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d-Gaussian Pell polynomials and their matrix representation

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We introduce a new generalization of Gaussian Pell polynomials. We call it *d*-Gaussian Pell polynomials. Then we present the sum, generating functions and Binet formulas of these polynomials. We give the matrix representation of *d*-Gaussian Pell polynomials. We introduce the matrix as binary representations according to the Riordan matrix representation. Using the Riordan method, we obtain the factorizations of the Pascal matrix involving the polynomial. In addition, we present the inverse of matrices of the polynomial.

Keywords: *d*-Gaussian Pell polynomials; generating function; Riordan matrix.

Mathematics Subject Classification 2020: 11B39, 11B83

1. General Appearance

Number sequences and their polynomials have attracted the attention of many scientists for many years, as they find application in nature and in many sciences. Of course, the Fibonacci numbers are the best known of the sequences of numbers [1, 6, 7, 13, 14, 24, 25, 27, 28]. Many generalizations of number sequences were then described and studied [2–5, 8, 10, 17, 22, 29]. One of the most important of these generalizations is those about Gaussian [9, 11, 12, 18–21]. Some techniques used in the paper are also available in [15, 16]. One of the latest works in this area is [23]

where it is introduced d-Fibonacci and d-Lucas polynomials. In this paper, we give new generalizations of the Pell polynomials. We find the matrix representations of these polynomials and introduce these matrices as binary representations according to the Riordan matrix representation. Using the Riordan method, we obtain the factorizations of the Pascal matrix including these polynomials. Also, we present the inverse of matrices of these polynomials.

2. Generalization of Gaussian Pell Polynomials

Let us give some basic definitions for this paper. The Pell numbers P_n are defined by

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3$$

with $P_1 = 1$ and $P_2 = 2$ [10].

The Pell polynomials are defined by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x)$$

with $P_0(x) = 0$ and $P_1(x) = 1$ [8].

Definition 2.1. Let $p_i(x)$ be polynomials with a real coefficient for $i = 1, \ldots, d+1$. Then d-Fibonacci polynomials ARE defined by the following recurrence relation:

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x) \tag{2.1}$$

with $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$ [23].

d-Lucas polynomials are defined by the following recurrence relation:

$$L_{n+1}(x) = F_{n+1}(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x) \tag{2.2}$$

with $L_n(x) = 0$ for $n < 0$ and $L_0(x) = 2$ and $L_1(x) = p_1(x)$ [23].

Now, we present a new generalization of Gaussian Pell polynomials.

Definition 2.2. Let $p_i(x)$ be a real coefficient for $i = 1, \ldots, d+1$. Then d-Gaussian Pell polynomials are defined by

$$\mathrm{GP}_n(x) = p_1(x)\mathrm{GP}_{n-1}(x) + p_2(x)\mathrm{GP}_{n-2}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d-1}(x) \tag{2.3}$$

with $\mathrm{GP}_0(x) = i$ and $\mathrm{GP}_n(x) = 0$ for $n < 0$.

We give a few terms of d-Gaussian Pell polynomials in Table 1.

From Eq. (2.3), the characteristic equation of d-Gaussian Pell polynomials are given by

$$r^{d+1} - p_1(x)r^d - p_2(x)r^{d-1} - \cdots - p_{d+1}(x) = 0.$$

The roots of this characteristic equation are $\{\alpha_1(x), \alpha_2(x), \ldots, \alpha_{d+1}(x)\}$.

