EXPLICIT FORMS OF THE TRACE FORMULA FOR GL(2)[1, 2]

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Abstract. There has recently been renewed interest in the trace formula – in particular, that of the initial case of GL(2) – due to counting applications in the function field case. For these applications, one needs a very precise form of the trace formula, with all terms computed explicitly. Our aim in this work is to compute the trace formula for GL(2) over a number field in as full detail as was done for the function field case and to give an accessible exposition, being motivated by these applications to counting, but also by pure curiosity as to the optimal form of this plastic formula. We also explain a correction argument in our context here of GL(2). The idea is to introduce a global summand which does not change the formula globally but changes the local weighted orbital integrals at the hyperbolic terms, so that their limit at the identity becomes a unipotent contribution to the trace formula. This gives a harmonious and pleasing form to the formula. Finally, we put the trace formula in an invariant form; thus all its terms are distributions whose value at a test function $f^y(x) = f(y^{-1}xy)$ is independent of y in $\mathrm{GL}(2, \mathbb{A})$.

Contents

1 Introduction **4**

I Statement of the Trace Formula **7**

2 Notation and definitions **7**

2.1 The field. 7

2.2 The group. 7

2.3 Central characters. 8

2.4 The Weyl group. 8

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2.5 The Bruhat decomposition. 8
2.6 The Iwasawa decomposition. 8
2.7 Haar measures. 9
2.8 Elements in GL(2, F). 10
2.9 Centralizers. 10
2.10 Quadratic extensions. 11
2.11 Test functions. 11
2.12 Orbital integrals. 11
2.13 Convolution operators. 13

3 Induced representations **13**
3.1 Definition of induced representations. 13
3.2 The space H($\mu(s)$). 14
3.3 Sesquilinear pairing. 15

4 Intertwining operators **15**
4.1 Definition of intertwining operators. 15

5 Normalized intertwining operators **17**
5.1 Sections of H(μ_υ). 17
5.2 Functional equation. 18
5.3 Properties of the normalized operator. 20

6 Automorphic forms **21**
6.1 Definition of automorphic forms. 21
6.2 One-dimensional automorphic forms. 21
6.3 Cusp forms. 21
6.4 Discrete spectrum. 24
6.5 Poisson summation formula ([Lang 94]). 25

7 Trace Formula **26**

II Continuous Spectrum **28**

8 B-series **28**
8.1 Siegel domains. 28
8.2 B-automorphic forms. 29
8.3 B-series. 29

9 Mellin transform **31**
9.1 Mellin's inversion formula. 31
9.2 Definition of the Mellin transform. 32

10 Global intertwining operators **35**
10.1 Local-global relations. 35
10.2 The intertwining operator. 35
10.3 Unitarity. 36
10.4 Proof of Theorem 6.2. 36

11 Truncation operator **38**
11.1 Properties of truncation operator. 38

12 Eisenstein series **40**

13 The kernel on the continuous spectrum **43**
13.1 The kernel of $r(fdg)$ in $L^2_{c,\omega}(G)$. 43
13.3 A formula for the kernel. 45

14 The kernel on the cuspidal spectrum **46**

15 The truncated kernel over the entire spectrum **48**

III Computation of the Terms in the Trace Formula **49**

16 The elliptic part of the trace formula **50**

17 The hyperbolic terms in the trace formula **50**

18 The correction **52**
18.2 A limit. 55

19 The unipotent terms in the trace formula **55**
19.1 Computation of unipotent terms. 56

20 The geometric side of the corrected trace formula **59**

21 Contribution from the continuous spectrum **59**
21.1 The term (21.0.1). 61
21.2 The term (21.0.2). 61
21.3 The term (21.0.3). 62

21.6 Completion of the proof of Proposition 21.0.1. 64

IV Invariant Trace Formula **65**

22 Invariant trace formula for GL(2) **65**
22.1 Equivalence classes. 65
22.2 Study of invariance. 65
22.3 Continuous spectrum. 67
22.4 Invariant trace formula. 68

1 Introduction

There has recently been a revival of interest in the trace formula, starting with that of GL(2), following the computation in [DF 13] of the number of rank n local systems with principal unipotent local monodromy, equivalently, certain Galois representations of dimension n, or cuspidal (automorphic) representations of GL(n) unramified outside a set S of cardinality greater than 1 and Steinberg twisted by an unramified character at the places in S, over a function field. This result uses a simple form of the trace formula in the compact quotient case. To remove the restriction $|S| > 1$, i.e., to count all cuspidal nowhere ramified representations of GL(n) and those with $|S| = 1$, one has to use the full trace formula. In the case of $n = 2$ and $|S| = 0$, this was done in the seminal masterpiece [Dr 81] of Drinfeld. In the case of $n = 2$ and $|S| = 1$, this was done in [Fl 15], which details the proof also when S is empty. Both [Dr 81] and [Fl 15] use the complete trace formula for GL(2) over a function field. This trace formula for GL(2) over a function field is given in [Fl 14] in full detail, where the exact computation of all terms is crucial to get a precise count, and verify in [Fl 15] a case of a conjecture made in [DF 13].

The trace formula for GL(2) over a number field was introduced by Selberg and computed in full by Jacquet and Langlands [JL 70] for the group GL(2) over the ring $\mathbb{A}$ of adèles of any global field F. Jacquet and Langlands gave not only a clear idea of the proofs, but also striking new applications to comparison with automorphic representations of the multiplicative group of any quaternion algebra, as well as a comparison of the Tamagawa number of GL(2) with that of its inner forms. The proof in [Fl 14] differs from that in [JL 70]; in particular, a new approach for the study of Eisenstein series is exposed in [Fl 14]. In light of this, to study the relations between the two approaches, we thought is helpful to review [JL 70, Chapter 16] well enough, say, to use for a graduate course.

There are several worthy expositions to the sketch of [JL 70]. The exposition given by Gelbart and Jacquet [GJ 79] is valuable, even though it does not compute the formula completely but only concentrates on the application of vanishing of most terms once the test function has two components coming from the quaternion algebra. Another useful exposition is by Knapp [Kn 97], which concentrates on analytic details in the context of $\mathrm{SL}(2,\mathbb{R})$ and the discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$, and summarizes the adèle case for $F = \mathbb{Q}$; see also Gelbart [Ge 96].

Our aim in this work is to compute the trace formula for GL(2) over a number field in as full detail as in [Fl 14]. We hope the exposition is detailed enough for a graduate course and for a future comparison with [Fl 14]. Further, we need this base case to compare with work of [Fl 16] on the trace formula for reductive groups of any rank. We hope this will help to generalize the results of [Fl 14] from the case $n = 2$ to that of general n. However, it seems easier to prove the trace formula for GL(2) than to deduce it from the higher rank case, which relies on the theory of Eisenstein series for higher rank groups. We concentrate on the computational aspects of the trace formula, rather than the theoretical aspects.

In addition to applications to counting automorphic representations of special kind, working on an explicit closed form of the formula – rather than something most of whose terms vanish for certain test functions – is "curiosity driven" in the sense of Deligne [De 13]. We wonder whether the formula, which does not involve any truncation parameter, can be obtained without using truncation.

All developments of the trace formula beyond [JL 70] are due to Arthur, who extended the formula to the context of a reductive linear algebraic group over a number field; see [Ar 05] for an excellent exposition. In [Ar 78] and [Ar 80], he deals with the question of absolute convergence, which we ignore, and introduces the truncation, which we use. We do not think that reproducing the discussion of absolute convergence in the case of GL(2) will clarify things further, but we hope that computing the terms in the formula will lead to applications.

We also explain the correction argument introduced by the second author in [Fl 82] in our context here of GL(2). The idea is to introduce a global term which does not change the formula globally, but which changes the local weighted orbital integrals at the regular hyperbolic terms so that their limit at the identity becomes the non-invariant unipotent contribution to the trace formula. We then compute the difference of the non-invariant weighted orbital integrals and the logarithmic derivatives in the spectral side of the trace formula at $f^y(x) = f(y^{-1}xy)$ and at f. After applying the Poisson summation formula, we can incorporate the logarithmic derivatives with the non-invariant weighted orbital integrals to create invariant distributions.

The correction argument of [Fl 82] was generalized by Arthur in [Ar 88.1], and the invariant trace formula was developed in [Ar 81], [Ar 88.2], and [Ar 88.3]. In Section 18, we explain the correction; and in Section 22, we obtain the invariant trace formula in our context of GL(2).

As stated above, our aim in this work is to give an accessible exposition of the trace formula for GL(2), being motivated by the applications to the counting of automorphic representations of certain type found in [Dr 81], [DF 13], [Fl 15] and [Fl 17]. Since we do not deal with applications in this work, let us mention some of them here.

The original application of [JL 70, Chapter 16] is a comparison with cuspidal representations of the multiplicative group G' of a quaternion algebra, using test functions f on $\mathrm{GL}(2,\mathbb{A})$ matching those that come from $G'(\mathbb{A})$. Deligne and Kazhdan observed that when dealing only with automorphic representations that have two cuspidal (the local notion of "cuspidal" is defined in [BZ 76]) components, a simple form of the trace formula suffices, and the full trace formula considered here is not needed. A simple proof of this comparison for division algebras of any rank is found in [Fl 90] and, for simple algebras and representations with at least one cuspidal component, in [FK 88]. Thus the full trace formula that we develop here is needed – in the context of this application – only to transfer automorphic representations with no cuspidal components. These are the representations studied in [Dr 81], [DF 13], [Fl 15], and [Fl 17].

Other comparison, or lifting, applications in rank one using the trace formula are the metaplectic correspondence of [Fl 80], base change for GL(2) (a simple and short proof is in [Fl 88]), the symmetric square lifting from SL(2) to PGL(3), and base change for U(3) [Fl 06].

Another application is a comparison of Tamagawa numbers, for GL(2) and its inner forms in [JL 70] (see 16.1.8 and the paragraph following it), and for any reductive connected F-group G and its quasi-split inner form G^* by Kottwitz [Ko 88], proving Weil's conjecture on Tamagawa numbers. Here, one uses the trace formula for G and its quasi-split inner form G^*. With a test function with a cuspidal component, only traces of cuspidal representations are left in the spectral sides of trace formulas. With matching functions, all terms in the difference of the geometric sides of the trace formulae cancel, except for the contribution from the identity multiplied by the Tamagawa number. In other words, a comparison of trace formulae gives that $(\tau(G) - \tau(G^*))f(e)$, which is a continuous sum of traces by the Plancherel formula, equals the discrete sum $\sum_\pi \mathrm{tr}\,\pi(f)$ (minus the analogous sum for G^*). This implies that both sums vanish, and the Tamagawa number $\tau(G)$ of G equals that of G^*. In this application, the term $f(e)$ plays a key

role, so we cannot use a simple form of the trace formula for test functions which vanish on the singular set. The full formula is needed.

Chapter I
Statement of the Trace Formula

In this chapter, we state the trace formula for GL(2) over a number field and gather definitions of terms which appear in the formula.

2 Notation and definitions

2.1 The field. Let F be a global field, i.e., a number field or the field of rational functions on a smooth projective absolutely irreducible curve over a finite field $\mathbb{F}_q$ of q elements. Whereas in this paper we concentrate on the number fields, much of the theory applies to the function fields; for the case of function fields, see [Fl 14]. However, the statement of the trace formula in Section 7 applies only to number fields.

For any place v of F, let F_v be the completion of F at v. We denote by $\mathbb{A}$ the ring of adèles of F. Its multiplicative group $\mathbb{A}^\times$ is the group of idèles of F. Let $F^\times$ and $F_v^\times$ be the multiplicative groups of F and F_v, respectively. When v is non-Archimedean, the ring $\mathcal{O}_v$ of integers in F_v is a local ring. Denote by $\boldsymbol{\pi}_v$ a generator of the maximal ideal in $\mathcal{O}_v$. Let q_v be the cardinality of the residue field $\mathbb{F}_v = \mathcal{O}_v/(\boldsymbol{\pi}_v)$. Denote by $|\cdot|_v$ the absolute value on F_v, normalized by $|\boldsymbol{\pi}_v|_v = q_v^{-1}$. If $F_v = \mathbb{R}$, set $|x|_v = x \cdot \operatorname{sgn} x$. Set $|z|_v = z\bar{z}$ for $z \in F_v = \mathbb{C}$ and $|x| = \prod_v |x_v|_v$ for $x = (x_v) \in \mathbb{A}^\times$. Embed F_v in $\mathbb{A}$ as the component at v and F in $\mathbb{A}$ diagonally. Then $|x| = 1$ for $x \in F^\times$. Let $\mathbb{V}$ be the set of places of F. Embed $\mathbb{R}_{>0}^\times$ in $\mathbb{A}^\times$ by mapping t to the idèle with Archimedean components t and other components 1. Denote the image of this embedding by F_∞^+.

2.2 The group. Let G be the affine group scheme GL(2) over $\mathbb{Z}$. It associates to an algebra $\mathfrak{A}$ the group of invertible 2×2 matrices with entries in $\mathfrak{A}$. Thus $G(F)$ is GL(2, F) and $G(\mathbb{A})$ is GL(2, $\mathbb{A}$). We denote by A the subgroup of diagonal matrices in G and by Z the center of G. Let N be the subgroup of unipotent upper triangular matrices in G. The group $B = AN = NA$ of upper triangular matrices in G is a minimal parabolic subgroup of G. We write G_v for $G(F_v)$, Z_v for $Z(F_v)$, N_v for $N(F_v)$, A_v for $A(F_v)$, B_v for $B(F_v)$, etc. We denote by K_v the standard maximal

compact subgroup of G_υ. Let $K = \prod_\upsilon K_\upsilon \subset G(\mathbb{A})$. If υ is non-Archimedean, then $K_\upsilon = G(\mathcal{O}_\upsilon)$; if F_υ is $\mathbb{R}$, then $K_\upsilon = O(2)$; and if F_υ is $\mathbb{C}$, then $K_\upsilon = U(2)$.

2.3 Central characters. A unitary character ω of $F^\times\backslash\mathbb{A}^\times$ takes values on the unit circle $\mathbb{C}^1$ in $\mathbb{C}^\times$. Denote by ω_υ the restriction of ω to $F_\upsilon^\times$. Then $\omega = \otimes_\upsilon \omega_\upsilon$. It defines a character of $Z(F)\backslash Z(\mathbb{A})$. A character ω_υ of $F_\upsilon^\times$ is called **unramified** if υ is non-Archimedean and $\omega_\upsilon|\mathcal{O}_\upsilon^\times = 1$.

2.4 The Weyl group. A **torus** is an algebraic group isomorphic to $\mathbb{G}_m^k$, where k a positive integer. A torus is F**-split** or **split over** F if it is isomorphic to $\mathbb{G}_m^k$ over F as an algebraic group. Every maximal torus in GL(2) which is not F-split splits over a quadratic field extension E of F. Let T be a maximal torus in G over F. Denote by $N(T)$ the normalizer of T in G. The centralizer of T in G is T. The **Weyl group** $W(T, G)$ of the torus T in G is $N(T)/T$.

2.5 The Bruhat decomposition. The **Bruhat decomposition** of G by B is defined by $G = \cup_{w\in W} BwB = B \cup NwB$, where $W = W(A, G)$ is the Weyl group of A in G. In our case $G = \mathrm{GL}(2)$, the Weyl group is

$$W = \left\{ I, w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Set

$$u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

2.6 The Iwasawa decomposition. The **Iwasawa decomposition** of the group G_υ is defined by $G_\upsilon = A_\upsilon N_\upsilon K_\upsilon$. If $g_\upsilon = a_\upsilon n_\upsilon k_\upsilon$, then a_υ is determined uniquely by g_υ up to $A_\upsilon \cap K_\upsilon$. Hence the **Harish-Chandra** δ-function

$$\delta_\upsilon(g_\upsilon) = \left|\frac{a'}{a''}\right|_\upsilon \text{ if } a_\upsilon = d(a', a''), \text{ where } d(a', a'') = \begin{pmatrix} a' & 0 \\ 0 & a'' \end{pmatrix},$$

is well-defined. Globally, $\delta(g) = \prod_\upsilon \delta_\upsilon(g_\upsilon)$ if $g = (g_\upsilon) \in G(\mathbb{A})$. Almost all factors in the product equal 1.

If $x = (x_\upsilon)$, $y = (y_\upsilon)$, are adèles, define (x, y) to be any adèle with components (x_υ, y_υ), where

$$|(x_\upsilon, y_\upsilon)|_\upsilon = \begin{cases} \max(|x_\upsilon|_\upsilon, |y_\upsilon|_\upsilon) & \text{if } \upsilon \text{ is non-Archimedean,} \\ (|x_\upsilon|_\upsilon^2 + |y_\upsilon|_\upsilon^2)^{1/2} & \text{if } \upsilon \text{ is Archimedean.} \end{cases}$$

Put $|(x,y)| = \prod_v |(x_v,y_v)|_v$. We are interested in the quantity $\ln \delta(\omega u(x))$, which is $-2\ln|(1,x)| = -2\sum_v \ln|(1,x_v)|_v$. Indeed, $\omega u(x_v) = ndk$, where $d = d(1/(1,x_v),(1,x_v))$. If v is non-Archimedean and $|x|_v > 1$ ($x \in F_v$), then $\omega u(x) = u(-x^{-1})\cdot d(-x^{-1},-x)\cdot \omega u(-x^{-1})w^{-1}$. A similar explicit computation applies in the Archimedean case.

2.7 Haar measures. Fix a Haar measure dg_v on $Z_v\backslash G_v$ for all v such that $\prod_v |(Z_v\cap K_v)\backslash K_v|$ converges. The product measure $dg = \otimes_v dg_v$ is a left-invariant measure on $Z(\mathbb{A})\backslash G(\mathbb{A})$ and defines a measure – which we denote by $d\dot{g}$ if we wish to emphasize the quotient – on the homogeneous space $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. Here, we use the atomic measure on discrete sets. Thus, for all $f \in C_c(Z(\mathbb{A})\backslash G(\mathbb{A}))$, setting $\widetilde{f}(g) = \sum_{\gamma\in Z(F)\backslash G(F)} f(\gamma g)$, we have $\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(g)dg = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \widetilde{f}(g)d\dot{g}$. Each dg_v and each dg is unique up to a non-zero scalar multiple. We denote the volume of a set X_v in G_v by $\mathrm{vol}(X_v) = |X_v| = \int_{X_v} dg_v$.

A Haar measure $dx = \otimes_v dx_v$ on $\mathbb{A}$ satisfying $|\mathcal{O}_v|_v = 1$ for almost all v and $|F\backslash\mathbb{A}| = 1$ defines a measure on $N(F)\backslash N(\mathbb{A})$ via the mapping $x \mapsto u(x)$.

Let $\mathbb{A}^1$ be the group of idèles of norm 1 and A^1 be the group of diagonal matrices with entries in $\mathbb{A}^1$. We denote by A_∞^+ (respectively, Z_∞^+) the group of diagonal (respectively, scalar) matrices with entries in F_∞^+. We transfer the usual measure on $\mathbb{R}_{>0}^\times$ to F_∞^+ via the natural isomorphism $\mathbb{R}_{>0}^\times \simeq F_\infty^+$. Let $t \mapsto |t|$ signify $F_\infty^+ \to \mathbb{R}_{>0}^\times$.

Fix a Haar measure $d^\times a$ on $\mathbb{A}^\times$. Give $F^\times\backslash\mathbb{A}^\times$ the quotient measure. We use the isomorphism $F^\times\backslash\mathbb{A}^\times \simeq F_\infty^+ \times F^\times\backslash\mathbb{A}^1$ to get a Haar measure on $F^\times\backslash\mathbb{A}^1$. We also use the isomorphism $d(a) = \left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right) \longmapsto a$ to get a measure on $Z(\mathbb{A})\backslash A(\mathbb{A})$. We select measures dk_v on K_v and dk on K such that for all smooth compactly supported complex valued functions f on $Z(\mathbb{A})\backslash G(\mathbb{A})$,

$$\begin{aligned}\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(g)\,dg &= \int_{N(\mathbb{A})}\int_{Z(\mathbb{A})\backslash A(\mathbb{A})}\int_K f(nak)\delta(a)^{-1}\,dk\,da\,dn\\ &= \int_K\int_{\mathbb{A}^\times}\int_{\mathbb{A}} f(u(x)d(a)k)dx\,\frac{d^\times a}{|a|}\,dk\end{aligned}$$

where $g = nak \in Z(\mathbb{A})\backslash G(\mathbb{A})$, $n \in N(\mathbb{A})$, $a \in Z(\mathbb{A})\backslash A(\mathbb{A})$, and $k \in K$. Note that $\mathrm{vol}(K)$ is not necessarily 1.

Put $\nu_v(x) = |x|_v$. Let μ_{1v} be a character of $F_v^\times$ into $\mathbb{C}^\times$. Suppose that v is non-Archimedean. Define

$$L(\mu_{1v}) = \begin{cases} 1/(1-\mu_{1v}(\boldsymbol{\pi}_v)), & \text{if } \mu_{1v} \text{ is unramified,}\\ 1, & \text{if } \mu_{1v} \text{ is ramified.}\end{cases}$$

Let $L(s,\mu_{1v}) = L(\mu_{1v}\nu_v^s)$.

If $F_v = \mathbb{R}$, $\mu_{1v} : \mathbb{R}^\times \to \mathbb{C}^\times$ is $\mu_{1v}(x) = x^{-N}|x|^s$ with $N \in \{0, 1\}$ and $s \in \mathbb{C}$, let $L(\mu_{1v})$ be $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. If $F_v = \mathbb{C}$, $\mu_{1v} : \mathbb{C}^\times \to \mathbb{C}^\times$ is $\mu_{1v}(z) = z^{-N}|z|^s$, let $L(\mu_{1v})$ be $\Gamma_{\mathbb{C}}(s) = 2 \cdot 2\pi^{-s}\Gamma(s)$.

Let μ_1 be a character of $F^\times\backslash\mathbb{A}^\times$ into $\mathbb{C}^\times$. The restriction μ_{1v} of μ_1 to the subgroup $F_v^\times$ of $\mathbb{A}^\times$ is a character of $F_v^\times$ into $\mathbb{C}^\times$. The global L-function $L(s, \mu_1) = L(\mu_1 \nu^s)$ is defined to be the product of $L(s, \mu_{1v}) = L(\mu_{1v}\nu_v^s)$ over all v. If μ_1 is unitary, the product converges for $\mathrm{Re}(s) > 1$ and has analytic continuation as a meromorphic function to the entire complex s-plane. This extended function is entire if μ_1 does not factorize through ν and has a simple pole at $s = 1$ if $\mu_1 = 1_F$, where 1_F is the trivial character of $\mathbb{A}^\times/F^\times$. For details, see [Lang 94].

