n}}{A} + \frac{\beta^{(m-4)n}}{B} + \frac{\theta^{(m-4)n}}{C}\right)$$
$$= \frac{\alpha^{(m-1)n}}{A} + \frac{\beta^{(m-1)n}}{B} + \frac{\theta^{(m-1)n}}{C} + \frac{\alpha^n\beta^{(m-2)n}}{B} + \frac{\alpha^n\theta^{(m-2)n}}{C} + \frac{\beta^n\alpha^{(m-2)n}}{A} + \frac{\beta^n\theta^{(m-2)n}}{C}$$
$$+\frac{\theta^n\alpha^{(m-2)n}}{A} + \frac{\theta^n\beta^{(m-2)n}}{B} - \frac{\gamma^n\alpha^{(m-3)n}}{A} - \frac{\gamma^n\beta^{(m-3)n}}{B} - \frac{\gamma^n\theta^{(m-3)n}}{C} - \frac{\delta^n\alpha^{(m-3)n}}{A}$$
$$-\frac{\delta^n\beta^{(m-3)n}}{B} - \frac{\delta^n\theta^{(m-3)n}}{C} - \frac{\varphi^n\alpha^{(m-3)n}}{A} - \frac{\varphi^n\beta^{(m-3)n}}{B} - \frac{\varphi^n\theta^{(m-3)n}}{C}$$
$$+\frac{\alpha^{(m-4)n}}{A} + \frac{\beta^{(m-4)n}}{B} + \frac{\theta^{(m-4)n}}{C}$$
$$= T_{(m-1)n} + \frac{\alpha^{(m-4)n}}{A}[\alpha^{2n}\beta^n + \alpha^{2n}\theta^n - \alpha^n\gamma^n - \alpha^n\delta^n - \alpha^n\varphi^n + 1]$$
$$+\frac{\beta^{(m-4)n}}{B}[\alpha^n\beta^{2n} + \theta^n\beta^{2n} - \gamma^n\beta^n - \delta^n\beta^n - \varphi^n\beta^n + 1]$$
$$+\frac{\theta^{(m-4)n}}{C}[\theta^{2n}\alpha^n + \theta^{2n}\beta^n - \theta^n\gamma^n - \theta^n\delta^n - \theta^n\varphi^n + 1]$$
$$= T_{(m-1)n} + \frac{\alpha^{(m-4)n}}{A}[\alpha^n(\alpha\beta)^n + \alpha^n(\alpha\theta)^n - (\alpha\gamma)^n - (\alpha\delta)^n - (\alpha\varphi)^n + 1]$$

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$$+\frac{\beta^{(m-4)n}}{B}[\beta^n(\alpha\beta)^n+\beta^n(\theta\beta)^n-(\gamma\delta)^n-(\delta\beta)^n-(\varphi\beta)^n+1]$$
$$+\frac{\theta^{(m-4)n}}{C}[\theta^n(\alpha\theta)^n+\vartheta^n(\theta\beta)^n-(\theta\gamma)^n-(\theta\delta)^n-(\theta\varphi)^n+1]$$

Since $\alpha\gamma=1,\beta\varphi=1,\theta\delta=1,\alpha\beta=\delta,\alpha\theta=\varphi,\beta\theta=\gamma$, we have
$[\alpha^n(\alpha\beta)^n+\alpha^n(\alpha\theta)^n-(\alpha\gamma)^n-(\alpha\delta)^n-(\alpha\varphi)^n+1]=0$
$[\beta^n(\alpha\beta)^n+\beta^n(\theta\beta)^n-(\gamma\delta)^n-(\delta\beta)^n-(\varphi\beta)^n+1]=0$
$[\theta^n(\alpha\theta)^n+\vartheta^n(\theta\beta)^n-(\theta\gamma)^n-(\theta\delta)^n-(\theta\varphi)^n+1]=0$. From here, it is proven that
$T_{(m-2)n}C_n+T_{(m-3)n}S_n+T_{(m-4)n}=T_{(m-1)n}$.

New Recurrence Relations for the Generalized Tribonacci Sequence

In this section, we will define the Generalized Tribonacci sequence. Afterwards, we will implement the method which is used for the Tribonacci sequence in the Generalized Tribonacci sequence to obtain a new relation and attempt to provide proof for the relation.