Table 1. Same values of d-Gaussian Pell polynomials.

n	$\mathrm{GP}_n(x)$
0	i
1	$p_1(x)i$
2	${p_1}^2(x)i + p_2(x)i$
3	${p_1}^3(x)i + 2p_1(x)p_2(x)i + p_3(x)i$
4	${p_1}^4(x)i + 3{p_1}^2(x)p_2(x)i + 2p_1(x)p_3(x)i + {p_2}^2(x)i + p_4(x)i$

Theorem 2.3. *Generating function of* $\mathrm{GP}_n(x)$ *is given as follows*:

$$G(x,t) = \sum_{n=0}^{\infty} \mathrm{GP}_n(x) r^n = \frac{i}{(1 - p_1(x)r - p_2(x)r^2 - \cdots - p_{d+1}(x)r^{d+1})}.$$

Proof. We have

$$G(x,r) = \mathrm{GP}_0(x) + \mathrm{GP}_1(x)r + \mathrm{GP}_2(x)r^2 + \cdots + \mathrm{GP}_n(x)r^n + \cdots . \tag{2.4}$$

Let us multiply Eq. (2.4) by $p_1(x)r, p_2(x)r^2, \ldots, p_{d+1}(x)r^{d+1}$, respectively. So, the following equations are obtained:

$$\begin{aligned}
G(x,t) &= \mathrm{GP}_0(x) + \mathrm{GP}_1(x)r + \mathrm{GP}_2(x)r^2 + \cdots + \mathrm{GP}_n(x)r^n + \cdots ,\\
p_1(x)rG(x,t) &= p_1(x)r\mathrm{GP}_0(x) + p_1(x)r^2\mathrm{GP}_1(x) + p_1(x)r^3\mathrm{GP}_2(x) + \cdots ,\\
p_2(x)r^2G(x,t) &= p_2(x)r^2\mathrm{GP}_0(x) + p_2(x)r^3\mathrm{GP}_1(x) + p_2(x)r^4\mathrm{GP}_2(x) + \cdots ,\\
&\vdots\\
p_{d+1}(x)r^{d+1}G(x,t) &= p_{d+1}(x)\mathrm{GP}_0(x) + p_{d+1}(x)r^{d+2}\mathrm{GP}_1(x)\\
&\quad + p_{d+1}(x)r^{d+3}\mathrm{GP}_2(x) + \cdots .
\end{aligned}$$

If we take the necessary calculations to take advantage of the recurrence relation, we obtain the following equation:

$$\begin{aligned}
&G(x,r)(1 - p_1(x)r - p_2(x)r^2 - \cdots - p_{d+1}(x)r^{d+1})\\
&\quad = \mathrm{GP}_0(x) + \mathrm{GP}_1(x)r - p_1(x)r\mathrm{GP}_0(x)\\
&G(x,r) = \frac{i}{(1 - p_1(x)r - p_2(x)r^2 - \cdots - p_{d+1}(x)r^{d+1})}.
\end{aligned}$$

□

Binet formula of $\mathrm{GP}_n(x)$ has the following form:

$$\mathrm{GP}_n(x) = \sum_{i=1}^{d+1} T_i(x)[\alpha_i(x)]^n.$$

Let us write the following equations for each value of n for the equation:

$$\begin{aligned}
\mathrm{GP}_0(x) &= \sum_{i=1}^{d+1} T_i(x), \\
\mathrm{GP}_1(x) &= \sum_{i=1}^{d+1} T_i(x)\alpha_i(x), \\
\mathrm{GP}_2(x) &= \sum_{i=1}^{d+1} T_i(x)[\alpha_i(x)]^2, \\
&\vdots \\
\mathrm{GP}_n(x) &= \sum_{i=1}^{d+1} T_i(x)[\alpha_i(x)]^n.
\end{aligned}$$

If we multiply both sides of the last three equations by $r, r^2, \ldots, r^n$, respectively, we obtain

$$\begin{aligned}
\mathrm{GP}_0(x) &= \sum_{i=1}^{d+1} T_i(x), \\
r\mathrm{GP}_1(x) &= \sum_{i=1}^{d+1} T_i(x)\alpha_i(x)r, \\
r^2\mathrm{GP}_2(x) &= \sum_{i=1}^{d+1} T_i(x)[\alpha_i(x)]^2 r^2, \\
&\vdots \\
r^n\mathrm{GP}_n(x) &= \sum_{i=1}^{d+1} T_i(x)[\alpha_i(x)]^n r^n.
\end{aligned}$$