We further normalize the Haar measure on $\mathbb{A}^\times$ as follows. Let $\zeta = \prod_v \zeta_v$ be a non-trivial additive character of $F\backslash\mathbb{A}$. For each place v, let dx_v be the self-dual Haar measure on F_v with respect to the Fourier transform defined using ζ_v; see Subsection 6.5. Using this normalization, we prove in (6.5.1) the Poisson summation formula. Then the Haar measure we take on $F_v^\times$ is $d^\times x_v = L(1, 1_v)dx_v/|x_v|_v$. Note that on $\mathbb{A}$, the Haar measure $dx = \otimes_v dx_v$ is self dual. On $\mathbb{A}^\times$, the Tamagawa measure is

$$d^\times x = \frac{1}{\lambda_{-1}} \otimes_v d^\times x_v, \quad \text{where } L(s, 1_F) = \frac{\lambda_{-1}}{s-1} + \lambda_0 + \cdots .$$

The map $x \mapsto |x|$ allows us to identify $\mathbb{A}^1\backslash\mathbb{A}^\times$ with $\mathbb{R}^\times_{>0}$. The Tamagawa measure τ on $\mathbb{A}^1$ is such that the quotient measure $d^\times x/\tau$ is dt/t. As shown in Tate's thesis (see [Lang 94, Chapter XIV]), $|F^\times\backslash\mathbb{A}^1| = 1$.

2.8 Elements in $\mathrm{GL}(2, F)$**.** In this subsection, F is either a global or a local field. An element $g \in \mathrm{GL}(2, F)$ is called **regular** if its eigenvalues are distinct. If the eigenvalues of g are distinct and lie in $F^\times$, g is called **hyperbolic**. Each such element is conjugate in $G(F)$ to an element of $A(F)$. An element g is called **elliptic** if it is regular and its eigenvalues do not lie in F. Each elliptic element γ generates a quadratic field extension E over F. Its centralizer G_γ in G is a maximal torus over F which splits over E. A **singular** element of $\mathrm{GL}(2, F)$ is one whose eigenvalues are equal. Each singular element is conjugate to an element of $Z(F)N(F)$. The singular elements are: (1) the scalars in $Z(F)$, (2) the unipotent non-scalar elements - those which have a conjugate in $N(F)$-$Z(F)$, and (3) the products of these.

2.9 Centralizers. The **centralizer** of $\gamma \in G$ in G is denoted by G_γ. If $z \in Z$, then $G_z = G$; if $a \in A - Z$, then G_a is the F-split torus A. If $\gamma \in G(F)$ is elliptic, then G_γ is a torus non-split over F. The latter is an algebraic group T

over F such that there exist a quadratic field extension E of F and an isomorphism $T \simeq \mathbb{G}_m^2$ over E. In particular, $G_\gamma(E) \simeq E^{\times 2}$, $G_\gamma(F) \simeq E^\times$. If $x \neq 0$, then $G_{u(x)} = ZN$.

2.10 Quadratic extensions. Let Q be a set of representatives of the set of isomorphism classes of quadratic extensions of F. Then Q parametrizes the set of conjugacy classes of non-split maximal F-tori in G. Thus $E \mapsto T_E$, where T_E is an F-torus in G which splits over E, is an isomorphism of sets, namely, a bijection.

2.11 Test functions. For every place v of F, let $C^\infty_{c,\omega_v}(G_v)$ be the space of smooth functions $f_v : G_v \to \mathbb{C}$ with $f_v(zg) = \omega_v^{-1}(z) f_v(g)$, $z \in Z_v$, which are compactly supported modulo Z_v (thus $\mathrm{supp}(f_v) \subset C_v Z_v$, where C_v is a compact subset of G_v).

If v is non-Archimedean, we call f_v **smooth** (see [BZ 76]) if it is locally constant, thus bi-invariant under some open compact subgroup of G_v; namely, f_v is $\mathbf{K_v}$-finite, i.e., $\{g \mapsto f_v(k_1 g k_2); k_1, k_2 \in K_v\}$ is finite dimensional. If v is Archimedean, we call f_v **smooth** if it is infinitely differentiable and K_v-finite, i.e., $\{g \mapsto f_v(k_1 g k_2); k_1, k_2 \in K_v\}$ is infinite dimensional.

Using a Haar measure dg_v on $Z_v \backslash G_v$ we write $\mathbb{H}_{\omega_v}$ for the space $C^\infty_{c,\omega_v}(G_v)$ viewed as a convolution algebra. This $\mathbb{H}_{\omega_v}$ is called the **Hecke algebra at** v.

At v where ω_v is unramified, let f_v^0 be the element of $C^\infty_{c,\omega_v}(G_v)$ supported on $Z_v K_v$ equals $|K_v / K_v \cap Z_v|^{-1}$ on K_v. It exists, is unique, and is the **unit element** of the convolution Hecke algebra $\mathbb{H}_{\omega_v} = C^\infty_{c,\omega_v}(G_v)$ with respect to the implicitly chosen Haar measure on $Z_v \backslash G_v$.

Let $C^\infty_{c,\omega}(G(\mathbb{A}))$ be the span of $\otimes_v f_v$, where $f_v \in C^\infty_{c,\omega_v}(G_v)$ for all v, and $f_v = f_v^0$ for almost all v. Its elements f are called **test functions**. Using a Haar measure $dg = \otimes_v dg_v$, we write $\mathbb{H}_\omega$ for the space $C^\infty_{c,\omega}(G(\mathbb{A}))$ viewed as a convolution algebra. It is called the **(global) Hecke algebra**.

2.12 Orbital integrals. Let G_γ be the centralizer of $\gamma \in G_v = G(F_v)$ in G. The **orbital integral** of $f_v \in \mathbb{H}_{\omega_v} = C^\infty_{c,\omega_v}(G_v)$ at $\gamma \in G_v$ is defined by

$$\mathrm{OI}(\gamma, f_v dg_v) = \int_{G_{\gamma,v} \backslash G_v} f_v(g^{-1} \gamma g) d\dot{g}_v, \quad G_{\gamma,v} = G_\gamma(F_v).$$

Here $d\dot{g}_v$ is the quotient dg_v / dt_v of the Haar measure dg_v by a measure dt_v on $G_{\gamma,v}$.

For $g \in G_v$ and $a \in A_v$, set

$$f_v^{K_v}(g) = \int_{K_v} f_v(k_v^{-1} g k_v) dk_v, \quad f_{vN_v}^{K_v}(a) = \delta_v(a)^{1/2} \int_{N_v} f_v^{K_v}(a n_v) dn_v.$$

In particular, for $\alpha \in F_v^\times - \{1\}$, the **orbital integral** of f_v at $d(\alpha)$ is

$$\mathrm{OI}(d(\alpha), f_v dg_v) = \int_{A_v \backslash G_v} f_v(g_v^{-1} d(\alpha) g_v) d\dot{g}_v = \int_{N_v} f_v^{K_v}(n_v^{-1} d(\alpha) n_v) dn_v.$$

The Jacobian $\Delta_v(\gamma)$ of any γ in G_v with eigenvalues a, b is $|(a-b)^2/ab|_v^{1/2}$. Thus an element $\gamma \in G_v$ is regular precisely when $\Delta_v(\gamma) \neq 0$. The **normalized orbital integral** of f_v at a regular $\gamma \in G_v$ is defined by

$$\mathrm{OI}'(\gamma, f_v dg_v) = \Delta_v(\gamma)\, \mathrm{OI}(\gamma, f_v dg_v).$$

The reason for introducing the Jacobian $\Delta_v(\gamma)$ is that as $a \to e$ in A_v, where e denotes the identity in the group A_v, the normalized orbital integral $\mathrm{OI}'(a, f_v dg_v)$ has a finite limit, which we denote by $\mathrm{OI}'(e, f_v dg_v)$; see Subsection 18.1.

The **weighted orbital integral** of f_v at $a \in A_v^{\mathrm{reg}}$, where A_v^{reg} is the set of regular elements in A_v, is defined by

$$\mathrm{WOI}(a, f_v dg_v) = \int_{A_v \backslash G_v} f_v(g^{-1} a g) \mathbf{w}(g) d\dot{g}_v$$

where $\mathbf{w}(g) = \ln \delta_v(g) + \ln \delta_v(wg)$ depends only on $n \in N_v$ if $g = ank \in G_v$, $a \in A_v$, $k \in K_v$. The weight $\mathbf{w}(g)$ can be explicitly computed using the Iwasawa decomposition (Subsection 2.6). Stated differently, for $\alpha \in F_v^\times - \{1\}$, the **weighted orbital integral** of f_v at $d(\alpha)$ is

$$\mathrm{WOI}(d(\alpha), f_v dg_v) = \int_{N_v} f_v^{K_v}(n_v^{-1} d(\alpha) n_v) \ln \delta_v(w n_v) dn_v.$$

The **normalized weighted orbital integral** of f_v at $d(\alpha)$, $\alpha \in F_v^\times - \{1\}$, is defined by

$$\mathrm{WOI}'(d(\alpha), f_v dg_v) = \Delta_v(d(\alpha))\, \mathrm{WOI}(d(\alpha), f_v dg_v).$$

Recall that the orbital integral of f_v at $u(1)$ is

$$\mathrm{OI}(u(1), f_v dg_v) = \int_{Z_v N_v \backslash G_v} f_v(g_v^{-1} u(1) g_v) d\dot{g}_v.$$

By the Iwasawa decomposition $g_v = n_v a_v k_v$, with $dg_v = \delta(a_v)^{-1} da_v dn_v dk_v$ and $d^\times \alpha_v = L(1, 1_v) \frac{d\alpha_v}{|\alpha_v|_v}$, we obtain

$$\mathrm{OI}(u(1), f_v dg_v) = \int_{F_v^\times} f_v^{K_v}(u(a_v)) |a_v|_v \, d^\times a_v = L(1, 1_v) \int_{F_v} f_v^{K_v}(u(x_v)) dx_v.$$

Define the **normalized orbital integral** of f_v at $d(1)$ (or $u(1)$)

$$\mathrm{OI}'(d(1), f_v dg_v) = \mathrm{OI}'(u(1), f_v dg_v) = L(1, 1_v)^{-1}\, \mathrm{OI}(u(1), f_v dg_v).$$

We shall see in Subsection 18.0.1 that the right side is the limit as $\alpha \to 1$ in $F_v^\times$ of $\mathrm{OI}'(d(\alpha), f_v dg_v)$, which explain the notation $d(1)$ on the left-hand side.

In Section 18, we introduce the normalized corrected weighted orbital integral $\mathrm{CWOI}'(d(\alpha), fdg)$ and study its limit as $\alpha \to 1$.

2.13 Convolution operators. For non-Archimedean v, a representation (π_v, V) of G_v is called **smooth** (algebraic in [BZ 76]) if every vector ξ in V is fixed by some compact open subgroup of G_v. A representation is called **admissible** if it is smooth and the space $V^Y = \{\xi \in V; \pi_v(y)\xi = \xi \text{ for all } y \in Y\}$ is finite dimensional for every compact open subgroup Y of G_v.

If π_v is admissible and $f_v \in \mathbb{H}_{\omega_v}$, then $\dim_{\mathbb{C}} V^{U_v} < \infty$ for every open subgroup U_v of G_v, and the convolution operator $\pi_v(f_v dg_v) = \int_{Z_v \backslash G_v} f_v(g_v)\pi_v(g_v)dg_v$, which acts on the space V of π_v, has **finite rank** (its image is finite dimensional). Hence it has a trace, denoted by $\mathrm{tr}\,\pi_v(f_v dg_v)$, which depends on a choice of a Haar measure dg_v. If dg_v is fixed, it is often deleted from the notation. Similar conclusions hold in the Archimedean case (see, e.g., [BJ 79]) using the theory of $(\mathfrak{g}, K)$-modules. We do not discuss this here.

Analogously, in the global case, given a test function $f \in \mathbb{H}_\omega$, a Haar measure dg, and an admissible representation (π, V_π) of $G(\mathbb{A})$ with central character ω (thus a representation of finite length whose irreducible subquotients are $\otimes_v \pi_v$, all π_v are admissible and almost all are unramified), the convolution operator $\pi(fdg) = \int f(g)\pi(g)dg$ ($g \in Z(\mathbb{A})\backslash G(\mathbb{A})$) is an endomorphism of the space V_π of the representation π.

Integrating π against f produces an operator $g \in G(\mathbb{A})$, smoother than $\pi(g)$ in the sense that it is of **trace class**, i.e., it has finite trace.

If $\pi = \otimes_v \pi_v$, $f = \otimes_v f_v$, then $\mathrm{tr}\,\pi(fdg) = \prod_v \mathrm{tr}\,\pi_v(f_v dg_v)$. As f_v is the unit element f_v^0 for almost all v, almost all factors in the product over v are 1.

3 Induced representations

3.1 Definition of induced representations. Let μ_{1v} and μ_{2v} be characters, not necessarily unitary, of $F_v^\times$, with $\mu_{1v}\mu_{2v} = \omega_v$. Let $\mu_v = (\mu_{1v}, \mu_{2v})$ be the associated character of B_v that is trivial on N_v. Thus $\mu_v(d(a,b)n) = \mu_{1v}(a)\mu_{2v}(b)$. Recall that $\delta_v(g) = \delta_v(ank) = |a_1/a_2|_v$ if $a = d(a_1, a_2)$. Put $(\pi_{\mu_v}, \mathrm{H}(\mu_v))$ for the **induced** or **principal series representation** $\mu_v^{G_v} = i_{B_v}^{G_v}(\delta_v^{1/2}\mu_v)$. It is the representation of G_v by right translation: $(i_{B_v}^{G_v}(\mu_v)(g)\psi)(g_1) = \psi(g_1 g)$, on the space $\mathrm{H}(\mu_v)$ of functions ψ for which there exists an open compact subgroup U_ψ of G_v

such that

$$\psi(d(a,b)u(x)gu) = \mu_{1v}(a)\mu_{2v}(b)|a/b|_v^{1/2}\psi(g)$$

for all $a, b \in F_v^\times$, $x \in F_v$, $g \in G_v$, $u \in U_\psi$. If μ_v is unramified, the space $\mathrm{H}(\mu_v)$ contains a unique member $\psi^0 = \psi^0_{\mu_v}$ whose value on K_v is $|K_v/K_v \cap Z_v|^{-1}$.

Analogously, starting with global μ_1, μ_2, we form the induced representation $(\pi_\mu, \mathrm{H}(\mu)) = i_B^G(\delta^{1/2}\mu)$, where i_B^G signifies $i_{B(\mathbb{A})}^{G(\mathbb{A})}$. The factor δ is introduced to preserve unitarity: if μ is unitary, then so is π_μ; see Subsection 3.3.

3.2 The space $\mathrm{H}(\mu(s))$. Let s be a complex number. Let $\mu(s)$ be the character $(\mu_1\nu^s, \mu_2\nu^{-s})$, where $\nu(x) = |x|$, of $A(\mathbb{A})$. It extends to $B(\mathbb{A})$ trivially on $N(\mathbb{A})$. Let $\mathrm{H}(\mu(s))$ be the space of the induced representation $i_B^G(\mu(s))$, normalized by the factor $\delta^{1/2}$. Thus $\mathrm{H}(\mu(s))$ is the space of complex-valued functions ψ for which there exists an open compact subgroup U_ψ of $G(\mathbb{A})$ satisfying

$$\psi(au(x)gu) = \mu_1(b)\mu_2(c)|b/c|^{s+\frac{1}{2}}\psi(g)$$

for all $a = d(b,c) \in A(\mathbb{A})$, $x \in \mathbb{A}$, $g \in G(\mathbb{A})$, and $u \in U_\psi$. Moreover, ψ is K_v-finite at the Archimedean places.

The function $\psi \in \mathrm{H}(\mu(s))$ is determined uniquely by its restriction $\psi|A^1K$ to A^1K. This $\psi|A^1K$ lies in the space $\mathrm{H}_K(\mu)$ of complex-valued functions ψ' on A^1K for which there exists an open compact subgroup $U_{\psi'}$ of K with

$$\psi'(aku) = \mu(a)\psi'(k), \quad \text{for all } a \in A^1,\ k \in K,\ u \in U_{\psi'}.$$

Conversely, any function $\psi' \in \mathrm{H}_K(\mu)$ extends uniquely to an element in $\mathrm{H}(\mu(s))$. The space $\mathrm{H}_K(\mu)$ is independent of s. The spaces $\mathrm{H}(\mu(s))$ depend on the parameter $s \in \mathbb{C}$; they form a vector bundle over $\mathbb{C}$ which is trivial via the trivialization $\mathrm{H}(\mu(s)) \simeq \mathrm{H}_K(\mu)$, $\psi \mapsto \psi|A^1K$.

The group $G(\mathbb{A})$ acts on $\mathrm{H}(\mu(s))$ by the right translation. Denote the action by $i_B^G(\mu(s))$. Then $(i_B^G(\mu(s))(g)\psi)(g_1) = \psi(g_1g)$ for all $g, g_1 \in G(\mathbb{A})$ and $\psi \in \mathrm{H}(\mu(s))$. The representation $i_B^G(\mu(s))$ is continuous, but not always unitary. It is unitary when s is purely imaginary and μ is unitary; see Subsection 3.3.

To repeat, the family $\mathrm{H}(\mu(s))$ is a holomorphic vector bundle with base $\mathbb{C}$. Indeed, the sections over an open set U of $\mathbb{C}$ are the functions $\psi(g,s)$ on $G(\mathbb{A}) \times U$ satisfying $\psi(nagu, s) = (\delta^{1/2}\mu(s))(a) \cdot \psi(g,s)$. This bundle is trivial, since every ψ in $\mathrm{H}(\mu(s))$ is determined uniquely by its restriction to the set A^1K; so all $\mathrm{H}(\mu(s))$ are isomorphic to $\mathrm{H}_K(\mu)$, which is independent of s, and isomorphic to each other.

Thus the family $\mathrm{H}(\mu)$ over all characters μ of $A(F)\backslash A(\mathbb{A})$ whose restriction to $Z(\mathbb{A})$ is ω is a trivial vector bundle over a complex manifold of dimension 1

with infinitely many connected components. In each component, the character $a \mapsto \mu(a)$ of A^1 is fixed.

3.3 Sesquilinear pairing. The space $\mathrm{H}_K(\mu)$ is a Hilbert space with inner product

$$(\psi_1, \psi_2) = \int_{A(F)\cdot Z(\mathbb{A})\cap A^1\backslash A^1} \int_K \psi_1(ak)\ \overline{\psi}_2(ak)da\,dk,$$

which extends to a **sesquilinear pairing** $\mathrm{H}(\mu(s)) \times \mathrm{H}(\overline{\mu}^{-1}(-\overline{s})) \to \mathbb{C}$. Indeed, if $\psi_1 \in \mathrm{H}(\mu(s))$ and $\psi_2 \in \mathrm{H}(\overline{\mu}^{-1}(-\overline{s}))$, then

$$\psi_1(d(a,b)u(x)k) = \mu_1(a)\mu_2(b)|a/b|^{s+\frac{1}{2}}\psi_1(k)$$
$$\psi_2(d(a,b)u(x)k) = \overline{\mu}_1^{-1}(a)\overline{\mu}_2^{-1}(b)|a/b|^{-\overline{s}+\frac{1}{2}}\psi_2(k).$$

Hence $\psi_1(ank)\overline{\psi}_2(ank) = \delta(a)\psi_1(k)\overline{\psi}_2(k)$ and

$$(\psi_1, \psi_2) = \int_{N(\mathbb{A})A(\mathbb{A})\backslash G(\mathbb{A})} \psi_1(g)\overline{\psi}_2(g)dg,$$

since $dg = \delta(a)^{-1}dndadk$; see Subsection 2.7. This sesquilinear form is non-degenerate and satisfies

$$(i_B^G(\mu(s))(g)\psi_1, i_B^G(\overline{\mu}^{-1}(-\overline{s}))(g)\psi_2) = (\psi_1, \psi_2)$$

for all $\psi_1 \in \mathrm{H}(\mu(s))$, $\psi_2 \in \mathrm{H}(\overline{\mu}^{-1}(-\overline{s}))$. We conclude that when μ is unitary ($\mu\overline{\mu} = 1$) and s is purely imaginary, then $i_B^G(\mu(s))$ is unitary.

4 Intertwining operators

4.1 Definition of intertwining operators. For a function ψ_v on G_v, introduce the integral

$$(M\psi_v)(g) = \int_{N_v} \psi_v(wng)dn.$$

Let μ_v be a character of A_v whose restriction to Z_v is ω_v. If $\mu_v = (\mu_{1v}, \mu_{2v})$, put $\widetilde{\mu}_v = (\mu_{2v}, \mu_{1v})$. The restriction of the operator M to the Hilbert space $\mathrm{H}(\mu_v)$ defines an operator $M(\mu_v) : \mathrm{H}(\mu_v) \to \mathrm{H}(\widetilde{\mu}_v)$. It converges if μ_{1v}/μ_{2v} is (1) ramified, or (2) unramified, thus of the form ν_v^s with $s \in \mathbb{C}$, and $\mathrm{Re}(s) > 0$; see Proposition 4.1.2 below. It is called the **local intertwining operator**, as it intertwines the representations $i_B^G(\mu_v)$ and $i_B^G(\widetilde{\mu}_v)$.

Proposition 4.1.1. *For all $g \in G_v$ and μ_v, we have*

$$M(\mu_v)\cdot i_B^G(\mu_v)(g) = i_B^G(\widetilde{\mu}_v)(g)\cdot M(\mu_v).$$

Proof. For every $g_1 \in G_v$ and $\psi \in \mathrm{H}(\mu_v)$, we have

$$\begin{aligned}\big((M(\mu_v)\cdot i_B^G(\mu_v)(g))\psi\big)(g_1) &= (M(\mu_v)(i_B^G(\mu_v)(g)\psi))(g_1)\\ &= \int_{N_v}((i_B^G(\mu_v)(g))\psi)(wng_1)dn = \int_{N_v}\psi(wng_1g)dn\end{aligned}$$

On the other hand,

$$\begin{aligned}\big((i_B^G(\widetilde{\mu}_v)(g)\cdot M(\mu_v))\psi\big)(g_1) &= i_B^G(\widetilde{\mu}_v)(g)\cdot(M(\mu_v)\psi)(g_1)\\ &= (M(\mu_v)\psi)(g_1g) = \int_{N_v}\psi(wng_1g)dn.\end{aligned}$$

□

Recall that the unit element $\psi^0_{\mu_v}$ of $\mathrm{H}(\mu_v)$ is the unique element of $\mathrm{H}(\mu_v)$ which is $|K_v/K_v\cap Z_v|^{-1}$ on K_v.

Suppose v is non-Archimedean and μ_v is unramified. Multiplying μ_v or $i_B^G(\mu_v)$ by $\omega_v^{-1/2}$, where $\mu_{1v}\mu_{2v} = \omega_v$, we assume that $\omega_v = 1$ and $\mu_{1v} = \mu_{2v}^{-1} = |\cdot|_v^s$.

Proposition 4.1.2. *Suppose μ_v is unramified. If* $\mathrm{Re}(s) > 0$*, then*

$$M(\mu_v)\psi^0_{\mu_v} = m(\mu_v)\psi^0_{\mu_v},$$

where $m(\mu_v) = \frac{L(\mu_{1v}/\mu_{2v})}{L(\nu_v\mu_{1v}/\mu_{2v})}$.

Note that $m(\mu_v)$ is defined for all μ_v (not necessarily unramified) in (5.2.1) below.