Definition: The Generalized Tribonacci sequence is defined by
$V_n=T_{n-1}a+(T_{n-1}+T_{n-2})b+T_nc,\ n\geq 3$
with $V_0=a,V_1=b,V_2=c$.
The first few terms as follows:
$$(V_n)=(a,b,c,a+b+c,a+2b+2c,2a+3b+4c,4a+6b+7c,7a+11b+13c,13a+20b+24c,24a+37b+44c,\dots)$$

Theorem: For the m^{th} term coinciding with the corner A_k in an n-gon with the relation
$$V_{(m-1)n+k}=V_{(m-2)n+k}C_n+V_{(m-3)n+k}S_n+V_{(m-4)n+k},\ n\geq 3.$$

***Proof*:**
$$V_{(m-2)n}C_n+V_{(m-3)n}S_n+V_{(m-4)n}$$
$$=\left[T_{(m-2)n-1}.a+\left(T_{(m-2)n-1}+T_{(m-2)n-2}\right)b+T_{(m-2)n}.c\right]C_n$$
$$+\left[T_{(m-3)n-1}.a+\left(T_{(m-3)n-1}+T_{(m-3)n-2}\right)b+T_{(m-3)n}.c\right]S_n$$
$$+\left[T_{(m-4)n-1}.a+\left(T_{(m-4)n-1}+T_{(m-4)n-2}\right)b+T_{(m-4)n}.c\right]$$
$$V_{(m-2)n}C_n+V_{(m-3)n}S_n+V_{(m-4)n}$$
$$=\left[T_{(m-2)n-1}.C_n+T_{(m-3)n-1}S_n+T_{(m-4)n-1}\right]a$$
$$+\left[T_{(m-2)n-1}.C_n+T_{(m-3)n-1}.S_n+T_{(m-4)n-1}\right]b$$
$$+\left[T_{(m-2)n-2}.C_n+T_{(m-3)n-2}.S_n+T_{(m-4)n-2}\right]b$$
$$+\left[T_{(m-2)n}.C_n+T_{(m-3)n}.S_n+T_{(m-4)n}\right]c$$
$$V_{(m-2)n}C_n+V_{(m-3)n}S_n+V_{(m-4)n}=T_{(m-1)n-1}.a+T_{(m-1)n-1}.b+T_{(m-1)n-2}.b+T_{(m-1)n}.c$$
$$V_{(m-2)n}C_n+V_{(m-3)n}S_n+V_{(m-4)n}=V_{(m-1)n}$$
And the proof is completed.
Example:
Finally, we apply the relation for the 4th term in corner A of the triangle like Figure 4.

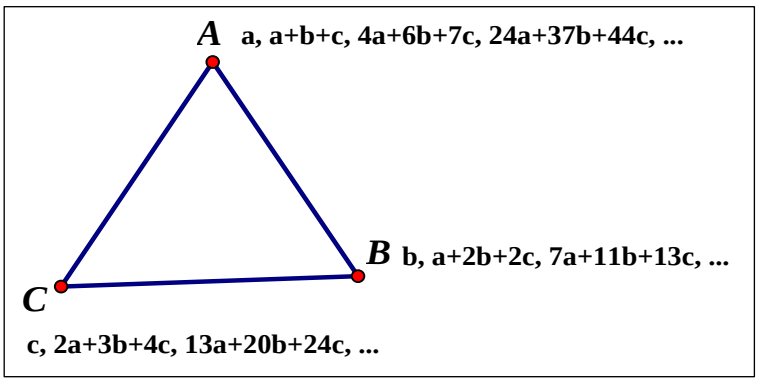


Fig. 4: *The Generalized Tribonacci Terms on the Corners of a Triangle Clockwise.*

Here, $k = 0, n = 3$ and $m = 4$ Therefore;
$V_9 = V_6C_3 + V_3S_3 + V_0$
$V_9 = (4a + 6b + 7c).7 + (a + b + c)(-5) + a$, if we do the necessary calculations, we find that
$V_9 = 24a + 37b + 44c$
Indeed, the 4th term of corner A in the triangle is $24a + 37b + 44c$.
Example:
If we choose $V_0 = 3, V_1 = 1, V_2 = 3$, then $V_n = C_n$. Thus, the m[th] term coinciding with corner A_k in an n-gon becomes

$$C_{(m-1)n+k} = C_{(m-2)n+k}C_n + C_{(m-3)n+k}S_n + C_{(m-4)n+k}$$

CONCLUSION

We placed the terms of the Tribonacci sequence on the corners of polygons clockwise with each corner receiving a term with each cycle. We have obtained a new link by implementing the method that was used for the Tribonacci sequence and use it for the Generalized Tribonacci sequence. Finally, we gave the new relations.

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