If we add up the left side of the equation, we have

$$\frac{i}{(1 - p_1(x)r - p_2(x)r^2 - \cdots - p_{d+1}(x)r^{d+1})}.$$

If we add up the right side of the equation

$$\sum_{i=1}^{d+1} T_i(x)(1 + \alpha_i(x)r + [\alpha_i(x)]^2 r^2 + \cdots) = \sum_{i=1}^{d+1} T_i(x)\left(\frac{1}{1 - \alpha_i(x)r}\right).$$

So, we get the following equation:

$$\frac{i}{(1 - p_1(x)r - p_2(x)r^2 - \cdots - p_{d+1}(x)r^{d+1})} = \sum_{i=1}^{d+1}\left(\frac{T_i(x)}{1 - \alpha_i(x)r}\right).$$

More precisely, the coefficients allow us to give the explicit form of d-Gaussian Pell polynomials. Actually;

Theorem 2.4. *For $n \geq 0$, the following equality is true*:

$$\mathrm{GP}_n(x) = i \sum_{\substack{n_1,n_2,\ldots,n_{d+1} \\ n_1+2n_2+\cdots+(d+1)n_{d+1}=n}} \times \left[\binom{n_1+n_2+\cdots+\ n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} {p_1}^{n_1}(x){p_2}^{n_2}(x)\ldots{p_{d+1}}^{n_{d+1}}(x) \right].$$

Proof. Let us use the generating function for proof:

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{GP}_n(x)r^n &= \frac{i}{(1-p_1(x)r-p_2(x)r^2-\cdots-p_{d+1}(x)r^{d+1})} \\
&= i\sum_{n=0}^{\infty} \left(p_1(x)r+p_2(x)r^2+\cdots+p_{d+1}(x)r^{d+1}\right)^n \\
&= i\sum_{n=0}^{\infty} \left[\sum_{n_1+n_2+\cdots+n_{d+1}=n} \binom{n}{n_1,n_2,\ldots,n_{d+1}} \right. \\
&\quad \left. \times {p_1}^{n_1}(x){p_2}^{n_2}(x)\ldots{p_{d+1}}^{n_{d+1}}(x) \right] r^{n_1+2n_2+\cdots+(d+1)n_{d+1}} \\
&= i\sum_{n=0}^{\infty} \left[\sum_{\substack{n_1,n_2,\ldots,\ n_{d+1} \\ n_1+2n_2+\cdots+(d+1)n_{d+1}=n}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} {p_1}^{n_1}(x) \right. \\
&\quad \left. \times {p_2}^{n_2}(x)\ldots{p_{d+1}}^{n_{d+1}}(x) \right] r^n.
\end{aligned}$$

Thus, the proof is obtained. □

Theorem 2.5. *Let $\mathrm{SGP}_n(x)$ be sum of the d-Gaussian Pell polynomials.Then we have*

$$\mathrm{SGP}_n(x) = \sum_{n=0}^{\infty} \mathrm{GP}_n(x) = \frac{i}{1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x)}.$$

Proof. We get the following equation:

$$\mathrm{SGP}_n(x) = \sum_{n=0}^{\infty} \mathrm{GP}_n(x) = \mathrm{GP}_0(x) + \mathrm{GP}_1(x) + \mathrm{GP}_2(x) + \cdots \mathrm{GP}_n(x) + \cdots .$$