Proof. To compute $M(\mu_v)\psi^0_{\mu_v}$, note that since it is right-K_v-invariant, it is a multiple of $\psi^0_{\widetilde{\mu}_v}$ and determined by its value at e. We have

$$\big(M(\mu_v)\psi^0_{\mu_v}\big)(e) = \int_{N_v}\psi^0_{\mu_v}(wne)dn = \int_{F_v}\psi^0_{\mu_v}(wu(x))dx.$$

Since

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix} = \begin{pmatrix}x^{-1}&-1\\0&x\end{pmatrix}\begin{pmatrix}-1&0\\-x^{-1}&-1\end{pmatrix},$$

if $|x| > 1$,

$$\begin{aligned}\big(M(\mu_v)\psi^0_{\mu_v}\big)(e) &= \int_{|x|\le 1}\psi^0_{\mu_v}(e)dx + \int_{|x|>1}\psi^0_{\mu_v}\begin{pmatrix}x^{-1}&-1\\0&x\end{pmatrix}dx\\ &= \psi^0_{\mu_v}(e)\left(1 + \int_{|x|>1}|x|^{-2s}\frac{dx}{|x|} = 1 + \sum_{n\ge 1}\int_{|x|=q_v^n}q_v^{-2sn}\frac{dx}{|x|}\right).\end{aligned}$$

Changing variables $x = \boldsymbol{\pi}_v^{-n}y$ and observing that

$$\int_{|y|=1}dy = 1 - \int_{|y|<1}dy = 1 - q_v^{-1}\int_{|y|\le 1}dy = 1 - q_v^{-1},$$

we obtain

$$\begin{aligned}\big(M(\mu_v)\psi^0_{\mu_v}\big)(e) &= \psi^0_{\mu_v}(e)\left(1+\sum_{n\geq 1} q_v^{-2sn}\int_{|y|=1} dy\right)\\ &= \psi^0_{\mu_v}(e)\left(1+\left(1-\frac{1}{q_v}\right)\sum_{n\geq 1} q_v^{-2sn}\right)\\ &= \psi^0_{\mu_v}(e)\left(\frac{1-q_v^{-2s-1}}{1-q_v^{-2s}}\right)\\ &= \psi^0_{\mu_v}(e)m(\mu_v).\end{aligned}$$

The sum converges if $\mathrm{Re}(s) > 0$. □

Corollary 4.1.3. *Let v be non-Archimedean. Suppose* $\mathrm{Re}(s) > 0$ *if μ_v is unramified and $\mu_{1v}/\mu_{2v} = \nu_v^s$. Then the local intertwining operator $M(\mu_v)$ converges.*

Definition 4.2. Define $R(\mu_v) = m(\mu_v)^{-1}M(\mu_v)$. It is called the **normalized intertwining operator**. If v is non-Archimedean and μ_v is unramified with $\mathrm{Re}(s) > 0$, then $R(\mu_v)\psi^0_{\mu_v} = \psi^0_{\mu_v}$.

5 Normalized intertwining operators

5.1 Sections of $\mathrm{H}(\mu_v)$. The group G_v operates by right matrix multiplication on $F_v \times F_v$. The stabilizer of the line $F_v(0, 1)$ is the group B_v.

Proposition 5.1.1. *For all $\Phi \in C_c^\infty(F_v \times F_v)$, the integral*

$$\psi(g, \mu_v) = \frac{\mu_{1v}\nu_v^{1/2}(\det g)}{L(\nu_v\mu_{1v}/\mu_{2v})}\int_{F_v^\times} \Phi((0, t)g)(\mu_{1v}/\mu_{2v})(t)|t|_v d^\times t \tag{5.1.1}$$

converges if μ_{1v}/μ_{2v} is ramified or $\mu_{1v}/\mu_{2v} = \nu_v^{2s}$ with $\mathrm{Re}(s) > -1/2$. *For such μ, the function ψ defines an element of* $\mathrm{H}(\mu_v)$ *which extends to a regular function on the entire s-plane. Every element of* $\mathrm{H}(\mu_v)$ *can be represented by $\psi(g, \mu_v)$ in the domain of convergence.*

Proof. This follows from Tate's theory of the local zeta function ([Lang 94, Section XIV]. Clearly, for $f \in C_c^\infty(F_v)$, the integral

$$Z(f, \mu_v, s) = Z(f, \mu_v\nu_v^s) = \int_{F_v^\times} f(t)\mu_v(t)|t|_v^s\, d^\times t$$

converges if μ_v is ramified. If μ_v is unramified, we may assume $\mu_v = 1$. For f the characteristic function of $|t|_v \leq 1$, this integral is

$$\int_{|t|_v \leq 1} |t|_v^s \, d^\times t = \sum_{n \geq 0} \int_{|t|_v = q_v^{-n}} q_v^{-ns} d^\times t = \sum_{n \geq 0} q_v^{-ns} = \frac{1}{1 - q_v^{-s}} = L(\nu_v^s),$$

which converges if $\mathrm{Re}(s) > 0$. Then $Z(f, \mu_v, s)/L(\nu_v^s)$ extends to an entire function of s. For the integral in our proposition, $f(t) = \Phi((0,t)g) \in C_c^\infty(F_v^\times)$. For $s \in \mathbb{C}$, write $\mu_v \nu_v^{1+2s}$ for $\nu_v \mu_{1v}/\mu_{2v}$, where μ_v is a unitary character, which is 1 if μ_{1v}/μ_{2v} is unramified. Since $\nu_v \mu_{1v}/\mu_{2v} = \nu_v^{1+2s}$ in the unramified case, we obtain $L(\nu_v \mu_{1v}/\mu_{2v})$, which explains the factor in the denominator in front of the integral. Here we have convergence for $1 + 2\,\mathrm{Re}(s) > 0$, or $\mathrm{Re}(s) > -1/2$. Using the factor in the numerator in front of the integral of the proposition, it is easy to check that $\psi(g, \mu_v) \in \mathrm{H}(\mu_v)$. □

5.2 Functional equation. Let $\zeta_v : F_v \to \mathbb{C}^\times$ be a non-trivial additive character. For $f \in C_c^\infty(F_v)$, define the Fourier transform $\widehat{f}$ by

$$\widehat{f}(t) = \int_{F_v} f(x)\zeta_v(xt)dx.$$

The functional equation for the quotient $\frac{Z(f,\mu_v,s)}{L(s,\mu_v)} = \frac{Z(f,\mu_v\nu_v^s)}{L(\mu_v\nu_v^s)}$ asserts that for all $s \in \mathbb{C}$, $f \in C_c^\infty(F_v)$, and characters $\mu_v : F_v^\times \to \mathbb{C}^\times$, there exists an exponential function $\varepsilon(s, \mu_v, \zeta_v)$ (of the form ce^{ds}, complex $c \neq 0$ and d) satisfying

$$\frac{Z(\widehat{f}, \mu_v^{-1}, 1-s)}{L(1-s, \mu_v^{-1})} = \varepsilon(s, \mu_v, \zeta_v)\frac{Z(f, \mu_v, s)}{L(s, \mu_v)}.$$

Define the Fourier transform of $\Phi \in C_c^\infty(F_v \times F_v)$ to be

$$\widehat{\Phi}((x,y)) = \int_{F_v}\int_{F_v} \Phi((u,v))\zeta_v\big((u,v)w \cdot {}^t(x,y)\big)du\,dv,$$

where ${}^t(x,y)$ is the transpose of the row vector (x,y).

Proposition 5.2.1. *The function $\psi(\mu_v) : g \mapsto \psi(g, \mu_v)$ of* (5.1.1) *is mapped by the operator $M(\mu_v)$ to the product of*

$$m(\mu_v) = \frac{L(0, \mu_{1v}/\mu_{2v})}{L(1, \mu_{1v}/\mu_{2v})\varepsilon(0, \mu_{1v}/\mu_{2v}, \zeta_v)} \tag{5.2.1}$$

and

$$\omega_v(-1)\frac{\mu_{2v}\nu_v^{1/2}(\det g)}{L(1, \mu_{2v}/\mu_{1v})}\int_{F_v^\times} \widehat{\Phi}((0,t)g)(\mu_{2v}/\mu_{1v})(t)|t|_v d^\times t. \tag{5.2.2}$$

Note that $m(\mu_v)$ is defined for all (not necessarily unramified) μ_v.

Proof. Applying the operator $M(\mu_v)$ to $\psi(\mu_v)$ of (5.1.1), changing variables $(x,t)\mapsto(x/t,-t)$, and using the fact that $\mu_{1v}\mu_{2v}=\omega_v$, we obtain

$$\begin{aligned}(M(\mu_v)\psi(\mu_v))(g) &= \int_{F_v}\psi(\omega u(x)g,\mu_v)dx\\ &= \omega_v(-1)\frac{\mu_{1v}\nu_v^{1/2}(\det g)}{L(\nu_v\mu_{1v}/\mu_{2v})}\int_{F_v}\int_{F_v^\times}\Phi((t,x)g)(\mu_{1v}/\mu_{2v})(t)d^\times t dx\\ &= \omega_v(-1)\frac{\mu_{1v}\nu_v^{1/2}(\det g)}{L(1,\mu_{1v}/\mu_{2v})}\int_{F_v^\times}f(t)(\mu_{1v}/\mu_{2v})(t)\,d^\times t \qquad (5.2.3)\\ &= \omega_v(-1)\frac{\mu_{1v}\nu_v^{1/2}(\det g)}{L(1,\mu_{1v}/\mu_{2v})}Z(f,\mu_{1v}/\mu_{2v},0),\end{aligned}$$

where $f(t)=\int_{F_v}\Phi((t,x)g)dx$. Now by the functional equation of Subsection 5.2 with $s=0$,

$$(M(\mu_v)\psi(\mu_v))(g)=m(\mu_v)\omega_v(-1)\frac{\mu_{1v}\nu_v^{1/2}(\det g)}{L(1,\mu_{2v}/\mu_{1v})}Z(\widehat{f},\mu_{2v}/\mu_{1v},1).$$

So it suffices to show that

$$\begin{aligned}Z(\widehat{f},\mu_{2v}/\mu_{1v},1) &= \int_{F_v^\times}\widehat{f}(t)(\mu_{2v}/\mu_{1v})(t)|t|_v d^\times t\\ &= (\mu_{2v}/\mu_{1v})(\det g)\int_{F_v^\times}\widehat{\Phi}((0,t)g)(\mu_{2v}/\mu_{1v})(t)|t|_v d^\times t.\end{aligned}$$

To show the second equality, we compute

$$\begin{aligned}\widehat{f}(t) &= \int_{F_v}f(u)\zeta_v(tu)du=\int_{F_v}\int_{F_v}\Phi((u,v)g)dv\zeta_v(tu)du\\ &= \int_{F_v}\int_{F_v}\Phi((u,v)g)\zeta_v((u,v)gg^{-1}w\cdot{}^t(0,t))du\,dv\\ &= \int_{F_v}\int_{F_v}\Phi((u,v)g)\zeta_v((u,v)gw\cdot{}^tg(\det g)^{-1}\cdot{}^t(0,t))du\,dv.\end{aligned}$$

Change variables $(u,v)g\mapsto(u,v)$ to obtain

$$\begin{aligned}\widehat{f}(t) &= |\det g|_v^{-1}\int_{F_v}\int_{F_v}\Phi((u,v))\zeta_v((u,v)w\cdot{}^tg(\det g)^{-1}\cdot{}^t(0,t))du\,dv\\ &= |\det g|_v^{-1}\widehat{\Phi}((0,t)g(\det g)^{-1}).\end{aligned}$$

Hence

$$\begin{aligned}\int_{F_v^\times} \widehat{f}(t)\left(\frac{\mu_{2v}}{\mu_{1v}}\right)(t)|t|_v d^\times t &= |\det g|_v^{-1} \int_{F_v^\times} \widehat{\Phi}((0,t)g(\det g)^{-1})(\mu_{2v}/\mu_{1v})(t)|t|_v d^\times t \\ &= |\det g|_v^{-1} \int_{F_v^\times} \widehat{\Phi}((0,t)g)(\mu_{2v}/\mu_{1v})(t\det g)|t|_v |\det g|_v d^\times t \\ &= (\mu_{2v}/\mu_{1v})(\det g) \int_{F_v^\times} \widehat{\Phi}((0,t)g)(\mu_{2v}/\mu_{1v})(t)|t|_v d^\times t,\end{aligned}$$

as required. □

Note that we obtain (5.2.2) (5.1.1) by replacing μ_v with $\widetilde{\mu}_v$ and Φ with $\widehat{\Phi}$, and by multiplying by $\omega_v(-1)$.

5.3 Properties of the normalized operator. As at the end of Section 4, define the operator $R(\mu_v) : \mathrm{H}(\mu_v) \to \mathrm{H}(\widetilde{\mu}_v)$ (on $|\mathrm{Re}(s)| > -1/2$ if μ_{1v}/μ_{2v} is unramified of the form ν_v^s) by mapping (5.1.1) to (5.2.2).

Since $\widetilde{\widetilde{\mu}}_v = \mu_v$, $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$, and $\omega_v^2(-1) = 1$, we have

$$R(\widetilde{\mu}_v)R(\mu_v) = \mathrm{Id}\,. \tag{5.3.1}$$

Hence $R(\mu_v)$ extends to a meromorphic function of μ_v for all μ_v.

For all μ_v, with $\mathrm{Re}(s) > 0$ if μ_v is unramified, we have $M(\mu_v) = m(\mu_v)R(\mu_v)$. Since $m(\mu_v)$ is meromorphic on the entire manifold of μ_v, this also gives the analytic continuation of the operator $M(\mu_v(s))$ in the domain $\mathrm{Re}(s) > -1/2$.

Recall from Proposition 4.1.2 that the operator $R(\mu_v)$ is normalized by mapping the unit element $\psi^0_{\mu_v}$ to the unit element $\psi^0_{\widetilde{\mu}_v}$ when μ_v is unramified and ζ_v has conductor $\mathcal{O}_v$.

A well known result of [JL 70], generalized in [BZ 76] to GL(n), asserts the following proposition in both the Archimedean and non-Archimedean cases.

Proposition 5.3.1. *The induced G_v-module $\mathrm{H}(\mu_v)$ is irreducible unless $\mu_v = (\mu_{1v}\nu_v^s,\ \mu_{1v}\nu_v^{-s})$ and $s = 1/2$ or $-1/2$. Put $s = 1/2$. Then $\mathrm{H}(\widetilde{\mu}_v)$ consists of ψ_v with $\psi_v(bg) = \mu_{1v}(\det b)\psi_v(g)$ $(b \in B_v)$. It has length 2 and a unique subspace, which is one-dimensional, whose members are the complex multiples of the function $\psi_v(g) = \mu_{1v}(\det g)$. The space $\mathrm{H}(\mu_v)$ also has length 2 and a unique subspace of codimension 1, which is the kernel of $M(\mu_v)$. The image of $M(\mu_v)$ is the one-dimensional subspace $\mathbb{C}\cdot\mu_{1v}(\det g)$ of $\mathrm{H}(\widetilde{\mu}_v)$. Hence, for the sesquilinear pairing $(\cdot,\cdot) : \mathrm{H}(\widetilde{\mu}_v) \times \mathrm{H}(\widetilde{\widetilde{\mu}}_v^{-1}) \to \mathbb{C}$ defined by $(\psi_1, \psi_2) = \int_{K_v} \psi_1(k)\overline{\psi}_2(k)\,dk$, we have*

$$(R(\mu_v)\psi_{\mu_v}, \psi_{\widetilde{\widetilde{\mu}}_v^{-1}}) = c'(\psi_{\mu_v}, \mu_v \circ \det)\overline{(\psi_{\widetilde{\widetilde{\mu}}_v^{-1}}, \mu_v \circ \det)}$$

for some $c' \in \mathbb{C}^\times$.

6 Automorphic forms

6.1 Definition of automorphic forms. Throughout this paper, a unitary character ω of $\mathbb{A}^\times/F^\times$, that is, of $Z(\mathbb{A})/Z(F)$, is fixed. A complex-valued smooth measurable function ϕ on $G(\mathbb{A})$ is **slowly increasing** if $|\phi(g)| \leq C\delta(g)^N$ for some $C > 0$ and $N \in \mathbb{Z}_{>0}$ and all g with $\delta(g) > 1$; ϕ is **rapidly increasing** if for each $N \in \mathbb{Z}_{>0}$, there exists $C > 0$ with $\delta(g)^N|\phi(g)| \leq C$ for all g with $\delta(g) > 1$. An **automorphic form** on $G(\mathbb{A})$ with central character ω is a complex-valued smooth measurable function ϕ on $G(\mathbb{A})$ that is right K-finite and $Z(\mathfrak{g})$-finite, where $Z(\mathfrak{g})$ is the center of the universal enveloping algebra of the complexified Lie algebra of G, such that

$$\phi(\gamma zg) = \omega(z)\phi(g) \quad \text{for all } \gamma \in G(F),\ z \in Z(\mathbb{A})$$

and ϕ is slowly increasing. By an automorphic form we mean one with central character ω.

We are interested in the space $L^2_\omega(G)$ of $\mathbb{C}$-valued functions ϕ on $G(\mathbb{A})$ satisfying $\phi(\gamma zg) = \omega(z)\phi(g)$ for all $\gamma \in G(F)$, $z \in Z(\mathbb{A})$, and which are **square-integrable modulo the center**, namely,

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} |\phi(g)|^2 dg < \infty$$

and in its subspace of cusp forms, described in Section 6.3 below.

There exist automorphic forms, such as the Eisenstein series (see below), which are not in $L^2_\omega(G)$.

Denote the inner product in $L^2_\omega(G)$ by $(\cdot,\cdot)$. The space $L^2_\omega(G)$ is a $G(\mathbb{A})$-module under the right regular representation r. Thus, for all $\phi \in L^2_\omega(G)$ and $g, h \in G(\mathbb{A})$, $(r(g)\phi)(h) = \phi(hg)$.

6.2 One-dimensional automorphic forms. Let $\mu : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^1$ be a character with $\mu^2 = \omega$. Then $g \mapsto \mu(\det g)$ is an example of an automorphic form. The space $L^2_{1,\omega}(G)$ spanned by these automorphic forms for all such μ is a submodule of $L^2_\omega(G)$.

6.3 Cusp forms. The constant term ϕ_N of a function ϕ on $N(F)\backslash G(\mathbb{A})$ is $\phi_N(g) = \int_{N(F)\backslash N(\mathbb{A})} \phi(ng)dn$, $g \in G(\mathbb{A})$. A **cusp form** is an automorphic form ϕ whose constant term ϕ_N is 0. Every cusp form lies in $L^2_\omega(G)$; see [JL 70, Proposition 10.7].

The space $L^2_{0,\omega}(G)$ of cusp forms is a submodule of $L^2_\omega(G)$. It decomposes as a direct sum of irreducible representations of $G(\mathbb{A})$, called **cuspidal representations**.

Recall that a bounded operator T on a separable Hilbert space H is called a **Hilbert-Schmidt** operator if there exists an orthonormal basis $\{e_n; n \geq 1\}$ of H such that $\sum_{n\geq 1} \|Te_n\|^2 < \infty$. If $\{f_n; n \geq 1\}$ is another orthonormal basis, $\sum_{n\geq 1} \|Tf_n\|^2 = \sum_{n\geq 1} \|Te_n\|^2$. The adjoint of a Hilbert-Schmidt operator is also Hilbert-Schmidt. Every Hilbert-Schmidt operator is **compact**, namely, it is the limit of a sequence $\{T_n; n \geq 1\}$ of finite rank operators.

The following theorem, which we prove below, shows that these representations occur with finite multiplicities in $L^2_{0,\omega}(G)$.

Theorem 6.1. *For every $f \in C^\infty_{c,\omega}(G(\mathbb{A}))$, the convolution operator $r(fdg)$ on $L^2_{0,\omega}$ is Hilbert-Schmidt, hence compact.*

Corollary 6.3.1. *The $G(\mathbb{A})$-module $L^2_{0,\omega}(G)$ is the orthonormal Hilbert space direct sum of irreducible representations, each occurring with finite multiplicity.*

Proof. To decompose $L^2_{0,\omega}(G)$ into discrete irreducible closed invariant submodules, it suffices, by Zorn's lemma, to show that every non-zero invariant closed submodule Y of $L^2_{0,\omega}(G)$ contains an irreducible submodule.

The operator $r(fdg)$ is self-adjoint on Y if $f(x^{-1}) = \overline{f}(x)$ and is non-zero if f is non-zero and supported on a sufficiently small neighborhood of the identity. By Theorem 6.1, $r(fdg)$ is compact. Hence it has a non-zero eigenvalue λ, which has finite multiplicity, m. Let ϕ be a non-zero eigenvector of λ and Φ the closed submodule generated by ϕ. We claim that Φ decomposes as a direct sum of at most m irreducible closed submodules. Indeed, if not, there exist $m+1$ mutually orthogonal closed submodules in Φ. Let $\mathrm{pr}_0, \mathrm{pr}_1, \cdots, \mathrm{pr}_m$ be the orthogonal projections onto these submodules. Then $r(fdg)$ has eigenvalue λ on each of the linearly independent vectors $\mathrm{pr}_0\,\phi, \ldots, \mathrm{pr}_m\,\phi$. Thus λ has multiplicity greater m, which is impossible. Any of these m subspaces is the sought after irreducible submodule of Y.

Consequently we may write $L^2_{0,\omega}(G)$ as the orthogonal Hilbert space direct sum of irreducible submodules. Let Y be a submodule in this decomposition. Choose $f \in C^\infty_{c,\omega}(G)$ with $f(x^{-1}) = \overline{f}(x)$, so that $r(fdg)|Y \neq 0$. Now $r(fdg)$ is a compact self-adjoint operator on Y. Hence $r(fdg)$ has a non-zero eigenvalue λ on a non-zero submodule of Y. On each irreducible submodule of $L^2_{0,\omega}(G)$ that is equivalent to the irreducible Y, the operator $r(fdg)$ must act as the scalar λ. If Y occurs with infinite multiplicity, then λ occurs with infinite multiplicity as an eigenvalue of

$r(fdg)$. But this contradicts the compactness of the self-adjoint operator $r(fdg)$ on $L^2_{0,\omega}(G)$. □

Proof of Theorem 6.1. For $c > 0$, compact $C \subset \mathbb{A}$, and compact $C' \subset \mathbb{A}^\times$, consider the **Siegel domain**

$$\begin{aligned} S &= S(c, C, C') \\ &= \{g = zu(x)d(ta, 1)k,\ z \in Z(\mathbb{A}),\ x \in C,\ t \in F_\infty^+ \text{ with } t > c,\ a \in C',\ k \in K\}. \end{aligned}$$

For sufficiently large C, C', and sufficiently small c, we have $G(\mathbb{A}) = G(F) \cdot S$; see [PR 93, Proposition 5.7].

The image S^N under $G(\mathbb{A}) \to N(F)\backslash G(\mathbb{A})$ of a Siegel domain $S \subset G(\mathbb{A})$ is called a **Siegel domain** in $N(F)\backslash G(\mathbb{A})$. For such S^N, let $L^2_\omega(S^N)$ be the space of functions $\phi : S^N \to \mathbb{C}$ with $\phi(zg) = \omega(z)\phi(g)$ $(z \in Z(\mathbb{A}),\ g \in S^N)$ and $\int_{Z(\mathbb{A})\backslash S^N} |\phi(g)|^2 dg < \infty$. When $G(\mathbb{A}) = G(F) \cdot S$, the space $L^2_\omega(G)$ is isomorphic to a closed subspace V of $L^2_\omega(S^N)$. In this case, there exists a bounded operator $A : L^2_\omega(G) \to V$ with bounded inverse A^{-1}.