If we multiply the last equation by $p_1(x), p_2(x), \ldots, p_{d+1}(x)$, respectively, then we obtain

$$\begin{aligned}
p_1(x)\ \mathrm{SGP}_n(x) &= p_1(x)\mathrm{GP}_0(x) + p_1(x)\mathrm{GP}_1(x) + \cdots + p_1(x)\mathrm{GP}_n(x) + \cdots, \\
p_2(x)\mathrm{SGP}_n(x) &= p_2(x)\mathrm{GP}_0(x) + p_2(x)\mathrm{GP}_1(x) + \cdots + p_2(x)\mathrm{GP}_n(x) + \cdots, \\
&\vdots \\
p_{d+1}(x)\mathrm{SGP}_n(x) &= p_{d+1}(x)\mathrm{GP}_0(x) + p_{d+1}(x)\mathrm{GP}_1(x) \\
&\quad + \cdots + p_{d+1}(x)\mathrm{GP}_n(x) + \cdots.
\end{aligned}$$

If the necessary mathematical operations are done, we get

$$\mathrm{SGP}_n(x)(1 - p_1(x) - p_2(x) - \cdots - p_{d+1}(x)) = i.$$

Thus, we have

$$\mathrm{SGP}_n(x) = \sum_{n=0}^{\infty} \mathrm{GP}_n(x) = \frac{i}{1 - p_1(x) - p_2(x) - \cdots - p_{d+1}(x)}. \qquad \square$$

Now, we give the d-Gaussian Pell polynomials matrix P_d for $\mathrm{GP}_n(x)$.

Definition 2.6. The d-Gaussian Pell polynomials matrix P_d is given by

$$P_d = Q_d i = \begin{bmatrix} p_1(x)i & p_2(x)i & \cdots & p_{d+1}(x)i \\ i & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & i & 0 \end{bmatrix}, \tag{2.5}$$

where

$$Q_d = \begin{bmatrix} p_1(x) & p_2(x) & \cdots & p_{d+1}(x) \\ 1 & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{bmatrix}. \text{ [23]}$$

Now, we can give matrix representation for $\mathrm{GP}_n(x)$ in the next theorem.

Theorem 2.7. *We have the following representation for* $\mathrm{GP}_n(x)$ *as follows*:

$$P_d^n = \begin{bmatrix} \mathrm{GP}_n(x) & p_2(x)\mathrm{GP}_{n-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d}(x) \\ \mathrm{GP}_{n-1}(x) & p_2(x)\mathrm{GP}_{n-2}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d-1}(x) \\ \vdots & \vdots \\ \mathrm{GP}_{n-d}(x) & p_2(x)\mathrm{GP}_{n-d-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-2d+2}(x) \end{bmatrix}$$
$$\begin{matrix} \cdots & p_{d+1}(x)\mathrm{GP}_{n-1}(x) \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-2}(x) \\ \cdots & \vdots \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-d-1}(x) \end{matrix}\Bigg], \tag{2.6}$$

where $P_d^{n+1} = P_d^n Q_d$.

Proof. To prove the theorem, let us use mathematical induction over n. If we take $n = 1$ on the right side of Eq. (2.6), we get the following matrix:

$$P_d = \begin{bmatrix} p_1(x)i & p_2(x)i & \cdots & p_{d+1}(x)i \\ i & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & i & 0 \end{bmatrix}. \tag{2.7}$$

From the recurrence relation of $\mathrm{GP}_n(x)$, it will be seen that the matrices in (2.5) and (2.7) are equal. Now, assume the matrix (2.6) satisfies for n. That is, we have

$$P_d^n = \begin{bmatrix} \mathrm{GP}_n(x) & p_2(x)\mathrm{GP}_{n-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d}(x) \\ \mathrm{GP}_{n-1}(x) & p_2(x)\mathrm{GP}_{n-2}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d-1}(x) \\ \vdots & \vdots \\ \mathrm{GP}_{n-d}(x) & p_2(x)\mathrm{GP}_{n-d-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-2d+2}(x) \end{bmatrix}$$
$$\begin{matrix} \cdots & p_{d+1}(x)\mathrm{GP}_{n-1}(x) \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-2}(x) \\ \cdots & \vdots \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-d-1}(x) \end{matrix}\Bigg].$$

Let us show that it is true for $n+1$. We know that

$$P_d^{n+1} = P_d^n Q_d$$

$$= \begin{bmatrix} \mathrm{GP}_n(x) & p_2(x)\mathrm{GP}_{n-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d}(x) \\ \mathrm{GP}_{n-1}(x) & p_2(x)\mathrm{GP}_{n-2}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d-1}(x) \\ \vdots & \vdots \\ \mathrm{GP}_{n-d}(x) & p_2(x)\mathrm{GP}_{n-d-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-2d+2}(x) \end{bmatrix}$$

$$\begin{matrix} \cdots & p_{d+1}(x)\mathrm{GP}_{n-1}(x) \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-2}(x) \\ \cdots & \vdots \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-d-1}(x) \end{matrix}\Bigg]$$

$$= \begin{bmatrix} p_1(x) & p_2(x) & \cdots & p_{d+1}(x) \\ 1 & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{GP}_{n+1}(x) & p_2(x)\mathrm{GP}_n(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d+1}(x) \\ \mathrm{GP}_n(x) & p_2(x)\mathrm{GP}_{n-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d}(x) \\ \vdots & \vdots \\ \mathrm{GP}_{n-d+1}(x) & p_2(x)\mathrm{GP}_{n-d}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-2d+1}(x) \end{bmatrix}$$

$$\begin{matrix} \cdots & p_{d+1}(x)\mathrm{GP}_n(x) \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-1}(x) \\ \cdots & \vdots \\ \cdots & p_{d+1}(x)\mathrm{GP}_{n-d}(x) \end{matrix}\Bigg].$$

□

Corollary 2.8. *For $n, m \geq 0$, the following equality is provided*:

$$\mathrm{GP}_{n+m}(x) = \mathrm{GP}_n(x)\mathrm{GP}_m(x) + (p_2(x)\mathrm{GP}_{n-1}(x)\mathrm{GP}_{m-1}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-d}(x)\mathrm{GP}_{m-1}(x)) + \cdots + p_{d+1}(x)\mathrm{GP}_{n-1}(x)\mathrm{GP}_{m-d}(x).$$

Proof. From the product of P_d^n and P_d^m, we get

$$P_d^n P_d^m = P_d^{n+m}.$$

The result is the first row and the first column of matrix P_d^{n+m}. □

Lemma 2.9. *For $n \geq 1$ the following equality is true*:

$$\mathrm{GP}_{n-1}(x) = F_n(x)i.$$

Proof. Let us show the proof inductively over n. For $n = 1$, equality is true because $\mathrm{GP}_0(x) = i = F_1(x)i$. Let the equality be true for $n = k$. Thus,

$$\mathrm{GP}_{k-1}(x) = F_k(x)i.$$

Now, for $n = k + 1$, we have to show that the equation is true. So,

$$\begin{aligned}\mathrm{GP}_k(x) &= p_1(x)\mathrm{GP}_{k-1}(x) + p_2(x)\mathrm{GP}_{k-2}(x) + \cdots + p_{d+1}(x)\mathrm{GP}_{k-d-1}(x)\\ &= p_1(x)F_k(x)i + p_2(x)F_{k-1}(x)i + \cdots + p_{d+1}(x)F_{k-d}(x)i\\ &= F_{k+1}(x)i.\end{aligned}$$

□

Lemma 2.10. *For $n \geq 1$, the following equality is true*:

$$\mathrm{GP}_n(x) = ip_1(x)F_{n-1}(x) + ip_2(x)F_{n-2}(x) + \cdots + ip_{d+1}(x)F_{n-d-1}(x).$$