For $f \in C^\infty_{c,\omega}(G)$ and $\phi \in L^2_\omega(G)$, we have

$$\begin{aligned} (r(fdg)\phi)(x) &= \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(g)\phi(xg)dg = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \phi(g) f(x^{-1}g)dg \\ &= \int_{Z(\mathbb{A})N(F)\backslash G(\mathbb{A})} \phi(g) K_f^N(x, g)dg \end{aligned}$$

with $K_f^N(x, g) = \sum_{\gamma \in N(F)} f(x^{-1}\gamma g)$. If $\phi \in L^2_{0,\omega}(G)$, the change of variables $g \mapsto n_1 g$, $n_1 \in N(F)\backslash N(\mathbb{A})$, followed by the change of variables $n \mapsto nn_1^{-1}$ yields

$$\int_{Z(\mathbb{A})N(F)\backslash G(\mathbb{A})} \phi(g) \int_{N(\mathbb{A})} f(x^{-1}ng)dn\, dg = 0.$$

In this case,

$$(r(fdg)\phi)(x) = \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \widetilde{K}_f^N(x, g)\phi(g)dg$$

with $\widetilde{K}_f^N(x, g) = K_f^N(x, g) - \int_{N(\mathbb{A})} f(x^{-1}ng)dn$.

If $x \in S^N$, there exists another domain S_1^N such that $g \notin S_1^N$ implies $\widetilde{K}_f^N(x, g) = 0$. The term subtracted in $\widetilde{K}_f^N(x, g)$ is the Fourier transform of $y \mapsto f(x^{-1}u(y)g)$ at 0. Hence

$$\int_{Z(\mathbb{A})\backslash S} \int_{Z(\mathbb{A})\backslash S_1} |\widetilde{K}_f^N(x, g)|^2 dx\, dg < \infty.$$

Indeed, consider $F_{x,g}(y) = f(x^{-1}u(y)g)$ with $x \in S^N$, $g \in S^{N'}$ which determine $t, t' > c$ (> 0). Since f has compact support, $d(t)^{-1}u(y)d(t')$ lies in a compact. Hence $t' < c''$, and $|y|t'/t < c''$(for some $c'' = c''(f) > 0$). Thus $t' < |y|^{-1}t = t$

(as $y \in F - \{0\}$), from which it follows that the integrals over x, g in $S^N/Z(\mathbb{A})$, $S^{N'}/Z(\mathbb{A})$ range over compacts. Hence the sum over $\gamma \in N(F)$ ranges over a finite set, depending on the support of f. This means that $\widetilde{K}_f^N$ defines a Hilbert-Schmidt operator $B : L^2_\omega(S_1^N) \to L^2_\omega(S^N)$.

Enlarging S_1, we may assume that $G(\mathbb{A}) = G(F) \cdot S_1$. Then $r(fdg)|L^2_{0,\omega}(G)$ is the composition of the injection $L^2_{0,\omega}(G) \hookrightarrow L^2_\omega(S_1^N)$, the operator B, the projection $L^2_\omega(S^N) \to V$, and the operator $A^{-1} : V \to L^2_\omega(G)$. Hence the operator $r(fdg)|L^2_{0,\omega}(G)$ is Hilbert-Schmidt. □

Corollary 6.3.2. *The operator $r_0(fdg) = r(fdg)|L^2_{0,\omega}(G)$ is of trace class (has finite trace).*

Proof. The function f can be written as $\sum_i f_i^1 * f_i^2$, where $f_i^1, f_i^2 \in C^\infty_{c,\omega}(G)$ are smooth at the finite places and highly differentiable at ∞ (m-differentiable for large m). Hence $r_0(fdg)$ is a sum of products of Hilbert-Schmidt operators, so is of trace class. □

6.4 Discrete spectrum. The **discrete spectrum** $L^2_{d,\omega}(G)$ in $L^2_\omega(G)$ is the sum of all irreducible submodules of $L^2_\omega(G)$. We have $L^2_{0,\omega}(G) \subset L^2_{d,\omega}(G)$. Also, $L^2_{1,\omega}(G) \subset L^2_{d,\omega}(G)$. We now show that $L^2_{d,\omega}(G) = L^2_{1,\omega}(G) \oplus L^2_{0,\omega}(G)$. Define $H(\mu(i\mathbb{R}))$ to be the direct integral over s in $\mathbb{R}$ of the Hilbert spaces $H(\mu(is))$ with respect to usual measure on $\mathbb{R}$.

Let us recall the definition of a direct integral (see [Di 81]) of a family of Hilbert Spaces. Let Z be a Borel space. Let μ_Z be a measure on Z. Let $H_Z = \{H_z : z \in Z\}$ be a family of Hilbert spaces. A family $F_Z = \{\eta = (\eta_z \in H_z; z \in Z)\}$ of sections of the vector bundle $H_Z \to Z$, $H_z \longmapsto z$, is called **measurable**, and its members, the sections η, are called **measurable** if they satisfy the following conditions:

(a) a section $\xi(: z \longmapsto \xi_z \in H_z)$ of $H_Z \to Z$ lies in F_Z if and only if for each section $\eta \in F_Z$, the function $z \longmapsto \langle \xi_z, \eta_z \rangle$ is measurable;
(b) there exists a countable set of sections $\{\eta_\alpha; \alpha \in A\}$ in F_Z such that for every z in Z, the set $\{(\eta_\alpha)_z; \alpha \in A\}$ is dense in H_z.

Two sections are called **equivalent** if they differ only on a subset of measure zero.

Given such Z, H_Z, and F_Z, define the **direct integral** $H = \int_Z H_z d\mu_Z$ of the family H_Z to be the space of equivalence classes of the measurable sections $\eta : z \longmapsto \eta_z \in H_z$, the square of whose norm $\|\eta\|^2 = \int_Z \|\eta_z\|^2 d\mu_Z$ is finite. The space H_Z is a Hilbert space.

The key result underlying the trace formula is the following spectral decomposition theorem, proven in Sections 8-10.

Theorem 6.2. *For any character ω of $F_\infty^+ F^\times \backslash \mathbb{A}^\times$, the space $L^2_\omega(G)$ decomposes as the direct sum $L^2_{0,\omega}(G) \oplus L^2_{1,\omega}(G) \oplus L^2_{c,\omega}(G)$. Here $L^2_{0,\omega}(G)$ is the space of cusp forms, $L^2_{1,\omega}(G)$ is the direct sum of all one-dimensional subrepresentations $g \mapsto \chi(\det g)$, $\chi^2 = \omega$, in $L^2_\omega(G)$, and the continuous spectrum $L^2_{c,\omega}(G)$ is the orthogonal complement of $L^2_{0,\omega}(G) \oplus L^2_{1,\omega}(G)$. This $L^2_{c,\omega}(G)$ is isomorphic as a $G(\mathbb{A})$-module to the $G(\mathbb{A})$-representation H, the orthogonal direct sum of $\mathrm{H}(\mu(i\mathbb{R}))$ over a set of representatives $\mu = (\mu_1, \mu_2)$ for the set of equivalence classes of pairs of characters $\mu_i : F_\infty^+ F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ with $\mu_1\mu_2 = \omega$, the equivalence relation being generated by $\mu \sim \widetilde{\mu}$ and $\mu \sim (\mu_1 \nu^s, \mu_2 \nu^{-s})$. Thus $L^2_{c,\omega}(G) = \oplus_\mu L^2_{c,\omega}(G)_\mu$, and there exist $G(\mathbb{A})$-equivariant isomorphisms $S_\mu : L^2_{c,\omega}(G)_\mu \to \mathrm{H}(\mu(i\mathbb{R}))$, $S = \sum_\mu S_\mu : L^2_{c,\omega}(G) \to \mathrm{H} = \oplus_\mu \mathrm{H}(\mu(i\mathbb{R}))$.*

The trace formula for GL(2) is an analogue of the following proposition, which can be viewed as the trace formula for the additive group $\mathbb{G}_a$.

6.5 Poisson summation formula ([Lang 94]). Let $f \in C_c^\infty(\mathbb{A})$. Let $\zeta = \prod_v \zeta_v$ be a non-trivial character of $\mathbb{A}/F$ into $\mathbb{C}^1$. Let $\widehat{f}(x) = \int_{\mathbb{A}} f(y)\zeta(xy)\, dy$ be the Fourier transform of f at $x \in \mathbb{A}$. As in Subsection 2.7, since $\mathbb{A}$ can be identified with its Pontryagin dual group, there exists a unique normalization of the Haar measure, called self-dual, for which the Fourier inversion formula $\widehat{\widehat{f}}(x) = f(-x)$ holds.

Proposition 6.5.1 (Poisson summation formula)**.** *For every $a \in \mathbb{A}\backslash\{0\}$ and $f \in C_c^\infty(\mathbb{A})$,*

$$\sum_{\alpha \in F} f(a\alpha) = |a|^{-1} \sum_{\alpha \in F} \widehat{f}(\alpha/a).$$

Also, $\mathrm{vol}(\mathbb{A}/F) = 1$.

Proof. It suffices to show

$$\sum_{\alpha \in F} f(\alpha) = \sum_{\alpha \in F} \widehat{f}(\alpha). \tag{6.5.1}$$

Indeed, if $f^a(x) = f(ax)$ for a fixed $a \in \mathbb{A}\backslash\{0\}$, then

$$\begin{aligned} \widehat{f^a}(x) &= \int_{\mathbb{A}} f^a(y)\zeta(yx)dy = \int_{\mathbb{A}} f(ay)\zeta(xy)dy \\ &= |a|^{-1} \int_{\mathbb{A}} f(y)\zeta(yx/a)dy = |a|^{-1}\widehat{f}(x/a). \end{aligned}$$

Applying (6.5.1) with f^a instead of f, we get $\sum_{\alpha \in F} f^a(\alpha) = \sum_{\alpha \in F} \widehat{f^a}(\alpha)$. The formula of the proposition follows.

It remains to prove (6.5.1), i.e., the case of $a = 1$. For $x \in \mathbb{A}$ define $g(x) = \sum_{\alpha\in F} f(x+\alpha)$. Note that all non-trivial characters of $\mathbb{A}/F$ are given by $x \mapsto \zeta(\beta x)$, $\beta \in F^\times$; see [Lang 94]. Since g is a continuous function on the compact abelian group $\mathbb{A}/F$, it has a Fourier expansion in terms of the characters of $\mathbb{A}/F$ given by $g(x) = \sum_{\beta\in F} c_\beta \zeta(-\beta x)$. The coefficients c_β can be computed as follows:

$$\begin{aligned} c_\beta &= \frac{1}{\mathrm{vol}(\mathbb{A}/F)} \int_{\mathbb{A}/F} g(x)\zeta(\beta x)dx = \frac{1}{\mathrm{vol}(\mathbb{A}/F)} \int_{\mathbb{A}/F} \sum_{\alpha\in F} f(x+\alpha)\zeta(\beta x)dx \\ &= \frac{1}{\mathrm{vol}(\mathbb{A}/F)} \int_{\mathbb{A}/F} \sum_{\alpha\in F} f(x+\alpha)\zeta(\beta(x+\alpha))dx \\ &= \frac{1}{\mathrm{vol}(\mathbb{A}/F)} \int_{\mathbb{A}} f(x)\zeta(\beta x)dx = \frac{1}{\mathrm{vol}(\mathbb{A}/F)} \widehat{f}(\beta). \end{aligned}$$

On the other hand,

$$\sum_{\alpha\in F} f(\alpha) = g(0) = \sum_{\beta\in F} c_\beta = \frac{1}{\mathrm{vol}(\mathbb{A}/F)} \sum_{\beta\in F} \widehat{f}(\beta).$$

Applying this formula with $\widehat{f}$ instead of f and using the Fourier inversion formula $\widehat{\widehat{f}}(x) = f(-x)$, we obtain

$$\sum_{\alpha\in F} f(\alpha) = \frac{1}{(\mathrm{vol}(\mathbb{A}/F))^2} \sum_{\alpha\in F} f(-\alpha).$$

This implies that $\mathrm{vol}(\mathbb{A}/F) = 1$. □

7 Trace Formula

The trace formula for GL(2) over F is an equality, for each test function $f \in \mathbb{H}_\omega$, of two expressions for the trace of the restriction $r_d(fdg)$ of the convolution operator $r(fdg)$ on $L^2_\omega(G)$ to the discrete spectrum $L^2_{d,\omega}(G)$ of $L^2_\omega(G)$. This $L^2_{d,\omega}(G)$ is the sum of all irreducible invariant subspaces of $L^2_\omega(G)$. It is the direct sum of the cuspidal spectrum $L^2_{0,\omega}(G)$, and the residual spectrum $L^2_{1,\omega}(G)$, which consists of one-dimensional representations. Thus one side of the formula is just the sum of the traces $\mathrm{tr}\,\pi(fdg)$ of the convolution operators $\pi(fdg)$ associated with the equivalence classes π of the irreducible cuspidal representations with central character ω, each counted with its multiplicity $m(\pi)$ in $L^2_{0,\omega}(G)$, plus the sum of the traces $\chi(fdg) = \mathrm{tr}\,\chi(fdg)$ of the convolution operators of the one-dimensional representations $g \mapsto \chi(g) = \chi(\det g)$, where χ is a character of $F^\times\backslash\mathbb{A}^\times$ with $\chi^2 = \omega$. The other side is given by a sum of orbital integrals, weighted (normalized, corrected) orbital integrals, weighted traces of some representations. as follows.

Theorem 7.1. *For every test function $f \in C^\infty_{c,\omega}(G)$, and Haar measure dg on $Z(\mathbb{A})\backslash G(\mathbb{A})$, the trace of the convolution operator $r_d(fdg)$ on the discrete spectrum is equal to either side of the following equality.*

$$\sum_{\pi \in L^2_{0,\omega}(G)} m(\pi)\ \operatorname{tr}\pi(fdg) + \sum_{\chi \in L^2_{1,\omega}(G)} \chi(fdg) =$$

(i) $$\operatorname{vol}(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}); dg) f(e)$$

(ii) $$+ \frac{1}{2} \sum_{T \in \mathcal{Q}} \sum_{\substack{\gamma \in Z(F)\backslash T(F) \\ \gamma \notin Z(F)}} \operatorname{vol}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}); dt) \cdot \prod_v \mathrm{OI}'(\gamma, f_v dg_v)$$

(iii)′ $$- \frac{\lambda_{-1}}{2} \sum_{\alpha \neq 1} \sum_u \mathrm{WOI}'(d(\alpha), f_u dg_u) \cdot \prod_{v \neq u} \mathrm{OI}'(d(\alpha), f_v dg_v)$$

(iv)′ $$+ \lambda_{-1} \sum_u \mathrm{WOI}'(u(1), f_u dg_u) \prod_{v \neq u} \mathrm{OI}'(u(1), f_v dg_v)$$

(iv) $$+ \left(\lambda_0 - \lambda_{-1} \sum_{\{u; f_u \neq f_u^0\}} \frac{L'(1, 1_u)}{L(1, 1_u)^2} \right) \prod_v \mathrm{OI}'(u(1), f_v dg_v)$$

(v) $$- \frac{1}{4} \sum_{\mu_1^2 = \omega} \operatorname{tr}\ M(\mu_1, \mu_1) i_B^G((\mu_1, \mu_1), fdg)$$

(vi) $$- \frac{1}{4\pi} \sum_\mu \int_{-i\infty}^{i\infty} m(\mu(iy))^{-1} m'(\mu(iy)) \operatorname{tr}\ i_B^G(\mu(iy), fdg) dy$$

$$- \frac{1}{4\pi} \sum_u \sum_\mu \int_{-i\infty}^{i\infty} \operatorname{tr}[R_u(\mu_u(iy))^{-1} R_u'(\mu_u(iy)) i_B^G(\mu_u(iy), f_u dg_u)]$$

(vii) $$\cdot \prod_{v \neq u} \operatorname{tr}\ i_B^G(\mu_v(iy), f_v\ dg_v) dy$$

Here, u and v range over $\mathbb{V}$. The sums are finite, depending on f. Also, μ ranges over a set of representatives for the pairs $\mu = (\mu_1, \mu_2)$ of characters $\mu_i : \mathbb{A}^\times / F^\times F_\infty^+ \to \mathbb{C}^1$ up to $\mu \sim \widetilde{\mu} = (\mu_2, \mu_1)$.

The geometric terms (i), (ii), (iii)′, (iv)′, and (iv) are computed in Sections 16, 17, and 19; the spectral terms (v), (vi) and (vii) are calculated in Section 21. Recall that we fix a character ω of $F_\infty^+ F^\times \backslash \mathbb{A}^\times$. Let F be a number field. Recall that $\mu(s) = (\mu_1 \nu^s, \mu_2 \nu^{-s})$, $\nu(x) = |x|$.

The spectral side of the trace formula is the left-hand side minus (v), (vi), and (vii). The geometric side of the trace formula is the sum of (i), (ii), (iii)′, (iv)′, and (iv). For the group $\mathrm{GL}(n)$, the multiplicity one theorem of [JS 81a] asserts that the multiplicities $m(\pi)$ of the cuspidal representations in the cuspidal spectrum are 1.

A more elegant form of the trace formula is obtained on observing that the non-invariant part (iv)′ of the unipotent contribution is the limit of the terms in (iii)′, giving the missing term of $\alpha = 1$ in (iii)′. To see this, we introduce in Section 18 a correction argument and the distributions $\mathrm{CWOI}'(d(\alpha))$, $\alpha \in F^\times$. The latter are obtained by adding zero globally, which is obtained as a sum of local terms each regularizing the weighted orbital integrals which blow up at $\alpha = 1$. Putting this limit as the term of $\alpha = 1$ in (iii)′, we obtain the following theorem.

Theorem 7.2. *The sum of the terms* (iii)′ *and* (iv)′ *in the trace formula can be replaced by*

$$\text{(iii)} \qquad -\frac{\lambda_{-1}}{2}\sum_{\alpha\in F^\times}\sum_{u}\mathrm{CWOI}'(d(\alpha), f_u dg_u)\cdot\prod_{v\neq u}\mathrm{OI}'(d(\alpha), f_v dg_v).)$$

Observe that all terms of the trace formula are invariant, i.e., take the same value at $f \in C^\infty_{c,\omega}(G)$ and at f^y, where $f^y(x) = f(y^{-1}xy)$, for any $y \in G(\mathbb{A})$, except the terms (iii) and (vii). In the final Section 22, we apply the Poisson summation formula to (vii) to incorporate it into (iii) and obtain purely invariant terms in the trace formula.

Chapter II
Continuous Spectrum

In this chapter, we compute the kernel of the convolution operator on the continuous spectrum, and consequently give an expression for the kernel on the cuspidal spectrum.

8 B-series

In this section, we begin the proof of Theorem 6.2 and study inner products of automorphic forms in the continuous spectrum.

8.1 Siegel domains. Recall that F^+_∞ is the group of idèles whose components at the finite places are 1 and whose components at the infinite places are positive and equal to each other. The isomorphism $F^+_\infty \to \mathbb{R}^\times_{>0}$ is denoted $t \mapsto |t|$. Let $c > 0$. Fix a compact subset C of $\mathbb{A}$ and a compact subset C' of $\mathbb{A}^\times$. A **Siegel domain** $S = S(c, C, C')$ in $G(\mathbb{A})$ is a set of points of the form $g = zu(x)d(ta, 1)k$, where $x \in C$, $t \in F^+_\infty$ with $|t| > c$, $a \in C'$, $z \in \mathbb{A}^\times$, and k is in the standard

maximal compact subgroup K of $G(\mathbb{A})$. For sufficiently large c^{-1}, C, C' we have $G(\mathbb{A}) = G(F)S$; see [PR 93]. Then S contains a fundamental domain for $G(F)\backslash G(\mathbb{A})$.

Proposition 8.1.1. *For a Siegel domain $S = S(c, C, C')$ and $c' > 0$, the set $\{\gamma \in B(F)\backslash G(F) : \delta(\gamma g) \geq c' \text{ for some } g \in S\}$ is finite.*

Proof. Note that $\delta(p\gamma g) = \delta(\gamma g)$ for all $p \in B(F)$, so we can indeed deal with γ in $B(F)\backslash G(F)$. A proof when $F = \mathbb{Q}$ is given in [Kn 97, Proposition 7.8], where the key step, Lemma 7.7, relies on Lemma 6.7.

This computation is also done in [PR 93, Subsection 5.2], but in the context of a reductive connected group over $\mathbb{Q}$, reducing it to reduction theory for $G(\mathbb{R})$ and $G(\mathbb{Z})$ (which reduces to [Kn 97, Lemma 6.7] when $G = \mathrm{GL}(2)$). The general case of a group over F follows on applying the result over $\mathbb{Q}$ to the $\mathbb{Q}$-group $\mathrm{R}_{F/\mathbb{Q}}G$ obtained from G by restriction of scalars from F to $\mathbb{Q}$; see also [Ar 05, p. 37] for references to Borel and Harish-Chandra. □

8.2 B**-automorphic forms.** Assume the central character ω is 1 on F_∞^+. Thus it is a (unitary) character on $F^\times\backslash\mathbb{A}^1 = F_\infty^+F^\times\backslash\mathbb{A}^\times$. Let $\mu = (\mu_1, \mu_2)$ be a pair of characters on $F_\infty^+F^\times\backslash\mathbb{A}^\times$ with $\mu_1\mu_2 = \omega$.

A (μ, B) **-automorphic form** φ_μ is a function on $G(\mathbb{A})$ that is a linear combination of a (μ_1, B)-automorphic form and a (μ_2, B)-automorphic form. We often delete the word automorphic here. A (μ_1, B)-**form** φ_{μ_1} satisfies

$$\varphi_{\mu_1}(znd(a)g) = \omega(z)\mu_1(a)\varphi_{\mu_1}(g)$$

for all $n \in N(\mathbb{A})$, $g \in G(\mathbb{A})$, $a \in \mathbb{A}^1$ and $z \in Z(\mathbb{A})$, and is square integrable on $G(\mathbb{A})$ modulo $Z(\mathbb{A})N(\mathbb{A})B(F)$. For a (μ_2, B)-form, replace μ_1 by μ_2. Thus a $(\widetilde{\mu}, B)$-form is a (μ, B)-form. Note that $B(F) \subset N(\mathbb{A})Z(\mathbb{A})A^1$, and $F^\times\backslash\mathbb{A}^1$ is compact. Let $L^2_{\mu,\omega,B} = L^2_{\mu,\omega}(Z(\mathbb{A})N(\mathbb{A})B(F)\backslash G(\mathbb{A}))$ denote the space of such functions φ_μ. Denote by $(\cdot,\cdot)_B$ the inner product on $L^2_{\mu,\omega,B}$.

We also call φ on $G(\mathbb{A})$ B-**automorphic** if $\varphi(znag) = \omega(z)\varphi(g)$ ($z \in Z(\mathbb{A})$, $a \in A(F)$, $n \in N(\mathbb{A})$, $g \in G(\mathbb{A})$).

8.3 B**-series.** The B-series of $\varphi_\mu \in L^2_{\mu,\omega,B}$ is the sum

$$\Phi_\mu(g) = \sum_{\gamma \in B(F)\backslash G(F)} \varphi_\mu(\gamma g).$$

To simplify notation, we sometimes write φ for φ_μ and Φ for Φ_μ, but write φ_μ and Φ_μ when the dependence on μ needs to be emphasized.