Proof. The proof is easily done from Lemma 2.9 and Eq. (2.1). □

Lemma 2.11. *For $n \geq 1$, the following equality is true*:

$$\mathrm{GP}_n(x) = F_n(x)ip_1(x) + i(L_n(x) - F_{n+1}(x)).$$

Proof. From Lemma 2.9 and Eq. (2.2),

$$\begin{aligned}\mathrm{GP}_n(x) &= iF_{n+1}(x) = i(L_n(x) - p_2(x)F_{n-1}(x) - \cdots - p_{d+1}(x)F_{n-d}(x))\\ &= iL_n(x) - i(p_2(x)F_{n-1}(x) - \cdots - p_{d+1}(x)F_{n-d}(x)) \text{ from Eq. (2.1)}\\ &= iL_n(x) - i(F_{n+1}(x) - p_1(x)F_n(x))\\ &= F_n(x)ip_1(x) + i(L_n(x) - F_{n+1}(x)).\end{aligned}$$

□

3. The Infinite Pell Polynomials Matrix

Shapiro *et al.* described Riordan matrices and the Riordan group as a set of matrices $M = (m_{ij}),\ i, j \geq 0$ whose elements are complex numbers [26]. The Riordan group was examined by Shapiro *et al.* as a set of infinite lower-triangular integer matrices where each matrix is defined by pair of formal power series

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \quad \text{and} \quad f(z) = \sum_{n=0}^{\infty} f_n z^n$$

with $g_0 \neq 0$ and $f_1 \neq 1$. An infinite lower triangular matrix

$$D = [d_{n,k}]_{n,k\geq 0}$$

is called a Riordan array, if its ith column generating function is $g(x)(f(x))^i$ for $i \geq 0$ where the first column is indexed by 0. Generally, we assume $d_{0,0} = g_0 = 1$.

The matrix corresponding to the pairs g and f is denoted by (g, f). The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f).$$

The d-Gaussian Pell polynomials matrix polynomials are denoted by

$$G\mathcal{P}(x) = [G\mathcal{P}_{p_1,p_2,\ldots,p_{d+1},i,j}(x)]$$

and defined as follow:

$$G\mathcal{P}(x) = \begin{bmatrix} i & 0 & 0 & \cdots \\ p_1(x)i & i & 0 & \cdots \\ (p_1(x))^2 i + p_2(x)i & p_1(x)i & i & \vdots \\ r_1(x) & r_2(x) & p_1(x)i & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= (g_{G\mathcal{P}(x)}(t), f_{G\mathcal{P}(x)}(t)),$$

where $r_1(x) = {p_1}^3(x)i + 2p_1(x)p_2(x)i + p_3(x)i$ and $r_2(x) = {p_1}^2(x)i + p_2(x)i$. We can write the d-Gaussian Pell polynomial matrix as follows:

$$G\mathcal{P}(x) = \begin{bmatrix} \mathrm{GP}_0(x) & 0 & 0 & 0 & \cdots \\ \mathrm{GP}_1(x) & \mathrm{GP}_0(x) & 0 & 0 & \cdots \\ \mathrm{GP}_2(x) & \mathrm{GP}_1(x) & \mathrm{GP}_0(x) & 0 & \cdots \\ \cdots & \cdots & \mathrm{GP}_1(x) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Notice that the matrix $G\mathcal{P}(x)$ is a Riordan matrix.

Theorem 3.1. *The first column of $G\mathcal{P}(x)$ matrix has the form*

$$(i,\ p_1(x)i, {p_1}^2(x)i + p_2(x)i, r_1(x), \ldots)^T.$$

So, the generating function of the first column is as follows:

$$g_{G\mathcal{P}(x)}(t) = \sum_{n=0}^{\infty} G\mathcal{P}_{p_1,p_2,\ldots,p_{d+1},i,j}(x)t^n$$

$$= \frac{i}{(1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1})}.$$

Proof. Let us write the generating function of the first column of $G\mathcal{P}(x)$ matrix as follow:

$$i + \ (p_1(x)i)t + ({p_1}^2(x)i + p_2(x)i)t^2 + \cdots = \mathrm{GP}_0(x) + \mathrm{GP}_1(x)t + \mathrm{GP}_2(x)t^2 + \cdots.$$