Proposition 8.3.1. *If φ_μ is a (μ, B)-automorphic form compactly supported* mod $Z(\mathbb{A})N(\mathbb{A})B(F)$*, then the sum which defines Φ_μ is finite and hence converges.*

This follows from Proposition 8.1.1.

Proposition 8.3.2. *Let ϕ be an automorphic form in $L^2_\omega(G)$. Let Φ_μ be the B-series of $\varphi_\mu \in L^2_{\mu,\omega,B}$. Then $(\phi, \Phi_\mu) = (\phi_N, \varphi_\mu)_B$.*

Proof. We compute the scalar product:

$$\begin{aligned}(\phi, \Phi_\mu) &= \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi(g)\bar{\Phi}_\mu(g)dg = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi(g) \sum_{\gamma\in B(F)\backslash G(F)} \bar{\varphi}_\mu(\gamma g)dg \\ &= \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \sum_{\gamma\in B(F)\backslash G(F)} \phi(\gamma g)\bar{\varphi}_\mu(\gamma g)dg = \int_{B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \phi(g)\bar{\varphi}_\mu(g)dg \\ &= \int_{B(F)N(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \bar{\varphi}_\mu(g)\, dg \int_{N(F)\backslash N(\mathbb{A})} \phi(ng)dn = (\phi_N, \varphi_\mu)_B. \end{aligned}$$

□

Corollary 8.3.3. *If ϕ is cuspidal, then $(\phi, \Phi_\mu) = 0$ for any B-series Φ_μ.*

Proof. Indeed, $(\phi, \Phi_\mu) = (\phi_N, \varphi_\mu)_B$, and $(\phi_N, \varphi_\mu)_B = 0$ as $\phi_N = 0$. □

Recall that the Haar measure on $N(\mathbb{A})$ is normalized by $\mathrm{vol}(N(F)\backslash N(\mathbb{A})) = 1$.

Proposition 8.3.4. *Let Φ_μ be the B-series of φ_μ. Then the constant term of Φ_μ is*

$$\Phi_{\mu N}(g) = \varphi_\mu(g) + (M\varphi_\mu)(g), \tag{8.3.1}$$

where

$$(M\varphi_\mu)(g) = \int_{N(\mathbb{A})} \varphi_\mu(wng)dn \tag{8.3.2}$$

is a $(\widetilde{\mu}, B)$-automorphic form.

Proof. By the Bruhat decomposition for G,

$$\Phi_\mu(g) = \varphi_\mu(g) + \sum_{v\in N(F)} \varphi_\mu(wvg) \quad \text{with} \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus

$$\begin{aligned}\Phi_{\mu N}(g) &= \int_{N(F)\backslash N(\mathbb{A})} \varphi_\mu(ng)dn + \int_{N(F)\backslash N(\mathbb{A})} \sum_{v\in N(F)} \varphi_\mu(wvng)dn \\ &= \varphi_\mu(g) + (M\varphi_\mu)(g). \end{aligned}$$

□

If φ_μ is a (μ_1, B)-automorphic form, then $M\varphi_\mu$ is a (μ_2, B)-automorphic form. For this reason, we define (μ, B)-automorphic forms to be in the span of (μ_1, B) and (μ_2, B) forms, so that the space of (μ, B)-forms is the same as the space of $(\widetilde{\mu}, B)$-forms and is invariant under the intertwining operator M.

Note that the assumption that φ_μ is compactly supported on $G(\mathbb{A})$ modulo $Z(\mathbb{A})N(\mathbb{A})B(F)$ need not imply that $M\varphi_\mu$ is compactly supported.

Corollary 8.3.5. *Let $\Phi_{1\mu}$, $\Phi_{2\mu}$ be the B-series associated with functions $\varphi_{1\mu}$, $\varphi_{2\mu} \in L^2_{\mu,\omega,B}$. Then*

$$(\Phi_{1\mu}, \Phi_{2\mu}) = (\Phi_{1\mu N}, \varphi_{2\mu})_B = (\varphi_{1\mu}, \varphi_{2\mu})_B + (M\varphi_{1\mu}, \varphi_{2\mu})_B.$$

If μ and μ' are in different connected components, then $(\Phi_{1\mu}, \Phi_{2\mu'}) = 0$. Here, μ, μ' range over the manifold of the start of Section 8.2. They are in the same connected component when $\mu' = \mu(\nu^s, \nu^{-s})$, $s \in \mathbb{C}$.

Proposition 8.3.6. *For any two (μ, B)-automorphic forms $\varphi_{1\mu}$ and $\varphi_{2\mu}$, we have*

$$(M\varphi_{1\mu}, \varphi_{2\mu})_B = (\varphi_{1\mu}, M\varphi_{2\mu})_B.$$

Proof. We have

$$\begin{aligned}(\varphi_{1\mu}, \varphi_{2\mu})_B + (M\varphi_{1\mu}, \varphi_{2\mu})_B &= (\Phi_{1\mu}, \Phi_{2\mu}) \\ &= \overline{(\Phi_{2\mu}, \Phi_{1\mu})} = \overline{(\varphi_{2\mu}, \varphi_{1\mu})}_B + \overline{(M\varphi_{2\mu}, \varphi_{1\mu})}_B,\end{aligned}$$

$(\varphi_{1\mu}, \varphi_{2\mu})_B = \overline{(\varphi_{2\mu}, \varphi_{1\mu})}_B$, and $\overline{(M\varphi_{2\mu}, \varphi_{1\mu})}_B = (\varphi_{1\mu}, M\varphi_{2\mu})_B$. □

9 Mellin transform

9.1 Mellin's inversion formula. Let $\psi : \mathbb{R}^\times_{>0} \to \mathbb{C}$ be integrable. The function $\phi : \mathbb{C} \to \mathbb{C}$ given by

$$\phi(s) = \int_{\mathbb{R}^\times_{>0}} \psi(x)\, x^{s-1} dx$$

is called the **Mellin transform** of ψ. For $c \in \mathbb{R}$, ϕ is invertible via the **Mellin inversion formula**

$$\psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s)\, x^{-s} ds, \;\; x \in \mathbb{R}^\times_{>0}.$$

The Mellin transform is closely related to the Fourier transform and to the Laplace transform; see [Ti 76, p. 443].

9.2 Definition of the Mellin transform. The **Mellin transform** of a (μ, B)-automorphic form φ_μ is

$$\widehat{\varphi}_\mu(g,s) = \int_{F_\infty^+} \varphi_\mu(d(t)g)|t|^{-s-\frac{1}{2}}\, d^\times t$$

whenever the integral converges. By the Mellin inversion formula, the set of functions $\widehat{\varphi}_\mu(s) : g \mapsto \widehat{\varphi}_\mu(g,s)$ of $G(\mathbb{A})$ is a dense subspace of $\mathrm{H}(\mu(s))$ for each $s \in \mathbb{C}$ as φ_μ varies over $L^2_{\mu,\omega,B}$. The Mellin inversion formula also implies

$$\varphi_\mu(g) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \widehat{\varphi}_\mu(g,s)ds$$

with $x \in \mathbb{R}$, where the integral converges.

Note that $\widehat{\varphi}_\mu(d(t)g,s) = |t|^{s+1/2}\widehat{\varphi}_\mu\ (g,s)$, $t \in F_\infty^+$ by definition of $\widehat{\varphi}_\mu$; recall that $|t|$ is defined in Section 2.7.

Proposition 9.2.1. *The function $\widehat{\varphi}_\mu$ defines a section of the vector bundle* $\mathrm{H}(\mu(s))$.

Proof. We have

$$\begin{aligned}
\widehat{\varphi}_\mu(u(x)d(a',a'')g,s) &= \int_{F_\infty^+} \varphi_\mu(d(t)u(x)d(a',a'')g)|t|^{-s-\frac{1}{2}}\, d^\times t \\
&= \int_{F_\infty^+} \varphi_\mu(u(tx)d(a',a'')d(t)g)|t|^{-s-\frac{1}{2}}\, d^\times t \\
&= \int_{F_\infty^+} (\delta^{\frac{1}{2}}\mu)(d(a',a''))\varphi_\mu(d(t)g)|t|^{-s-\frac{1}{2}} d^\times t \\
&= (\delta^{\frac{1}{2}}\mu)(d(a',a''))\widehat{\varphi}_\mu(g,s).
\end{aligned}$$

As $\{d(a',a'')d(t); a' \in \mathbb{A}^1, a'' \in \mathbb{A}^\times\} = A(\mathbb{A})$, since $\mathbb{A}^\times = \mathbb{A}^1 \times F_\infty^+$, it follows that $\widehat{\varphi}_\mu(u(x)ag,s) = (\delta^{1/2}\mu(s))(a)\widehat{\varphi}_\mu(g,s)$, and $\widehat{\varphi}_\mu$ defines a section of $\mathrm{H}(\mu(s))$. □

Lemma 9.3. *If $\widehat{\varphi}_{i\mu}$ is the Mellin transform of the (μ, B)-automorphic form $\varphi_{i\mu}$, then*

$$\widehat{\varphi}_{1\mu}(d(a)k,s)\overline{\widehat{\varphi}}_{2\mu}(d(a)k,-\overline{s}) = \int_{F_\infty^+} \xi(z;a,k)|z|^{-s-\frac{1}{2}}d^\times z,$$

where

$$\xi(z;a,k) = \int_{F_\infty^+} \varphi_{1\mu}(d(zta)k)\,\overline{\varphi}_{2\mu}(d(ta)k)|t|^{-1}d^\times t.$$

Proof. We have

$$\begin{aligned}
&\widehat{\varphi}_{1\mu}(d(a)k, s)\overline{\widehat{\varphi}}_{2\mu}(d(a)k, -\overline{s}) \\
&\quad = \int_{F_\infty^+} \varphi_{1\mu}(d(za)k)|z|^{-s-\frac{1}{2}} d^\times z \cdot \int_{F_\infty^+} \overline{\varphi}_{2\mu}(d(ta)k)|t|^{s-\frac{1}{2}} d^\times t \\
&\quad = \int_{t\in F_\infty^+} \int_{z\in F_\infty^+} \varphi_{1\mu}(d(z^{-1}a)k)|z|^{s+\frac{1}{2}} d^\times z \cdot \overline{\varphi}_{2\mu}(d(ta)k)|t|^{s-\frac{1}{2}} d^\times t \\
&\quad = \int_{z\in F_\infty^+} \int_{t\in F_\infty^+} \varphi_{1\mu}(d(z^{-1}ta)k)|z|^{s+\frac{1}{2}} d^\times z \ \cdot \overline{\varphi}_{2\mu}(d(ta)k)|t|^{-s-\frac{1}{2}}|t|^{s-\frac{1}{2}} d^\times t \\
&\quad = \int_{z\in F_\infty^+} \left(\int_{t\in F_\infty^+} \varphi_{1\mu}(d(zta)k) \cdot \overline{\varphi}_{2\mu}(d(ta)k)|t|^{-1} d^\times t \right) |z|^{-s-\frac{1}{2}} d^\times z \\
&\quad = \int_{z\in F_\infty^+} \xi(z; a, k)|z|^{-s-\frac{1}{2}} d^\times z.
\end{aligned}$$

□

Proposition 9.3.1. *The inner product* $(\varphi_{1\mu}, \varphi_{2\mu})_B$ *on* $L^2_{\mu,\omega,B}$ *is*

$$(\varphi_{1\mu}, \varphi_{2\mu})_B = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} (\widehat{\varphi}_{1\mu}(s), \widehat{\varphi}_{2\mu}(-\overline{s}))\, ds.$$

The scalar product $(\widehat{\varphi}_{1\mu}(s), \widehat{\varphi}_{2\mu}(-\overline{s}))$ *on* $\mathrm{H}(\mu(s)) \times \mathrm{H}(\overline{\mu}^{-1}(-\overline{s}))$ *is as in Section* 3.3. Also, μ is unitary; thus $\overline{\mu}^{-1} = \mu$.

Proof. Using Lemma 9.3 and the Mellin inversion formula of Subsection 9.1, we have

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} (\widehat{\varphi}_{1\mu}(s), \widehat{\varphi}_{2\mu}(-\overline{s}))ds \\
&\quad = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{F^\times\backslash\mathbb{A}^1} \int_K \widehat{\varphi}_{1\mu}(d(a)k, s)\overline{\widehat{\varphi}}_{2\mu}(d(a)k, -\overline{s})\, dk\, d^\times a\, ds \\
&\quad = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} ds \int_{F_\infty^+} d^\times t \int_{F^\times\backslash\mathbb{A}^1} \int_K \xi(t, a, k)|t|^{-s-\frac{1}{2}} dk\, d^\times a \\
&\quad = \int_{F^\times\backslash\mathbb{A}^1} \int_K \varphi_{1\mu}(d(a)k)\overline{\varphi}_{2\mu}(d(a)k)\, d^\times a\, dk = (\varphi_{1\mu}, \varphi_{2\mu})_B.
\end{aligned}$$

□

Proposition 9.3.2. *The scalar product* $(M\varphi_{1\mu}, \varphi_{2\mu})_B$ *on* $L^2_{\mu,\omega,B}$ *for the functions* $M\widehat{\varphi}_{1\mu}(-s) \in \mathrm{H}(\widetilde{\mu}(-s))$, *and* $\widehat{\varphi}_{2\mu}(\overline{s}) \in \mathrm{H}(\overline{\widetilde{\mu}}^{-1}(\overline{s}))$, *is equal to*

$$\frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} (M\widehat{\varphi}_{1\mu}(-s), \widehat{\varphi}_{2\mu}(\overline{s}))ds.$$

Proof. We integrate on a line $x > 1/2$ on which $\widehat{M\varphi}_{1\mu}(-s)$ is convergent. For $\mathrm{Re}(s) > 1/2$,

$$\begin{aligned}
\widehat{M\varphi_\mu}(g,s) &= \int_{F_\infty^+} M\varphi_\mu(d(t)g)|t|^{-s-\frac{1}{2}}d^\times t \\
&= \int_{F_\infty^+} |t|^{-s-\frac{1}{2}}d^\times t \int_{\mathbb{A}} \varphi_\mu(\omega u(x)d(t)g)dx \\
&= \int_{F_\infty^+} |t|^{-s-\frac{1}{2}}d^\times t \int_{\mathbb{A}} \varphi_\mu(\omega d(t)u(x/t)g)dx \\
&= \int_{F_\infty^+} |t|^{-s+\frac{1}{2}}d^\times t \int_{\mathbb{A}} \varphi_\mu(d(t^{-1})\omega u(x)g)dx \\
&= \int_{F_\infty^+} |t|^{s-\frac{1}{2}}d^\times t \int_{\mathbb{A}} \varphi_\mu(d(t)\omega u(x)g)dx \\
&= \int_{N(\mathbb{A})} \widehat{\varphi}_\mu(\omega ng,-s)dn = (M\widehat{\varphi}_\mu)(g,-s).
\end{aligned}$$

□

Write $M(\mu)$ for the operator $M : \mathrm{H}(\mu) \to \mathrm{H}(\widetilde{\mu})$ defined by (8.3.2). Thus, for $\widehat{\varphi}_\mu(s) \in \mathrm{H}(\mu(s))$, we write $M(\mu(s))\widehat{\varphi}_\mu(s)$ for $M\widehat{\varphi}_\mu(s)$.

Corollary 9.3.3. *Let $\Phi_{1\mu}$ be a B-series of $\varphi_{1\mu}$, and $\Phi_{2\mu}$ a B-series of $\varphi_{2\mu}$. Then*

$$(\Phi_{1\mu},\Phi_{2\mu}) = \frac{1}{2\pi i}\int (\widehat{\varphi}_{1\mu}(s),\ \widehat{\varphi}_{2\mu}(-\overline{s}))\,ds + \frac{1}{2\pi i}\int (M(\mu(s))\widehat{\varphi}_{1\mu}(s),\widehat{\varphi}_{2\widetilde{\mu}}(\overline{s}))ds,$$

where the integrals are taken over any vertical line $\mathrm{Re}(s) > 1/2$ and the pairings are on $\mathrm{H}(\mu(s)) \times \mathrm{H}(\overline{\mu}^{-1}(-\overline{s}))$ and $\mathrm{H}(\widetilde{\mu}(-s)) \times \mathrm{H}(\overline{\widetilde{\mu}}^{-1}(\overline{s}))$.

Proof. Since $(\Phi_{1\mu},\Phi_{2\mu}) = (\varphi_{1\mu},\varphi_{2\mu})_B + (M\varphi_{1\mu},\varphi_{2\mu})_B$ by Corollary 8.3.5, this corollary follows from Propositions 9.2.1, 9.3.1, and 9.3.2. □

Proposition 9.3.4. *For $\mathrm{Re}(s) > 1/2$, $\psi_{1\mu} \in \mathrm{H}(\mu(s))$, and $\psi_{2\mu} \in \mathrm{H}(\overline{\widetilde{\mu}}^{-1}(\overline{s}))$,*

$$(M(\mu(s))\psi_{1\mu},\psi_{2\mu}) = (\psi_{1\mu},M(\overline{\widetilde{\mu}}^{-1}(\overline{s}))\psi_{2\mu}).$$

On the left-hand side, the pairing is on $\mathrm{H}(\widetilde{\mu}(-s)) \times \mathrm{H}(\overline{\widetilde{\mu}}^{-1}(\overline{s}))$. On the right-hand side, the pairing is on $\mathrm{H}(\mu(s)) \times \mathrm{H}(\overline{\mu}^{-1}(-\overline{s}))$.

Proof. The Mellin transform $\varphi_\mu \mapsto \widehat{\varphi}_\mu(s)$ provides an inner product preserving isomorphism $L^2_{\mu,\omega,B} \to \mathrm{H}(\mu(s))$, by Proposition 9.3.2. The identity of this proposition then follows from Proposition 8.3.6. □

Corollary 9.3.5. *The adjoint $M(\mu(s))^*$ of the operator $M(\mu(s))$ from $\mathrm{H}(\mu(s))$ to $\mathrm{H}(\widetilde{\mu(s)})$ is*

$$M(\mu(s))^* = M(\widetilde{\overline{\mu}}^{-1}(\overline{s})) = M(\widetilde{\overline{\mu(s)}}^{-1}).$$

When μ is unitary (thus $\mu\overline{\mu} = 1$) and s is imaginary, $M(\mu(s))$ is unitary, $R(\mu(s))$ is unitary, and so is each local $R(\mu_v(s))$ if μ_v is unitary and s is imaginary.

Proof. The last claim follows as $\overline{\mu}^{-1} = \mu$ and $\overline{s} = -s$ implies $\widetilde{\overline{\mu}}^{-1}(\overline{s}) = \widetilde{\mu}(-s)$; thus $M(\mu(s))^* = M(\widetilde{\mu}(-s)) = M(\mu(s))^{-1}$. Replacing μ by $\widetilde{\mu}$ yields $m(\mu) = L(1, \mu_{12}^{-1})/L(1, \mu_{12})$, where $\mu_{12} = \mu_1/\mu_2$ is replaced by $m(\overline{\mu})$ if $\overline{\mu}_{12} = \mu_{12}^{-1}$. Hence $R(\mu(s))$ is unitary. Each local μ_{0v} can be embedded as a component of a global $\mu_0\nu^s$. □

If φ_μ is replaced by $r(g)\varphi_\mu : h \mapsto \varphi_\mu(hg)$, then Φ_μ (respectively, $\widehat{\varphi}_\mu(s)$) is replaced by its right translate, i.e., by $r(g)\Phi$ (respectively, $i_B^G(\mu, g)\widehat{\varphi}_\mu(s)$).

10 Global intertwining operators

10.1 Local-global relations. Let μ be a character of $A(\mathbb{A})$ whose restriction to $Z(\mathbb{A})$ is ω. As in 8.3.4, define the operator $M(\mu)$ from $\mathrm{H}(\mu)$ to $\mathrm{H}(\widetilde{\mu})$ by

$$(M(\mu)\psi)(g) = \int_{N(\mathbb{A})} \psi(\omega ng)\, dn, \quad \psi \in \mathrm{H}(\mu).$$

We cannot define $M(\mu)$ to be the product $\otimes_v M(\mu_v)$, as the product does not converge. However, the product $\otimes_v R(\mu_v)$ does converge in the following sense. Denote by μ_{iv} and μ_v the restrictions of μ_i and μ, respectively, to $F_v^\times \hookrightarrow \mathbb{A}^\times$. Then $\mu_i = \otimes_v \mu_{iv}$ $(i = 1, 2)$. The space $\mathrm{H}(\mu)$ of the global induced representation $i_B^G(\mu)$ is spanned by the local products $\otimes_v \psi_v$ with $\psi_v \in \mathrm{H}(\mu_v)$ for all v and $\psi_v = \psi_v^0$ for almost all v, where μ_v is unramified. Then $R(\mu_v)\psi_v^0 = \psi_v^0$ for such v, and $R(\mu)$ is defined to be the operator $\otimes_v R(\mu_v)$ which takes $\otimes_v \psi_v$ to $\otimes_v R(\mu_v)\psi_v$, a well defined vector in $\mathrm{H}(\widetilde{\mu}_v)$.

10.2 The intertwining operator. Recall that $M(\mu_v) = m(\mu_v)R(\mu_v)$ for all μ_v. The product $m(\mu)$ over all v of

$$m(\mu_v) = \frac{L(0, \mu_{1v}/\mu_{2v})}{L(1, \mu_{1v}/\mu_{2v})\varepsilon(0, \mu_{1v}/\mu_{2v}, \zeta_v)},$$

where ζ_v is the component at v of a fixed non-trivial additive character ζ of $\mathbb{A}/F$, converges for all μ with $|\mu_1/\mu_2| = \nu^s$, $s > 1/2$. It is independent of ζ. Thus

$$M(\mu) = m(\mu)R(\mu), \quad m(\mu) = \prod_v m(\mu_v) = \frac{L(0, \mu_1/\mu_2)}{L(1, \mu_1/\mu_2)\varepsilon(0, \mu_1/\mu_2)}.$$

This gives the analytic continuation of $M(\mu)$ to $\mathrm{Re}(s) \geq 0$, since $m(\mu)$ has meromorphic continuation to the entire complex plane by the functional equation for the L-function $L(\mu_1)$, and since the operator $R(\mu)$ depends analytically on μ on this domain.

10.3 Unitarity. The functional equation for the L-function expresses $m(\mu)$ as

$$m(\mu) = \frac{L(1, \mu_2/\mu_1)}{L(1, \mu_1/\mu_2)}.$$

Since $R(\widetilde{\mu})R(\mu) = 1$, we conclude that $M(\widetilde{\mu})M(\mu) = 1$, and from $M(\mu)^* = M(\overline{\widetilde{\mu}}^{-1})$ (see Corollary 9.3.5), that $M(\mu)^* = M(\overline{\mu}^{-1})^{-1}$. If μ is unitary, thus $\overline{\mu}^{-1} = \mu$, then $M(\mu)^* = M(\mu)^{-1}$.

We summarize the analytic behavior of $M(\mu)$ as follows. Write $|\mu_1/\mu_2| = \nu^{2s}$ with s real. Then $M(\mu)$ is meromorphic in the half-plane $s \geq 0$, and its only poles there are simple and occur for $\mu_1 = \mu_0\nu^{1/2}$, $\mu_2 = \mu_0\nu^{-1/2}$, $\mu_0^2 = \omega$. As ω is 1 on $F_\infty^+ \subset \mathbb{A}^\times$, $\mu_0 = 1$ on F_∞^+ also. The residue there is a scalar multiple of the operator $R(\mu)$. By Proposition 5.3.1, the range of $R(\mu)$ is the one-dimensional space spanned by the function $g \mapsto \mu(\det g)$. Returning to $M(\mu)$ we obtain the following result.