From the generating function of $\mathrm{GP}_n(x)$, we have

$$\begin{aligned} G(x,t) &= \mathrm{GP}_0(x) + \mathrm{GP}_1(x)t + \mathrm{GP}_2(x)t^2 + \cdots + \mathrm{GP}_n(x)t^n + \cdots \\ &= \frac{i}{(1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1})}. \end{aligned}$$

Thus, the desired expression is obtained. So,

$$\begin{aligned} g_{G\mathcal{P}(x)}(t) &= \sum_{n=0}^{\infty} G\mathcal{P}_{p_1,p_2,\ldots,p_{d+1},i,j}(x)t^n \\ &= \frac{i}{(1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1})}. \end{aligned}$$

From the Riordan matrix, we have $f_{G\mathcal{P}(x)}(t) = t$,

$$\begin{aligned} G\mathcal{P}(x) &= (g_{G\mathcal{P}(x)}(t), f_{G\mathcal{P}(x)}(t)) \\ &= \left(\frac{i}{1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}, t \right). \end{aligned}$$

□

If the Gaussian Pell polynomial matrix $G\mathcal{P}(x)$ is finite, then the matrix is

$$G\mathcal{P}_f(x) = \begin{bmatrix} \mathrm{GP}_0(x) & 0 & 0 & \cdots \\ \mathrm{GP}_1(x) & \mathrm{GP}_0(x) & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \mathrm{GP}_n(x) & \mathrm{GP}_{n-1}(x) & \cdots & \mathrm{GP}_0(x) \end{bmatrix}$$

and

$$\det G\mathcal{P}_f(x) = |G\mathcal{P}_f(x)| = (\mathrm{GP}_0(x))^n = (i)^n.$$

We present two factorization of Pascal matrix including the d-Gaussian Pell Polynomials matrix. First, we define a matrix $K(x) = \frac{1}{i}(k_{i,j}(x))$ as follows:

$$\begin{aligned} k_{i,j}(x) = &\binom{i-1}{j-1} - p_1(x)\binom{i-2}{j-1} - p_2(x)\binom{i-3}{j-1} \\ &- \cdots - p_{d+1}(x)\binom{i-d-2}{j-1}. \end{aligned}$$

So, we obtain

$$K(x)=\begin{bmatrix} \frac{1}{i} & 0 & 0 & \cdots \\ \frac{1-p_1(x)}{i} & \frac{1}{i} & 0 & \cdots \\ \frac{1-p_1(x)-p_2(x)}{i} & \frac{2-p_1(x)}{i} & \frac{1}{i} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ m_1(x) & m_3(x) & \cdots & \vdots \\ m_2(x) & m_4(x) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{3.1}$$

where

$$m_1(x)=\frac{(1-p_1(x)-p_2(x)-\cdots-p_d(x))}{i},$$
$$m_2(x)=\frac{(d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-p_{d-1}(x))}{i},$$
$$m_3(x)=\frac{(1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x))}{i}$$

and

$$m_4(x)=\frac{((d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-p_d(x))}{i}.$$

By using the infinite d-Gaussian Pell polynomial matrix and the infinite $K(x)$ matrix as in (3.1), we present the first factorization of the infinite Pascal matrix with the following theorem.