Theorem 10.1. *Let $\mu = (\mu_1, \mu_2)$ with μ_i characters of $F^\times\backslash\mathbb{A}^\times$ and $\mu_1\mu_2 = \omega$. Let $\nu^{2s} = |\mu_1/\mu_2|$. As a function of μ, the operator $M(\mu)$ is meromorphic in the complex half-plane $\mathrm{Re}(s) \geq 0$. It satisfies the functional equation*

$$M(\widetilde{\mu})M(\mu) = \mathrm{Id}\,. \tag{10.3.1}$$

Its only pole in the half-plane $\mathrm{Re}(s) \geq 0$ is at $s = 1/2$ when $\mu = (\mu_0\nu^{1/2}, \mu_0\nu^{-1/2})$. The residue $\mathrm{Res}_{s=1/2} M(\mu(s)) = \lim_{s\to 1/2}(s - \frac{1}{2})M(\mu(s))$ satisfies

$$(\mathrm{Res}_{s=1/2} M(\mu(s))\widehat{\varphi}_{1\mu}(1/2), \widehat{\varphi}_{2\mu}(1/2)) = c(\widehat{\varphi}_{1\mu}(1/2), \mu_0\circ\det)\overline{(\widehat{\varphi}_{2\mu}(1/2), \mu_0 \circ \det)},$$

where c is a known constant.

The last assertion follows from Proposition 5.3.1.

10.4 Proof of Theorem 6.2. The orthogonal complement of $L^2_{0,\omega}(G)$ in $L^2_\omega(G)$ is spanned by the B-series, for all μ, by Proposition 8.3.2. Their inner product is given by Corollary 9.3.3. Using the analytic continuation of $M(\mu(s))$ to $\mathrm{Re}(s) \geq 0$ and the description of the only possible pole in the half-plane, at $s = 1/2$ and $\mu = \widetilde{\mu}$, of Theorem 10.1, we apply the Cauchy Residue Theorem and move the line of integration in Corollary 9.3.3 from $x+i\mathbb{R}$ to $i\mathbb{R}$, picking up the residues.

We obtain, for the B-series $\Phi_{1\mu}$, $\Phi_{2\mu}$, obtained from the (μ, B)-automorphic forms $\varphi_{1\mu}$, $\varphi_{2\mu}$, the formula

$$(\Phi_{1\mu}, \Phi_{2\mu}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}((\widehat{\varphi}_{1\mu}(iy), \widehat{\varphi}_{2\mu}(iy)) + (M(\mu(iy))\widehat{\varphi}_{1\mu}(iy), \widehat{\varphi}_{2\widetilde{\mu}}(-iy)))dy$$
$$+ c\cdot\delta(\mu, \widetilde{\mu})\cdot(\widehat{\varphi}_{1\mu}(1/2), \mu_1\circ\det)\overline{(\widehat{\varphi}_{2\mu}(1/2), \mu_1\circ\det)} \tag{10.4.1}$$

for an explicitly computable constant c. Here, $\delta(\mu, \widetilde{\mu})$ is 1 if $\mu_1 = \mu_2$ and it is 0 otherwise. Note that $\mu_1\circ\det \in \mathrm{H}(\mu(-1/2))$. Using the sesquilinear pairing as in Subsection 3.3, we have

$$\begin{aligned}(\widehat{\varphi}_{1\mu}(1/2), \mu_1\circ\det) &= \int_{B(F)N(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})}\varphi_{1\mu}(g)\overline{\mu}_1(\det g)dg\\ &= \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})}\Phi_{1\mu}(g)\overline{\mu}_1(\det g)dg,\end{aligned}$$

as $\Phi_{1\mu}$ is the B-series associated with $\varphi_{1\mu}$. So we may write the second term in (10.4.1) as

$$c\cdot\delta(\mu, \widetilde{\mu})\cdot(\Phi_{1\mu}, \mu_1\circ\det)\overline{(\Phi_{2\mu}, \mu_1\circ\det)}, \tag{10.4.2}$$

the scalar product being now in $L^2_\omega(G)$.

Recall that $M(\mu(iy))^* = M(\mu(iy))^{-1}$, namely, $M(\mu(iy))$ is unitary, as μ is unitary and iy is on the imaginary axis $\overline{s} = -s$. Introduce the element of the direct integral

$$(a(\varphi_\mu))(iy) = \frac{1}{2}(\widehat{\varphi}_\mu(iy) + M(\widetilde{\mu}(-iy))\widehat{\varphi}_{\widetilde{\mu}}(-iy)). \tag{10.4.3}$$

Then $M(\widetilde{\mu}(-iy))a(\varphi_{\widetilde{\mu}})(-iy) = (a(\varphi_\mu))(iy)$, and

$$\begin{aligned}4(a(\varphi_{1\mu})(iy), a(\varphi_{2\mu})(iy)) &= (\widehat{\varphi}_{1\mu}(iy), \widehat{\varphi}_{2\mu}(iy)) + (\widehat{\varphi}_{1\widetilde{\mu}}(-iy), \widehat{\varphi}_{2\widetilde{\mu}}(-iy))\\ &+ (\widehat{\varphi}_{1\mu}(iy), M(\widetilde{\mu}(-iy))\widehat{\varphi}_{2\widetilde{\mu}}(-iy)) + (M(\widetilde{\mu}(-iy))\widehat{\varphi}_{1\widetilde{\mu}}(-iy), \widehat{\varphi}_{2\mu}(iy)).\end{aligned}$$

Hence the sum of the first terms in $(\Phi_{1\mu}, \Phi_{2\mu})$ and $(\Phi_{1\widetilde{\mu}}, \Phi_{2\widetilde{\mu}})$ can be expressed as

$$\frac{2}{\pi}\int_{-\infty}^{\infty}(a(\varphi_{1\mu})(iy), a(\varphi_{2\mu})(iy))dy.$$

Note that the map $\varphi_{1\mu}\mapsto\Phi_{1\mu}$ is equivariant under right translation by $G(\mathbb{A})$: if $\varphi_{1\mu}$ is replaced by $h\mapsto\varphi_{1\mu}(hg)$, then $\Phi_{1\mu}$ is replaced by $r(g)\Phi_{1\mu}$.

We obtained an isometric $G(\mathbb{A})$-equivariant isomorphism

$$\Phi = \sum_\mu \Phi_\mu \mapsto (((a(\varphi_\mu))(iy))_\mu, (\delta(\mu, \widetilde{\mu})\cdot\widehat{\varphi}_\mu(1/2))_\mu),$$

from $L^2_{0,\omega}(G)^\perp$ to $\mathrm{H} \oplus (\oplus_{\mu_1} \mathbb{C} \cdot \mu_1(\det))$, where $\mathrm{H} = \oplus_\mu \mathrm{H}(\mu(i\mathbb{R}))$.

Note that the sum of (10.4.2) over all μ_1 with $\mu_1^2 = \omega$ is equal to the scalar product of the orthogonal projections of $\Phi_{1\mu}$, $\Phi_{2\mu}$ on

$$L^2_{1,\omega}(G) = \oplus_{\{\mu_i;\mu_1^2=\omega\}} \mathbb{C} \cdot \mu_1(\det).$$

Hence $c = 1/\operatorname{vol}(G(F)Z(\mathbb{A})\backslash G(\mathbb{A}))$.

This is summarized in the following corollary.

Corollary 10.4.1. *The $G(\mathbb{A})$-module $L^2_\omega(G)$ is the $G(\mathbb{A})$-equivariant direct sum of $L^2_{0,\omega}(G)$, $L^2_{1,\omega}(G)$, and $L^2_{c,\omega}(G)$. The map $S = \sum_\mu S_\mu : L^2_{c,\omega}(G) \to \mathrm{H}$, defined by $S_\mu : L^2_{c,\omega}(G)_\mu \to \mathrm{H}(\mu(i\mathbb{R}))$, $(S_\mu(\Phi_\mu))(y) = (a(\varphi_\mu))(iy)$ and $S_\mu(\Phi_{\mu'}) = 0$ if μ and μ' are not equivalent as defined in Theorem 6.2, is a $G(\mathbb{A})$-equivariant isomorphism of $L^2_{c,\omega}(G) = \oplus_\mu L^2_{c,\omega}(G)_\mu$ with $\mathrm{H} = \oplus_\mu \mathrm{H}(\mu(i\mathbb{R}))$.*

11 Truncation operator

11.1 Properties of truncation operator. We use a truncation operator to obtain the analytic continuation of Eisenstein series and to compute terms in the trace formula. If $T > 1$, let $\widehat{\tau}_T$ denote the characteristic function of $[T, \infty)$. For any function ϕ on $G(F)\backslash G(\mathbb{A})$, set

$$\Lambda^T \phi(g) = \phi(g) - \sum_{\gamma \in B(F)\backslash G(F)} \phi_N(\gamma g)\widehat{\tau}_T(\delta(\gamma g)). \tag{11.1.1}$$

The second term here is a B-series attached to a function with support in the set $\{g; \delta(g) > T\}$. Thus, if g is in a Siegel domain, the series has only finitely many terms; see Proposition 8.1.1.

Clearly, if ϕ is a cuspidal function, i.e., $\phi_N \equiv 0$, then $\Lambda^T \phi = \phi$. For the general case, we use the following proposition.

Proposition 11.1.1. *Suppose g in $G(\mathbb{A})$ satisfies $\delta(g) > 1$. Suppose $\gamma \in G(F)$ satisfies $\delta(\gamma g) > 1$. Then $\gamma \in B(F)$.*

Proof. By the Bruhat decomposition, $G = B \cup BwN$. Since $\delta(p\gamma g) = \delta(\gamma g)$ for $p \in B(F)$ by the product formula, if $\gamma \notin B(F)$, we may assume that $\gamma = wu(\nu)$, $\nu \in F^\times$. So it suffices to show that $\delta(g)\delta(wu(\nu)g) \leq 1$. To see this, write $g = u(x)dk$ with $d = d(a, b)$, $x \in \mathbb{A}$, $a, b \in \mathbb{A}^\times$. Then (see Subsection 2.6)

$$\begin{aligned} \delta(g)\delta(wu(\nu)g) &= \delta(dwu(x+\nu)d) = \delta(wu((x+\nu)b/a)w^{-1}) \\ &= |(1, (x+\nu)b/a)|^{-2} \leq 1. \end{aligned}$$

□

Proposition 11.1.1 shows that, for a given g, the series in (11.1.1) has at most one term if $T > 1$. In particular,

$$\Lambda^T\phi(g) = \begin{cases} \phi(g) - \phi_N(g)\widehat{\tau}_T(\delta(g)) & \text{if } \delta(g) > 1,\ T > 1, \\ \phi(g) - \phi_N(g) & \text{if } \delta(g) > T > 1. \end{cases} \tag{11.1.2}$$

Let S be a Siegel domain. Let Ω be the set of g in S such that $\delta(g) \leq T$. Then Ω is compact mod $Z(\mathbb{A})$. If $T > 1$, then $\Lambda^T\phi$ is given on $S - \Omega$ by the second formula in (11.1.2). Thus $\Lambda^T\phi(g)$ is bounded on $S - \Omega$ under very mild assumptions on ϕ. For instance, this is the case if ϕ is the convolution of a slowly increasing or square-integrable function on $G(F)\backslash G(\mathbb{A})$ with a C_c^∞-function. Then (11.1.2) is rapidly decreasing; see 6.1, and [Ar 80] for the situation for a general group.

Proposition 11.1.2. *If Ω' is a compact set, then*

$$\Lambda^T\phi(g) = \phi(g) \text{ for } g \in \Omega' \text{ and large } T\ (T = T(\Omega')). \tag{11.1.3}$$

Proof. To the contrary, suppose $g \in \Omega'$ and $\Lambda^T\phi(g) \neq \phi(g)$ with $T > 1$. Then there exists a finite non-empty set of elements $\gamma \in B(F)\backslash G(F)$ such that $\delta(\gamma g) > T > 1$ (see Proposition 8.1.1 again; this finite set depends on Ω' but not on T). Since the resulting elements γg must belong to a (common) compact set mod $B(F)N(\mathbb{A})Z(\mathbb{A})$, and since δ is continuous, $\delta(\gamma g) < T_0$. But if $T > T_0$, we get a contradiction. Therefore (11.1.3) must hold. □

Corollary 11.1.3. *As $T \to \infty$, $\Lambda^T\phi \to \phi$ uniformly on compact sets.*

Proposition 11.1.4. *The truncation operator Λ^T is a continuous Hermitian operator on $L^2_\omega(G)$:*

$$(\Lambda^T\phi_1, \phi_2) = (\phi_1, \Lambda^T\phi_2). \tag{11.1.4}$$

Proof. The left-hand side is (ϕ_1, ϕ_2) minus the scalar product of a B-series with ϕ_2. In particular, as in the proof of Proposition 8.3.2, the left-hand side equals

$$(\phi_1, \phi_2) - \int_{B(F)N(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \phi_{1,N}(g)\widehat{\tau}_T(\delta(g))\overline{\phi_{2,N}}(g)\, dg.$$

Similarly, the right-hand side equals

$$(\phi_1, \phi_2) - \int_{B(F)N(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \phi_{1,N}(g)\overline{\phi_{2,N}}(g)\widehat{\tau}_T(\delta(g))\, dg.$$

Since $\widehat{\tau}_T$ is real, the desired equality follows. □

Proposition 11.1.5. *For $T > 1$, we have*

$$((1 - \Lambda^T)\phi_1, \Lambda^T\phi_2) = 0, \tag{11.1.5}$$

i.e., Λ^T is an orthogonal projection in $L^2_\omega(G)$.

Proof. Indeed, $(1 - \Lambda^T)\phi_1$ is a B-series. So the left-hand side of (11.1.5) equals

$$\int_{N(\mathbb{A})B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \phi_{1,N}(g)\widehat{\tau}_T(\delta(g)) \left(\int_{N(F)\backslash N(\mathbb{A})} \Lambda^T\phi_2(ng)\, dn \right) dg.$$

But $\widehat{\tau}_T(\delta(g)) = 0$ unless $\delta(g) > T > 1$, in which case the inner integral is (by (11.1.2))

$$\int_{N(F)\backslash N(\mathbb{A})} (\phi_2(ng) - \phi_{2,N}(ng))dn = 0,$$

and (11.1.5) follows. (Note that ϕ_i need not belong to $L^2_\omega(G)$.) □

12 Eisenstein series

Let pr_0 be the orthogonal projection from $L^2_\omega(G)$ onto $L^2_{0,\omega}(G)$. To get an explicit formula for the kernel of the Hilbert-Schmidt operator $r_0(fdg) = \mathrm{pr}_0\, r(fdg)\, \mathrm{pr}_0$ we need an explicit description of the operator

$$S : L^2_{c,\omega}(G) \to \mathrm{H}, \quad \Phi \mapsto \{y \mapsto (a_1(\varphi_\mu))(iy)\}_\mu$$

of Section 10. This description is given in terms of Eisenstein series.

Definition 12.1. Let φ_μ be a section of the vector bundle $\mathrm{H}(\mu(s))$. The series

$$E(\varphi_\mu, s, g) = \sum_{\gamma \in B(F)\backslash G(F)} \varphi_\mu(\gamma g, s). \tag{12.1.1}$$

is called **Eisenstein series**. It converges only for $\mathrm{Re}(s) > 1/2$. For general $s \in \mathbb{C}$, it has to be defined by analytic continuation. It is slowly increasing, hence an automorphic form.

Let φ_μ be a (μ, B)-automorphic form. Then $\varphi_\mu(g) = (2\pi i)^{-1} \int_{x-i\infty}^{x+i\infty} \widehat{\varphi}_\mu(g, s)\, ds$, for $x > 1/2$. Interchanging summation and integration, we see that the B-series Φ_μ defined by φ_μ is given by

$$\Phi_\mu(g) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\widehat{\varphi}_\mu(s), s, g)ds, \tag{12.1.2}$$

provided $x > 1/2$. To exhibit a dense subspace of $L^2_{0,\omega}(G)^\perp$ as a "continuous sum of Eisenstein series", we must continue E analytically and shift the integration in (12.1.2) to the imaginary axis.

Since E is a B-series, its constant term can be computed as in (8.3.1) and (8.3.2), yielding

$$E_N(\varphi_\mu, s, g) = \varphi_\mu(g, s) + \int_{N(\mathbb{A})} \varphi_\mu(\omega n g, s) dn, \quad \mathrm{Re}(s) > 1/2.$$

Thus

$$E_N(\varphi_\mu, s, g) = \varphi_\mu(s)(g) + (M(\mu(s))\varphi_\mu(s))(g), \quad \mathrm{Re}(s) > 1/2. \tag{12.1.3}$$

To continue E analytically, we deal with sections obtained from identifying $\mathrm{H}(\mu(s))$ with $\mathrm{H}_\mu = \mathrm{H}(\mu(0))$. In other words, if $\varphi_\mu \in \mathrm{H}_\mu$, we define a section $\varphi_\mu(s)$ by

$$\varphi_\mu(g, s) = \varphi_\mu(g)\delta(g)^s, \tag{12.1.4}$$

where the Harish-Chandra modular δ-function is defined in Subsection 2.6.

We denote the corresponding Eisenstein series by $E(\varphi_\mu, s, g)$.

The next result concerns the analytic continuation of the Eisenstein series.

Theorem 12.1. *The function $\Lambda^T E(\varphi_\mu, s, g)$, and hence also $E(\varphi_\mu, s, g)$, has analytic continuation to $\mathrm{Re}(s) \geq 0$ as a meromorphic function, with singularities at most those of $M(\mu(s))$.*

Proof. Using the truncation operator (11.1.1), we can write the Eisenstein series as

$$E(\varphi_\mu, s, g) = \Lambda^T E(\varphi_\mu, s, g) + \sum_{\gamma \in B(F)\backslash G(F)} E_N(\varphi_\mu, s, \gamma g)\widehat{\tau}_T(\delta(\gamma g)). \tag{12.1.5}$$

The second term on the right-hand side is the B-series attached to a function with support in the set $\{g; \delta(g) > T\}$. Thus, for g in a Siegel domain, the second term has only finitely many terms. In particular, it represents a meromorphic function of s whose singularities are amongst those of $M(\mu(s))$ (by (12.1.3)).

The first term on the right-hand side of (12.1.5) is defined initially only for $\mathrm{Re}(s) > 1/2$. Since it is square-integrable (by (11.1.2) and the remarks immediately following it), it suffices to continue it analytically as an $L^2_\omega(G)$-valued function. Thus we need to examine the inner product

$$(\Lambda^T E(\varphi_{1\mu}, s_1), \Lambda^T E(\varphi_{2\mu}, \overline{s}_2)). \tag{12.1.6}$$

By (11.1.5) (which applies in this case), the inner product (12.1.6) is just $(E(\varphi_{1\mu}, s_1), \Lambda^T E(\varphi_{2\mu}, \bar{s}_2))$, which, since $\Lambda^T E$ is a difference of B-series, we compute (using Proposition 8.3.2) to be

$$\int_{N(\mathbb{A})B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} E_N(\varphi_{1\mu}, s_1, g)\{\overline{\varphi}_{2\mu}(g, \bar{s}_2) - \overline{E}_N(\varphi_{2\mu}, \bar{s}_2, g)\widehat{\tau}_T(\delta(g))\}dg.$$

Recall that E_N is given by (12.1.3) with $\varphi_\mu(g, s)$ satisfying

$$\varphi_\mu\left(\begin{pmatrix} ta & x \\ 0 & 1 \end{pmatrix} k, s\right) = |t|^{s+\frac{1}{2}}\mu_1(a)\varphi_\mu\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)$$

for $t \in F_\infty^+$, $a \in \mathbb{A}^1$, and $k \in K$. Recall that μ_1 is trivial on F_∞^+. Using the Iwasawa decomposition, we compute

$$\begin{aligned}(12.1.7)\quad &(\Lambda^T E(\varphi_{1\mu}, s_1), \Lambda^T E(\varphi_{2\mu}, \bar{s}_2)) \\ &= \left\{(\varphi_{1\mu}, \varphi_{2\mu})T^{s_1+s_2} - (M(\mu(s_1))\varphi_{1\mu}, M(\bar{\mu}(\bar{s}_2))\varphi_{2\mu})T^{-s_1-s_2}\frac{1}{s_1+s_2}\right\} \\ &\quad + \left\{(\varphi_{1\mu}, M(\bar{\mu}(\bar{s}_2))\varphi_{2\mu})T^{s_1-s_2} - (M(\mu(s_1))\varphi_{1\mu}, \varphi_{2\mu})T^{-s_1+s_2}\frac{1}{s_1-s_2}\right\}.\end{aligned}$$

Here, $\mathrm{Re}(s_1) > \mathrm{Re}(s_2) > 1/2$.

Note that the right-hand side of (12.1.7) is a meromorphic function of (s_1, s_2) that seems to have a singularity on the line $s_1+s_2 = 0$. However, by Corollary 9.3.5 and (10.3.1), the expression in the first bracket vanishes precisely along this line. Similarly the expression in the second bracket vanishes when $s_1 - s_2 = 0$. Thus we conclude that (12.1.7) is meromorphic in s_1, s_2 with singularities at most those of $M(\mu(s_1))$ and $M(\mu(s_2))$. In particular, for $s = s_1 = -s_2 \neq 0$,

$$\begin{aligned}&(\Lambda^T E(\varphi_{1\mu}, s), \Lambda^T E(\varphi_{2\mu}, -\bar{s})) = 2(\varphi_{1\mu}, \varphi_{2\mu})\ln T + (M(\widetilde{\mu}(-s))M'(\mu(s))\varphi_{1\mu}, \varphi_{2\mu}) \\ &(12.1.8)\qquad\qquad + \{(\varphi_{1\mu}, M(\overline{\widetilde{\mu}}(-\bar{s}))\varphi_{2\mu})T^{2s} - (M(\mu(s))\varphi_{1\mu}, \varphi_{2\mu})T^{-2s}\}\frac{1}{2s}.\end{aligned}$$

Now suppose that for $\varphi_{1\mu} = \varphi_{2\mu} = \varphi_\mu$, the right-hand side of (12.1.7) is analytic in the polydisc $|s_1 - s_0| < R$, $|s_2 - s_0| < R$, while $M(\mu(s))$ is holomorphic in some small disc centered at s_0. Then the double Taylor series of the left-hand side converges in this polydisc. Since

$$\begin{aligned}\left\|\frac{\partial^n}{\partial s^n}\Lambda^T E(\varphi_\mu, s)\right\|^2 &= \left(\frac{\partial^n}{\partial s^n}\Lambda^T E(\varphi_\mu, s), \frac{\partial^n}{\partial s^n}\Lambda^T E(\varphi_\mu, \bar{s})\right) \\ &= \frac{\partial^{2n}}{\partial s_1^n \partial s_2^n}(\Lambda^T E(\varphi_\mu, s_1), \Lambda^T E(\varphi_\mu, \bar{s}_2))|_{s_1=s_2=s},\end{aligned}$$

the Taylor series of $\Lambda^T E(\varphi_\mu, s)$ converges in a disc $|s - s_0| < R$, i.e., $\Lambda^T E(\varphi_\mu, s)$ is analytic in this disc. Thus $\Lambda^T E(\varphi_\mu, s)$ (and hence $E(\varphi_\mu, s, g)$) is meromorphic in $\mathrm{Re}(s) \geq 0$ with singularities at most those of $M(\mu(s))$. Note that in (12.1.7) and (12.1.8), we needed to identify $\mathrm{H}(\mu(s))$ with $\mathrm{H}(\mu(0))$ to define the scalar products and derivative $M'(\mu(s))$ of $M(\mu(s))$. □

Corollary 12.1.1. *If $\varphi_\mu(s)$ is a meromorphic section of the vector bundle $\mathrm{H}(\mu(s))$, then $E(\varphi_\mu, s, g)$ extends to a meromorphic function of s on $\mathrm{Re}(s) \geq 0$.*

Proposition 12.1.2. *Let φ_μ be a (μ, B)-automorphic form. Then*

$$E(M(\mu(s))\varphi_\mu, s, g) = E(\varphi_\mu, s, g). \tag{12.1.9}$$

for all $g \in G(\mathbb{A})$.