Theorem 3.2. *The factorization of the infinite Pascal matrix is as follows*:

$$P(x)=G\mathcal{P}(x)*K(x).$$

Proof. We get the following generating function from the first column of matrix $K(x)$,

$$g_{K(x)}(t)=\frac{1}{i}\left(\frac{1-p_1(x)t-p_2(x)t^2-\cdots-p_{d+1}(x)t^{d+1}}{1-t}\right).$$

From the Riordan matrix definition, we write $f_{K(x)}(t)$ as follows:

$$f_{K(x)}(t)=\frac{t}{1-t}.$$

Then we write matrix $K(x)$ as follow:

$$\begin{aligned} K(x) &= (g_{K(x)}(t), f_{K(x)}(t)) \\ &= \left(\frac{(1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1})}{i(1-t)}, \frac{t}{1-t} \right). \end{aligned}$$

By using the definition of infinite Pascal matrix and the infinite d-Gaussian Pell polynomials matrix, we obtain the Riordan representation as follows:

$$\begin{aligned} P &= \left(\frac{1}{1-t}, \frac{t}{1-t} \right), \\ G\mathcal{P}(x) &= \left(\frac{i}{1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}, t \right). \end{aligned}$$

Finally, we write $K(x)$ and $G\mathcal{P}(x)$ matrices instead of the desired equation by using the definition of Riordan matrix multiplication. Thus, the proof is completed. □

Now, we present other factorization of the Pascal matrix including the d-Gaussian Pell Polynomials matrix. For that, let us define an infinite $L(x) = \frac{1}{i}(l_{i,j}(x))$ as follows:

$$l_{i,j}(x) = \binom{i-1}{j-1} - p_1(x)\binom{i-2}{j} - p_2(x)\binom{i-3}{j+1} - \cdots - p_{d+1}(x)\binom{i-d-2}{j+d}.$$

We give the infinite $L(x)$ by

$$L(x) = \begin{bmatrix} \frac{1}{i} & 0 & 0 & \cdots \\ \frac{1-p_1(x)}{i} & \frac{1}{i} & 0 & \cdots \\ \frac{1-2p_1(x)-p_2(x)}{i} & \frac{2-p_1(x)}{i} & \frac{1}{i} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ n_1(x) & n_3(x) & \cdots & \vdots \\ n_2(x) & n_4(x) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \frac{1}{i}\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1-p_1(x) & 1 & 0 & \cdots \\ 1-2p_1(x)-p_2(x) & 2-p_1(x) & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ n_1(x) & n_3(x) & \cdots & \vdots \\ n_2(x) & n_4(x) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{3.2}$$

where

$$n_1(x) = \frac{\left(1 - dp_1(x) - \frac{d(d-1)}{2!}p_2(x) - \cdots - p_d(x)\right)}{i},$$
$$n_2(x) = \frac{(d-(d-1)p_1(x) - (d-2)p_2(x) - \cdots - p_{d-1}(x))}{i},$$
$$n_3(x) = \frac{\left(1 - (d+1)p_1(x) - \frac{d(d-1)}{2!}p_2(x) - \cdots - p_d(x)\right)}{i}$$

and

$$n_4(x) = \frac{((d+1) - dp_1(x) - (d-1)p_2(x) - \cdots - p_d(x))}{i}.$$

From the infinite d-Gaussian Pell polynomials matrix and the infinite $L(x)$ matrix as in (3.2), we introduce the second factorization of the infinite Pascal matrix with the following theorem.

Theorem 3.3. *The factorization of the infinite Pascal matrix is as follows*:

$$P(x) = G\mathcal{P}(x) * L(x).$$

Proof. The proof is like that of Theorem 3.2. □

Now, we can find the inverse of d-Gaussian Pell polynomials matrix by using from the definition of reverse element Riordan group in [17].

Corollary 3.4. *The inverse of d-Gaussian Pell polynomial is given by the following*:

$$G\mathcal{P}^{-1}(x) = \left(\frac{i}{1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}, t\right).$$

4. Conclusion

New generalized Gaussian Pell polynomials have been introduced and studied. We gave the matrix representations of d-Gaussian Pell polynomials. We also introduced these matrices as binary representations according to the Riordan matrix representation. Using the Riordan method, we found the factorizations of the Pascal matrix involving these polynomials. In addition, we gave the inverse of matrices of these polynomials.

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