Proof. Since $\varphi_\mu \in \mathrm{H}(\mu(s))$ and $(M(\mu(s))\varphi_\mu)(s) \in \mathrm{H}(\widetilde{\mu}(-s))$, the function $g \mapsto E_N(M(\mu(s))\varphi_\mu, s, g)$ is equal to the function of g, omitted from the next line,

$$\begin{aligned}(M(\mu(s))\varphi_\mu)(s) + M(\widetilde{\mu}(-s))(M(\mu(s))\varphi_\mu)(s) &= \varphi_\mu(s) + (M(\mu(s))\varphi_\mu)(s)\\ &= E_N(\varphi_\mu(s), g).\end{aligned}$$

Thus the difference between the two sides of (12.1.9) is a cuspidal function. Since any B-series (or its analytic continuation) is orthogonal to all cuspidal functions, this difference vanishes, as claimed. □

Remark. Analogous results can be proved similarly for any group whose derived group has F-rank one. The only difference is that the operator $M(\mu(s))$ may have a finite – but greater than 1 – number of poles. These poles need to be found. Yet, similarly, the operator $M(\mu(s))$ still controls the analytic behavior of the Eisenstein series.

13 The kernel on the continuous spectrum

13.1 The kernel of $r(fdg)$ in $L^2_{c,\omega}(G)$. Let $\Phi_{1\mu}(g)$ be a B-series attached to the (μ, B)-automorphic form $\varphi_{1\mu}$ (of compact support mod $Z(\mathbb{A})N(\mathbb{A})B(F)$). The orthogonal projection

$$S = \sum_\mu S_\mu : L^2_{c,\omega}(G) \to \mathrm{H} = \oplus_\mu \mathrm{H}(\mu(i\mathbb{R})), \quad (S_\mu(\Phi))(y) = (a(\varphi_\mu))(iy),$$

is defined at the end of Section 10.4. The set of μ is described in Theorem 6.2.

Proposition 13.1.1. *Define $E(\varphi_\mu, iy, g)$ by analytic continuation. For all $\Psi_\mu \in \mathrm{H}$,*

$$(\Psi_\mu, S\Phi_{1\mu}(iy)) = \frac{1}{2}\int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} E(\Psi_\mu, iy, g)\bar{\Phi}_{1\mu}(g)dg \tag{13.1.1}$$

for almost every $y \in \mathbb{R}$.

Proof. Let $\Phi_{2\mu}$ be the B-series attached to $\varphi_{2\mu}$. We then get from (12.1.2)

$$\begin{aligned}(\Phi_{1\mu}, \Phi_{2\mu}) &= \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \left\{ \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} E(\widehat{\varphi}_{1\mu}, s, g)ds \right\} \bar{\Phi}_{2\mu}(g)dg \\ &= \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} ds \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} E(\widehat{\varphi}_{1\mu}, s, g)\bar{\Phi}_{2\mu}(g)dg\end{aligned}$$

for $\mathrm{Re}(s) = x > 1/2$. Shifting the integration to the imaginary axis gives

$$\begin{aligned}(\Phi_{1\mu}, \Phi_{2\mu}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} E(\widehat{\varphi}_{1\mu}, iy, g)\bar{\Phi}_{2\mu}(g)dg \\ + c\cdot\delta(\mu, \widetilde{\mu})\cdot(\Phi_{1\mu}, \mu_1\circ\det)(\mu_1\circ\det, \Phi_{2\mu}).\end{aligned} \tag{13.1.2}$$

But (12.1.9) says that the integral in (13.1.2) remains unchanged upon replacement of $\widehat{\varphi}_{1\mu}(iy)$ with $(M(\mu(iy))\widehat{\varphi}_{1\mu})(iy)$ and then y with $-y$. Thus, with $a(\varphi_{1\mu})(iy)$ as in (10.4.3), (13.1.2) asserts that

$$\begin{aligned}(\Phi_{1\mu}, \Phi_{2\mu}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} E(a(\varphi_{1\mu}), iy, g)\bar{\Phi}_{2\mu}(g)dg \\ + c\cdot\delta(\mu, \widetilde{\mu})\cdot(\Phi_{1\mu}, \mu_1\circ\det)(\mu_1\circ\det, \Phi_{2\mu}).\end{aligned}$$

On the other hand, computing the inner product as in (10.4.1), we also have that

$$\begin{aligned}(\Phi_{1\mu}, \Phi_{2\mu}) = \frac{1}{\pi}\int_{-\infty}^{\infty}((a(\varphi_{1\mu}))(iy), (a(\varphi_{2\mu}))(iy))dy \\ + c\cdot\delta(\mu, \widetilde{\mu})\cdot(\Phi_{1\mu}, \mu_1\circ\det)(\mu_1\circ\det, \Phi_{2\mu}).\end{aligned}$$

Thus we conclude that

$$\begin{aligned}\int_{-\infty}^{\infty}((a(\varphi_{1\mu}))(iy), S\Phi_{2\mu}(iy))dy \\ = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} E(a(\varphi_{1\mu}), iy, g)\bar{\Phi}_{2\mu}(g)dg\right\}\end{aligned} \tag{13.1.3}$$

for all $a(\varphi_{1\mu}) \in \mathrm{H}$. This formula is then also valid for any square-integrable function $(a(\varphi_\mu))(iy)$ with values in H, since both sides remain unchanged when

$(a(\varphi_\mu))(iy)$ is replaced by $\frac{1}{2}((a(\varphi_\mu))(iy)+M(\widetilde{\mu}(-iy))a(\varphi_{\widetilde{\mu}})(iy))$. Thus we may take $(a(\varphi_\mu))(iy) = c(y)\varphi_\mu(iy)$ with c a scalar valued function and φ_μ a constant section, to conclude from (13.1.3) that (13.1.1) indeed holds. □

Extend the map $S : L^2_{c,\omega}(G) \to \mathrm{H}$ to $L^2_\omega(G) \to \mathrm{H}$ by setting it equal to 0 on $L^2_{0,\omega}(G)$ and $L^2_{1,\omega}(G)$. Then S^*S is the orthogonal projection of $L^2_\omega(G)$ onto $L^2_{c,\omega}(G)$. Recall that r is the right $G(\mathbb{A})$-action on $L^2_{c,\omega}(G)$, and $i = i_B^G$ is this action on H. Recall also that S intertwines r and i. This means that $Sr(fdg)S^* = i(fdg)$. Hence $S^*Sr(fdg)S^*S = S^*i(fdg)S$ for all test functions $f \in C^\infty_{c,\omega}(G)$; see Subsection 2.11.

If $\Phi_1 = \sum_\mu \Phi_{1\mu}$, $\Phi_2 = \sum_\mu \Phi_{2\mu} \in L^2_\omega(G)$, then

$$(S^*Sr(fdg)S^*S\Phi_1, \Phi_2) = (S^*i(fdg)S\Phi_1, \Phi_2) = (i(fdg)S\Phi_1, S\Phi_2)$$

$$= \frac{1}{\pi}\sum_\mu \int_{-\infty}^{\infty} (i_B^G(\mu(iy), fdg)S_\mu\Phi_{1\mu}(iy), S_\mu\Phi_{2\mu}(iy))dy. \tag{13.1.4}$$

Before applying the identity (13.1.1), we make the following observation.

Lemma 13.2. *Formula* (13.1.1) *is valid for all* $\Phi_1 \in L^2_\omega(G)$*, not only for in* $\Phi_1 \in L^2_{0,\omega}(G)^\perp$.

Proof. If $\Phi \in L^2_{0,\omega}(G)$, then $S\Phi \equiv 0$, by the definition of the extension of S to $L^2_\omega(G)$. Every Eisenstein series is orthogonal to every cuspidal function (in the domain of convergence of the Eisenstein series at first, but then for all s, by analytic continuation). Hence the right-hand side of (13.1.1) is also identically 0. □

13.3 A formula for the kernel. We can finally apply (13.1.1) to (13.1.4) to get, for an orthonormal basis $\{\varphi_\alpha^\mu\}$ of $\mathrm{H}(\mu(i\mathbb{R}))$, the equality

$$\begin{aligned}(S^*Sr(fdg)S^*S\Phi_{1\mu}, \Phi_{2\mu})&\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_\alpha (i_B^G(\mu(iy), fdg)S\Phi_{1\mu}(iy), \varphi_\alpha^\mu)(\varphi_\alpha^\mu, S\Phi_{2\mu}(iy))dy\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} dy \sum_\alpha \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Phi_{1\mu}(g)\bar{E}(i_B^G(\mu(iy), fdg)^*\varphi_\alpha^\mu, iy, g)dg\\ &\quad\cdot \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \bar{\Phi}_{2\mu}(h)E(\varphi_\alpha^\mu, iy, h)dh.\end{aligned}$$

Interchanging the order of integrations and summations then yields

$$(S^*Sr(fdg)S^*S\Phi_{1\mu}, \Phi_{2\mu}) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Phi_{1\mu}(g)\bar{\Phi}_{2\mu}(h)dg\,dh$$

$$\cdot\left\{\sum_\alpha \frac{1}{4\pi}\int_{-\infty}^{\infty} E(\varphi_\alpha^\mu, iy, h)\bar{E}(i_B^G(\mu(iy), fdg)^*\varphi_\alpha^\mu, iy, g)dy\right\}. \tag{13.3.1}$$

Let $\mathrm{pr_c}$ denote the projection S^*S onto $L^2_{c,\omega}(G)$. Then the kernel of $r_c(fdg) = \mathrm{pr_c}\, r(fdg)\, \mathrm{pr_c}$ is precisely the sum over μ of the expression in brackets in (13.3.1). Since $i_B^G(\mu(iy), fdg)^*\varphi_\alpha^\mu = \sum_\beta (i_B^G(\mu(iy), fdg)^*\varphi_\alpha^\mu, \varphi_\beta^\mu)\varphi_\beta^\mu$, we also have that the kernel $K_{f,c}(h, g)$ of the convolution operator on the continuous spectrum satisfies (13.3.2)

$$K_{f,c}(h, g) = \frac{1}{4\pi} \sum_\mu \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, \varphi_\alpha^\mu) E(\varphi_\alpha^\mu, iy, h)\bar{E}(\varphi_\beta^\mu, iy, g)dy.$$

Let $\mathrm{pr_1}$ be the the projection from $L^2_\omega(G)$ onto $L^2_{1,\omega}(G)$. Then the operator $r_1(fdg) = \mathrm{pr_1}\, r(fdg)\, \mathrm{pr_1}$ is defined by the kernel

$$K_{f,1}(h, g) = \frac{1}{\mathrm{vol}(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))} \sum_{\{\chi;\chi^2=\omega\}} \chi(h)\bar{\chi}(g) \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} f(g)\bar{\chi}(\det g)dg.$$

The operator $r(fdg)$ is still defined by the kernel

$$K_f(h, g) = \sum_{\gamma \in Z(F)\backslash G(F)} f(h^{-1}\gamma g).$$

Thus, for $h \in G(\mathbb{A})$ and $\phi \in L^2_{0,\omega}(G)$, we conclude that

$$(r_0(fdg)\phi)(h) = (r(fdg)|_{L^2_{0,\omega}(G)}\phi)(h) = (\mathrm{pr_0}\, r(fdg)\, \mathrm{pr_0}\, \phi)(h) = \int K_{f,0}(h, g)\phi(g)dg,$$

where $K_{f,0}(h, g) = K_f(h, g) - K_{f,c}(h, g) - K_{f,1}(h, g)$.

14 The kernel on the cuspidal spectrum

Proposition 14.0.1. *Let $f \in C^\infty_{c,\omega}(G)$. The kernel $K_{f,0}$ is square-integrable, continuous, and integrable over the diagonal.*

Proof. The operator $r_0(fdg)$ is a trace class operator, by Subsection 6.3.2. □

Proposition 14.0.1 allows us to write

$$\mathrm{tr}\, r_0(fdg) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K_{f,0}(g, g)dg. \tag{14.0.1}$$

We now give an explicit formula for the right-hand side of (14.0.1).

Proposition 14.0.2. *The function $y \mapsto K_{f,0}(x, y)$ lies in $L^2_{0,\overline{\omega}}(G(\mathbb{A}))$.*

Proof. For $\phi \in L^2_{0,\omega}(G(\mathbb{A}))^\perp$, we have $\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K_{f,0}(x, y)\phi(y)dy = 0$. In other words the function $y \mapsto \overline{K}_{f,0}(x, y)$ is orthogonal to $L^2_{0,\omega}(G(\mathbb{A}))^\perp$. Consequently $K_{f,0}(x, y) \in L^2_{0,\overline{\omega}}(G(\mathbb{A}))$. □

Let Λ_2^T be the truncation operator with respect to the second variable. By Proposition 14.0.2, $\Lambda_2^T K_{f,0} = K_{f,0}$. Hence

$$K_{f,0}(x,y) = \Lambda_2^T K_f(x,y) - \Lambda_2^T K_{f,c}(x,y) - \Lambda_2^T K_{f,1}(x,y). \tag{14.0.2}$$

As discussed in the paragraph following Proposition 11.1.1,

Proposition 14.0.3. *Each term in* (14.0.2) *is integrable over the diagonal.*

Thus

$$\text{tr } r_0(fdg) = \int \Lambda_2^T K_f(x,x)dx - \int \Lambda_2^T K_{f,c}(x,x)dx - \int \Lambda_2^T K_{f,1}(x,x)dx. \tag{14.0.3}$$

Proposition 14.0.4. *Let Ω be a compact subset of $Z(\mathbb{A})\backslash G(\mathbb{A})$. Then there exists a number $d_\Omega > 0$ such that if $\gamma \in G(F)$ and $n \in N(\mathbb{A})$ satisfy $x^{-1}\gamma n x \in \Omega$ for some x in $G(\mathbb{A})$ with $\delta(x) > d_\Omega$, then $\gamma \in B(F)$.*

Proof. Let sl(2) be the three-dimensional vector space of 2×2 matrices with trace 0. Consider the adjoint action of GL(2) on sl(2) given by

$$\text{Ad}(g) : X \mapsto gXg^{-1}, \quad g \in \text{GL}(2), X \in \text{sl}(2).$$

Since $\text{Ad}(zg) = \text{Ad}(g)$ for $g \in \text{GL}(2)$ and $z \in Z$, this action factorizes via $\text{PGL}(2) = Z\backslash \text{GL}(2)$. By the action of GL(2) on sl(2), we get an algebraic homomorphism $\text{Sym}^2 : \text{PGL}(2) \to \text{GL}(3)$; in fact, an isomorphism onto $\text{SO}(3) \subset \text{SL}(3) \subset \text{GL}(3)$. With the choice of basis

$$\left\{ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

this isomorphism is given explicitly by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{x} \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad+bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix}, \quad x = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $g = n_1 a_1 k_1 \in G(\mathbb{A})$. Then $a_1^{-1} n_1^{-1} \gamma n n_1 a_1 \in \Omega'$, where $\Omega' = K\Omega K$ is compact in $Z(\mathbb{A})\backslash \text{GL}(2, \mathbb{A})$.

Let

$$n_1 a_1 = \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad n = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Then

$$a_1^{-1} n_1^{-1} \gamma n n_1 a_1 = \begin{pmatrix} * & * \\ \frac{t_1}{t_2} c & * \end{pmatrix}.$$

If this lies in the compact subset $\Omega' \subset Z(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$, its image under Sym^2 lies in the compact image $\mathrm{Sym}^2\,\Omega'$ of Ω' in $\mathrm{GL}(3,\mathbb{A})$. Then there exists an $M>0$ such that $|1/x|\,|c^2t_1^2/t_2^2|$ is bounded by M. But $x=\det(\gamma)\in F^\times$, and so $|x|=1$. Hence $|c|<\sqrt{M}|t_2t_1^{-1}|$. As $c\in F$, $|c|=0$ or 1. So if $\delta(g)=|t_1/t_2|$ is large enough, c must vanish, which means $\gamma\in B(F)$. □

15 The truncated kernel over the entire spectrum

Proposition 15.0.1. *The truncated kernel $\Lambda_2^T K_f$ satisfies*

$$\Lambda_2^T K_f(x,x)=\sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\gamma x) - \sum_{\varepsilon\in B(F)\backslash G(F)}\int_{N(\mathbb{A})}\sum_{\alpha\in F^\times} f(x^{-1}\varepsilon^{-1}d(\alpha)n\varepsilon x)\widehat{\tau}_T(\delta(\varepsilon x))dn. \tag{15.0.1}$$

Proof. By (11.1.1), we may write

$$\begin{aligned}\Lambda_2^T K_f(x,x) &= \sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\gamma x)\\ &\quad - \sum_{\varepsilon\in B(F)\backslash G(F)}\left(\int_{N(F)\backslash N(\mathbb{A})} K_f(x,n\varepsilon x)dn\right)\widehat{\tau}_T(\delta(\varepsilon x))\\ &= \sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\gamma x)\\ &\quad - \sum_{\varepsilon\in B(F)\backslash G(F)}\left(\int_{N(F)\backslash N(\mathbb{A})}\left(\sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\gamma n\varepsilon x)\right)dn\right)\widehat{\tau}_T(\delta(\varepsilon x))\\ &= \sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\gamma x)\\ &\quad - \sum_{\varepsilon\in B(F)\backslash G(F)}\left(\int_{N(F)\backslash N(\mathbb{A})}\left(\sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\varepsilon^{-1}\gamma n\varepsilon x)\right)dn\right)\widehat{\tau}_T(\delta(\varepsilon x)),\end{aligned}$$

where in the last line we have replaced γ with $\varepsilon^{-1}\gamma$. Applying Proposition 14.0.4 to the support of f, we conclude that when T is sufficiently large, the last sum ranges only over γ in $Z(F)\backslash B(F)$. Such γ has the form $d(\alpha)\nu$, $\alpha\in F^\times$, $\nu\in N(F)$. Thus we may combine $\nu n\in N(F)\times N(F)\backslash N(\mathbb{A})$ into a single integral over $N(\mathbb{A})$

to obtain that $\Lambda_2^T K_f(x,x)$ is

$$\Lambda_2^T K_f(x,x) = \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma x) - \sum_{\varepsilon \in B(F)\backslash G(F)} \left(\int_{N(\mathbb{A})} \left(\sum_{\alpha \in F^\times} f(x^{-1}\varepsilon^{-1} d(\alpha) n \varepsilon x) \right) dn \right) \widehat{\tau}_T(\delta(\varepsilon x)),$$

as asserted. □

We now split the terms in (15.0.1) according to whether γ is the identity e, elliptic regular, hyperbolic regular, or unipotent regular, grouping them with the terms from the second sum with the same semisimple part. The result is the following corollary.

Corollary 15.0.2.

$$\Lambda_2^T K_f(x,x) = f(e), \tag{15.0.2a}$$

$$+ \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma x) \quad (\gamma \ F\text{-elliptic regular}) \tag{15.0.2b}$$

$$+ \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma x) \quad (\gamma \ F\text{-hyperbolic regular}) \tag{15.0.2c}$$

$$- \sum_{\varepsilon \in B(F)\backslash G(F)} \int_{N(\mathbb{A})} \sum_{\alpha \neq 1} f(x^{-1}\varepsilon^{-1} d(\alpha) n \varepsilon x) \widehat{\tau}_T(\delta(\varepsilon x)) dn,$$

$$+ \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma x) \quad (\gamma \text{ unipotent regular, i.e. } \gamma \neq e) \tag{15.0.2d}$$

$$- \sum_{\varepsilon \in B(F)\backslash G(F)} \int_{N(\mathbb{A})} f(x^{-1}\varepsilon^{-1} n \varepsilon x) \widehat{\tau}_T(\delta(\varepsilon x)) dn.$$

Chapter III
Computation of the Terms in the Trace Formula

In this chapter, we compute the various terms which appear in the trace formula, introducing the correction argument which makes the hyperbolic weighted orbital integrals continuous. We show that the limit of these terms as $a \to 1$ in A_v is a term from the regular unipotent contribution to the trace formula.

16 The elliptic part of the trace formula

We integrate $\Lambda_2^T K_f(x,x)$ over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. We may integrate each term of (15.0.2) separately. The integral of (15.0.2a) is simply

$$\text{vol}(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))f(e). \tag{16.0.1a}$$

Proposition 16.0.1. *The integral of* (15.0.2b) *over* $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ *is*

$$\sum_{\{T\}}\sum_{\gamma\in Z(F)\backslash T(F)^{\text{reg}}}\frac{\text{vol}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A});dt)}{[W(T,G)]}\int_{T(\mathbb{A})\backslash G(\mathbb{A})}f(x^{-1}\gamma x)d\dot{x}, \tag{16.0.2b}$$

where $d\dot{x}=dx/dt$ *and* dt *is a measure on* $T(\mathbb{A})$. *Here,* $\{T\}$ *is a set of representatives for the set of* $G(F)$*-conjugacy classes of* F*-elliptic tori* T *in* G, *and* $T(F)^{\text{reg}}=T(F)-Z(F)$. □

Proof. An F-elliptic torus is a maximal F-torus in G which is not F-split, namely, it is isomorphic to $\mathbb{G}_m^2$ over a quadratic extension of F, but not over F. Each elliptic regular γ in $G(F)$ has the form $\varepsilon^{-1}\gamma'\varepsilon$, with $\gamma'\in T(F)^{\text{reg}}$, $T\in\{T\}$, and $\varepsilon\in G_{\gamma'}(F)\backslash G(F)$, where $G_{\gamma'}$ is the centralizer of γ' in G. Since γ' is regular in $T(F)$, we have $G_{\gamma'}=T$. Each γ is obtained $[W(T,G)]$ times this way ($\varepsilon^{-1}w^{-1}\gamma_1'w\varepsilon=\varepsilon\gamma'\varepsilon$ with $\gamma_1'=w\gamma'w^{-1}\in T(F)$ for any $w\in W(T,G)$). Thus

$$\begin{aligned}\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}&\sum_{\{T\}}\frac{1}{[W(T,G)]}\sum_{\gamma\in Z(F)\backslash T(F)^{\text{reg}}}\sum_{\varepsilon\in T(F)\backslash G(F)}f(x^{-1}\varepsilon^{-1}\gamma\varepsilon x)dx\\&=\sum_{\{T\}}[W(T,G)]^{-1}\sum_{\gamma\in Z(F)\backslash T(F)^{\text{reg}}}\int_{Z(\mathbb{A})T(F)\backslash G(\mathbb{A})}f(x^{-1}\gamma x)dx\\&=\sum_{\{T\}}\frac{\text{vol}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A});dt)}{[W(T,G)]}\sum_{\gamma\in Z(F)\backslash T(F)^{\text{reg}}}\int_{T(\mathbb{A})\backslash G(\mathbb{A})}f(x^{-1}\gamma x)d\dot{x}.\end{aligned}$$
□

17 The hyperbolic terms in the trace formula

Proposition 17.0.1. *The integral of* (15.0.2c) *over* $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ *is*

$$(\ln T)|F^\times\backslash\mathbb{A}^1|\sum_{\alpha\neq 1}\text{OI}'(d(\alpha),fdg) \tag{17.0.1}$$

$$-\frac{\lambda_{-1}}{2}|F^\times\backslash\mathbb{A}^1|\sum_{\alpha\neq 1}\sum_{u\in\mathbb{V}}\text{WOI}'(d(\alpha),f_u dg_u)\cdot\prod_{v\neq u}\text{OI}'(d(\alpha),f_v dg_v). \tag{17.0.2}$$

Note that (17.0.2) *is Theorem* 7.1(iii)′. Here,

$$\mathrm{OI}'(d(\alpha), f_v dg_v) = \Delta_v(d(\alpha)) \int_{N_v} f_v^{K_v}(n_v^{-1} d(\alpha) n_v) dn_v,$$
$$\mathrm{WOI}'(d(\alpha), f_v dg_v) = \Delta_v(d(\alpha)) \int_{N_v} f_v^{K_v}(n_v^{-1} d(\alpha) n_v)(\ln \delta_v(w n_v)) dn_v.$$

Proof. Recall that by a hyperbolic regular element γ in $G(F)$, we mean an element with two distinct eigenvalues in $F^\times$. Each such element is conjugate in $G(F)$ to a diagonal element. Hence, in the first sum of (15.0.2c), we write the hyperbolic γ in the form $\gamma = \varepsilon^{-1} d(\alpha)\varepsilon$ with $\varepsilon \in A(F)\backslash G(F)$, $\alpha \in F^\times$, $\alpha \neq 1$. Here, A is the centralizer of $d(\alpha)$ in G, namely, the diagonal subgroup. But as ε varies over $A(F)\backslash G(F)$, each γ is obtained twice, as $\mathrm{Ad}(w)$ permutes $Z(F)\backslash A(F)$, namely,

$$w d(\alpha) w^{-1} = d(1, \alpha) = d(\alpha^{-1}) \bmod Z(F).$$

Thus the first term in (15.0.2c) is

$$\frac{1}{2} \sum_{\varepsilon \in A(F)\backslash G(F)} \sum_{\alpha \neq 1} f(x^{-1}\varepsilon^{-1} d(\alpha)\varepsilon x) = \frac{1}{2} \sum_{\varepsilon \in B(F)\backslash G(F)} \sum_{\nu \in N(F)} \sum_{\alpha \neq 1} f(x^{-1}\varepsilon^{-1}\nu^{-1} d(\alpha)\nu\varepsilon x).$$

The integral of (15.0.2c) is therefore the integral over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ of

$$\frac{1}{2} \sum_{\varepsilon \in B(F)\backslash G(F)} \sum_{\nu \in N(F)} \sum_{\alpha \neq 1} f(x^{-1}\varepsilon^{-1}\nu^{-1} d(\alpha)\nu\varepsilon x) - \sum_{\varepsilon \in B(F)\backslash G(F)} \int_{N(\mathbb{A})} \sum_{\alpha \neq 1} f(x^{-1}\varepsilon^{-1} d(\alpha) n\varepsilon x) \widehat{\tau}_T(\delta(\varepsilon x)) dn.$$

Changing variables $n \mapsto d(\alpha)^{-1} n^{-1} d(\alpha) n$ $(\alpha \neq 1)$, we see that the integral of (15.0.2c) over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ equals the integral over $B(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ of

$$\frac{1}{2} \sum_{\nu \in N(F)} \sum_{\alpha \neq 1} f(x^{-1}\nu^{-1} d(\alpha)\nu x) - \int_{N(\mathbb{A})} \sum_{\alpha \neq 1} f(x^{-1} n^{-1} d(\alpha) n x) \widehat{\tau}_T(\delta(x)) dn.$$

To compute this last integral, we have to integrate first over $N(F)\backslash N(\mathbb{A})$ and then over $N(\mathbb{A})A(F)Z(\mathbb{A})\backslash G(\mathbb{A})$. The integration over the homogeneous space $N(F)\backslash N(\mathbb{A})$, whose volume is 1, gives

$$\frac{1}{2} \sum_{\alpha \neq 1} \int_{N(\mathbb{A})} f(x^{-1} n^{-1} d(\alpha) n x) dn - \int_{N(\mathbb{A})} \sum_{\alpha \neq 1} f(x^{-1} n^{-1} d(\alpha) n x) \widehat{\tau}_T(\delta(x)) dn.$$

The subsequent integral over $N(\mathbb{A})A(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ is the same as the integral of

$$\sum_{\alpha \neq 1} f(x^{-1} d(\alpha) x) \left(\frac{1}{2} - \widehat{\tau}_T(\delta(x)) \right) \tag{17.0.3}$$

over $A(F)Z(\mathbb{A})\backslash G(\mathbb{A})$.

Note now that $f(x^{-1}d(\alpha)x)$ in the integral of (17.0.3) does not change when x is replaced by ωx, and w normalizes $A(F)$. But the factor $\frac{1}{2} - \widehat{\tau}_T(\delta(x))$ changes. Thus the integral of (15.0.2c) equals the integral over $A(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ of

$$\frac{1}{2}\sum_{\alpha\neq 1} f(x^{-1}d(\alpha)x)(1-\widehat{\tau}_T(\delta(x))-\widehat{\tau}_T(\delta(\omega x))).$$

Using the Iwasawa decomposition, we see that this integral is

$$\frac{1}{2}\sum_{\alpha\neq 1}\int_K\int_{N(\mathbb{A})} f(k^{-1}n^{-1}d(\alpha)nk) \cdot\left(\int_{F^\times\backslash\mathbb{A}^\times}(1-\widehat{\tau}_T(|a|)-\widehat{\tau}_T(|a|^{-1}\delta(\omega n)))\,d^\times a\right)dn\,dk. \tag{17.0.4}$$

But $T > 1$ and $\delta(\omega n) \leq 1$. Therefore, the integrand in the inner integral of in (17.0.4) vanishes, unless $T^{-1}\delta(\omega n) < |a| < T$, in which case it equals 1.

Thus the integral of (15.0.2c) over $G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ equals

$$|F^\times\backslash\mathbb{A}^1|\sum_{\alpha\neq 1}\int_K\int_{N(\mathbb{A})} f(k^{-1}n^{-1}d(\alpha)nk)\left(\ln T-\frac{1}{2}\ln\delta(\omega n)\right)dn\,dk.$$

Recall that $|F^\times\backslash\mathbb{A}^1| = 1$, $dk = \lambda_{-1}\otimes_v dk_v$, $dn = \otimes_v dn_v$, and $\delta(g) = \prod_v \delta_v(g_v)$ where

$$\delta\left(\begin{pmatrix} a_v & x_v \\ 0 & b_v\end{pmatrix}k_v\right) = \left|\frac{a_v}{b_v}\right|_v.$$

Hence $\ln\delta(\omega n) = \sum_u \ln\delta(\omega n_u)$, and the last integral is equal to (17.0.1)+(17.0.2). Indeed, since $\alpha \neq 1$ is rational, the product formula asserts that $\prod_v \Delta_v(d(\alpha)) = 1$.

Proposition 17.0.1 follows. □

18 The correction

Recall that the orbital integral OI, the normalized orbital integral OI′, and the weighted orbital integrals WOI and normalized weighted orbital integrals WOI′ are defined in Subsection 2.12. We first study their asymptotic behavior at $a \in A_v$ as $a \to e$.

Proposition 18.0.1. (1) *For $f_v \in C^\infty_{c,\omega}(G_v)$, we have*

$$\lim_{\alpha\to 1}\mathrm{OI}'(d(\alpha), f_v dg_v) = \mathrm{OI}'(d(1), f_v dg_v) = \int_{F_v} f_v^{K_v}(u(x_v))dx_v.$$

(2)

$$\int_{F_v^\times}\int_{K_v} f_v(k_v^{-1}u(a_v)k_v)|a_v|_v\, dk_v\, d^\times a_v = \mathrm{OI}(u(1), f_v dg_v). \tag{18.0.1}$$

Proof. (1) Consider

$$\mathrm{OI}'(d(\alpha), f_v dg_v) = \Delta_v(d(\alpha))\int_{N_v} f_v^{K_v}(n_v^{-1}d(\alpha)n_v)dn_v.$$

Let $n_v = u(x_v)$ with $x_v \in F_v$. Note that $d(\alpha)^{-1}u(-x_v)d(\alpha)u(x_v) = u(x_v(1-\alpha^{-1}))$. Changing variables $x_v \mapsto x_v(1-\alpha^{-1})^{-1}$, we get

$$\Delta_v(d(\alpha))|1-\alpha^{-1}|_v^{-1}\int_{F_v} f_v^{K_v}(d(\alpha)u(x_v))dx_v.$$

Since $\Delta_v(d(\alpha))|1-\alpha^{-1}|_v^{-1} = |\alpha-1|_v|\alpha|_v^{-1/2}/(|\alpha-1|_v|\alpha|_v^{-1}) = |\alpha|_v^{1/2}$, letting $\alpha \to 1$, we obtain $\lim_{\alpha\to 1}\mathrm{OI}'(d(\alpha), f_v dg_v) = \int_{F_v} f_v^{K_v}(u(x_v))dx_v$.

(2) By definition,

$$\mathrm{OI}(u(1), f_v dg_v) = \int_{Z_vN_v\backslash G_v} f_v(g_v^{-1}u(1)g_v)d\dot{g}_v.$$

By the Iwasawa decomposition, $g_v = n_v d_v k_v$ with $n_v \in N_v$, $d_v \in A_v$, and $k_v \in K_v$. Since $g_v \in Z_v\backslash G_v$, it follows that $d_v = d(a_v)$ for some $a_v \in F_v^\times$. Hence the Haar measure decomposition is given by

$$dg_v = \delta_v(d_v)^{-1}dn_v\, dd_v\, dk_v = dn_v\, dk_v\, \frac{d^\times a_v}{|a_v|_v}.$$

For $g_v \in Z_vN_v\backslash G_v$, g_v is of the form $d(a_v)k_v$, and thus

$$\mathrm{OI}(u(1), f_v dg_v) = \int_{F_v^\times} f_v^{K_v}(d(a_v)^{-1}u(1)d(a_v))\,\frac{d^\times a_v}{|a_v|_v} = \int_{F_v^\times} f_v^{K_v}(u(a_v^{-1}))\frac{d^\times a_v}{|a_v|_v}.$$

Changing variables $a_v \mapsto a_v^{-1}$ and recalling that $d^\times a_v = L(1,1_v)\frac{da_v}{|a_v|_v}$, we obtain $L(1,1_v)\int_{F_v} f_v^{K_v}(u(x_v))dx_v$. Finally, recall that

$$\mathrm{OI}'(d(1), f_v dg_v) = L(1,1_v)^{-1}\,\mathrm{OI}(u(1), f_v dg_v).$$

□

Remark. When $f_v = f_v^0$, the integral $\mathrm{OI}'(d(1), f_v dg_v)$ is equal to 1.

Definition 18.1. For $\alpha \in F_v^\times - \{1\}$, the **corrected normalized weighted orbital integral** $\mathrm{CWOI}'(d(\alpha), f_u dg_u)$, of f_u at $d(\alpha)$ is defined to be

$$\mathrm{WOI}'(d(\alpha), f_u dg_u) - 2\ln|1-\alpha^{-1}|_u \cdot \mathrm{OI}'(d(\alpha), f_u dg_u).$$

If $\alpha \in F^\times - \{1\}$ then $|1-\alpha^{-1}| = 1$ and $\sum_{\alpha\neq 1} \ln|1-\alpha^{-1}| \cdot \mathrm{OI}'(d(\alpha), fdg) = 0$. Noting that $\ln|1-\alpha^{-1}| = \sum_u \ln|1-\alpha^{-1}|_u$ and adding 0 to

$$\sum_{\alpha\neq 1}\sum_{u\in\mathbb{V}} \mathrm{WOI}'(d(\alpha), f_u dg_u) \prod_{v\neq u} \mathrm{OI}'(d(\alpha), f_v dg_v),$$

we obtain

$$\sum_{\alpha\neq 1}\sum_{u\in\mathbb{V}} \mathrm{CWOI}'(d(\alpha), f_u dg_u) \prod_{v\neq u} \mathrm{OI}'(d(\alpha), f_v dg_v).$$

Thus in (17.0.2), we may replace WOI' by CWOI'.

Proposition 18.1.1. *For $f_v \in C^\infty_{c,\omega}(G_v)$ and $\alpha \in F_v^\times - \{1\}$, we have*

$$\lim_{\alpha\to 1} \mathrm{CWOI}'(d(\alpha), f_v dg_v) = -2\int_{F_v} f_v^{K_v}(u(x_v)) \ln|x_v|_v dx_v.$$

Denote the last integral by $\mathrm{CWOI}'(d(1), f_v dg_v)$.

Proof. We prove the proposition for the non-Archimedean case; the proof for the Archimedean case is similar.

$$\begin{aligned}\mathrm{CWOI}'(d(\alpha), f_v dg_v) &= \mathrm{WOI}'(d(\alpha), f_v dg_v) - 2\ln|1-\alpha^{-1}|_v \,\mathrm{OI}'(d(\alpha), f_v dg_v)\\ &= -2\Delta_v(d(\alpha)) \int_{F_v} f_v^{K_v}(u(x_v)^{-1} d(\alpha) u(x_v)) \left(\ln|(1,x_v)|_v + \ln\left|1-\frac{1}{\alpha}\right|_v\right) dx_v,\end{aligned}$$

since $\ln\delta_v(\omega u(x_v)) = -2\ln|(1,x_v)|_v$; see Subsection 2.6. As in the proof of Proposition 18.0.1, we have

$$d(\alpha)^{-1} u(-x_v) d(\alpha) u(x_v) = u(x_v(1-\alpha^{-1})).$$

Changing variables $x_v \mapsto x_v(1-\alpha^{-1})^{-1}$, we obtain

$$\begin{aligned}&\mathrm{CWOI}'(d(\alpha), f_v dg_v)\\ &\quad = -2|\alpha|_v^{1/2} \int_{F_v} f_v^{K_v}(d(\alpha)u(x_v)) \left(\ln\left|\left(1, \frac{x_v}{(1-\alpha^{-1})}\right)\right|_v + \ln\left|1-\frac{1}{\alpha}\right|_v\right) dx_v.\end{aligned}$$

Note that

$$\ln|(1, x_v(1-\alpha^{-1})^{-1})|_v = \begin{cases} \ln|x_v(1-\alpha^{-1})^{-1}|_v & \text{if } |x_v|_v \geq |1-\alpha^{-1}|_v,\\ 0 & \text{if } |x_v|_v \leq |1-\alpha^{-1}|_v.\end{cases}$$

Thus

$$\begin{aligned}\mathrm{CWOI}'(d(\alpha), f_v dg_v) = -2|\alpha|_v^{1/2}\Bigg\{ &\ln|1-\alpha^{-1}|_v \int_{|x_v|_v\leq|1-\alpha^{-1}|_v} f_v^{K_v}(d(\alpha)u(x_v))dx_v.\\ &+ \int_{|x_v|_v\geq|1-\alpha^{-1}|_v} f_v^{K_v}(d(\alpha)u(x_v)) \ln|x_v|_v dx_v\Bigg\}.\end{aligned}$$

For $\alpha \in F_v^\times - \{1\}$ is close to 1, we get

$$\mathrm{CWOI}'(d(\alpha), f_v dg_v) = -2\ln|1-\alpha^{-1}|_v |1-\alpha^{-1}|_v f_v^{K_v}(e) - 2\int_{|x_v|_v \geq |1-\alpha^{-1}|_v} f_v^{K_v}(u(x_v)) \ln|x_v|_v dx_v.$$

By L'Hôpital's rule, as $\alpha \to 1$, the limit of the right-hand side is

$$-2\int_{F_v} f_v^{K_v}(u(x_v)) \ln|x_v|_v dx_v = -2\ \mathrm{CWOI}'(d(1), f_v dg_v). \qquad \square$$

18.2 A limit. If we are to add in (17.0.2), corresponding to $\alpha = 1$, the limit of the terms $\alpha \neq 1$ as $\alpha \to 1$ in F_u (for each $u \in \mathbb{V}$), the new term must be

$$\lambda_{-1} \sum_{u \in \mathbb{V}} \mathrm{CWOI}'(d(1), f_u dg_u) \prod_{v \neq u} \mathrm{OI}'(d(1), f_v dg_v).$$

Then we extend the sum over α in (17.0.2) to range over all $\alpha \in F^\times$, subtracting the last displayed expression which, as shown presently, equals (19.0.3).

19 The unipotent terms in the trace formula

Recall the definitions of OI', in particular, of $\mathrm{OI}'(d(1))$, from Subsection 2.12.

Proposition 19.0.1. *The integral of* (15.0.2d) *over* $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ *is*

$$(\ln T)|F^\times \backslash \mathbb{A}^1| \prod_v \mathrm{OI}'(d(1), f_v dg_v) \tag{19.0.1}$$

$$+ \left(\lambda_0 - \lambda_{-1} \sum_v \frac{L'(1, 1_v)}{L(1, 1_v)^2}\right) \prod_v \mathrm{OI}'(u(1), f_v dg_v) \tag{19.0.2}$$

$$+ \lambda_{-1} \sum_u \mathrm{CWOI}'(d(1), f_u dg_u) \prod_{v \neq u} \mathrm{OI}'(d(1), f_v dg_v). \tag{19.0.3}$$

Recall that λ_0 *and* λ_{-1} *are given by the Laurent expansion at* $s = 1$ *of*

$$L(s, 1_F) = \frac{\lambda_{-1}}{s-1} + \lambda_0 + \cdots.$$

Moreover, for $\alpha \in F_v^\times$,

$$\lim_{\alpha \to 1} \mathrm{OI}'(d(\alpha), f_v dg_v) = \mathrm{OI}'(d(1), f_v dg_v);$$

i.e., the unipotent term (19.0.1) *is the limit as* $\alpha \to 1$ *in* A_υ *of the hyperbolic term* (17.0.1). *Furthermore,*

$$\lim_{\alpha\to 1}(-\lambda_{-1}/2)\,\mathrm{CWOI}'(d(\alpha), f_u dg_u)\prod_{\upsilon\neq u}\mathrm{OI}'(d(\alpha), f_\upsilon dg_\upsilon)$$
$$= \lambda_{-1}\,\mathrm{CWOI}'(d(1), f_u dg_u)\cdot\prod_{\upsilon\neq u}\mathrm{OI}'(d(1), f_\upsilon dg_\upsilon);$$

i.e., the unipotent term (19.0.3) *is the limit as* $\alpha \to 1$ *of the hyperbolic term* (17.0.2).

Note that (19.0.2) is Theorem 7.1(iv), and (19.0.3) is Theorem 7.1(iii)-(iii)′, which is equivalent to Theorem 7.1(iv)′.

Proof. Every unipotent regular element $\gamma \in G(F)$ has the form $\gamma = \varepsilon^{-1}u(y)\varepsilon$ with $\varepsilon \in G(F)$, $y \in F^\times$. In fact, $\varepsilon \in B(F)\backslash G(F)$, since $B(F)$ is the normalizer of $N(F) - I$. Thus (15.0.2d) is equal to

$$\sum_{\varepsilon\in B(F)\backslash G(F)}\left(\sum_{y\neq 0} f(x^{-1}\varepsilon^{-1}u(y)\varepsilon x) - \int_{N(\mathbb{A})} f(x^{-1}\varepsilon^{-1}n\varepsilon x)\widehat{\tau}_T(\delta(\varepsilon x))dn\right).$$

The integral of this sum over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ is equal to the integral of

$$\sum_{y\neq 0} f(x^{-1}u(y)x) - \int_{\mathbb{A}} f(x^{-1}u(y)x)\widehat{\tau}_T(\delta(x))dy \tag{19.0.4}$$

over $B(F)Z(\mathbb{A})\backslash G(\mathbb{A})$. Put $x = na'k$ with $a' = d(a, 1)$, $n \in N(F)\backslash N(\mathbb{A})$, $a \in \mathbb{A}^\times/F^\times$, and $k \in K$. Then (19.0.4) equals

$$\sum_{y\neq 0} f(k^{-1}(a')^{-1}n^{-1}u(y)na'k) - \int_{\mathbb{A}} f(k^{-1}(a')^{-1}n^{-1}u(y)na'k)\widehat{\tau}_T(\delta(na'k))dy$$
$$= \sum_{y\neq 0} f(k^{-1}u(y/a)k) - \int_{\mathbb{A}} f(k^{-1}u(y/a)k)\widehat{\tau}_T(|a|)dy$$
$$= \sum_{y\neq 0} f(k^{-1}u(y/a)k) - \int_{\mathbb{A}} f(k^{-1}u(y)k)\widehat{\tau}_T(|a|)|a|dy.$$

□

19.1 Computation of unipotent terms. Let

$$Y(t) = \int_K f(k^{-1}u(t)k)dk \tag{19.1.1}$$

and observe that $Y \in C_c^\infty(\mathbb{A})$. Recall that $dx = dn\,dk\,\frac{d^\times a}{|a|}$. Changing variables $a \mapsto a^{-1}$ transforms the integral of (19.0.4) over $B(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ into

$$\int_{F^\times\backslash\mathbb{A}^\times} \left\{ \sum_{y\in F^\times} Y(ay) - \widehat{Y}(0)\widehat{\tau}_T(|a|^{-1})|a|^{-1} \right\} |a| d^\times a.$$

We split the domain of integration into the domains $|a| \geq 1$ and $|a| \leq 1$. On the domain of $|a| \geq 1$, $\widehat{\tau}_T(|a|^{-1}) = 0$, so

$$I_1 = \int_{|a|\geq 1} \sum_{y\in F^\times} Y(ay)|a| d^\times a.$$

The Poisson summation formula (see Subsection 6.5)

$$\sum_{y\in F} Y(ay) = |a|^{-1} \sum_{y\in F} \widehat{Y}(y/a),$$

applied on the domain $|a| \leq 1$ implies that the integral over $|a| \leq 1$ equals

$$\int_{|a|\leq 1} \left\{ |a|^{-1} \sum_{y\in F} \widehat{Y}(y/a) - Y(0) - \widehat{Y}(0)\widehat{\tau}_T(|a|^{-1})|a|^{-1} \right\} |a| d^\times a$$
$$= \int_{|a|\leq 1} \left\{ \sum_{y\in F^\times} \widehat{Y}(y/a) - Y(0)|a| + \widehat{Y}(0)(1 - \widehat{\tau}_T(|a|^{-1})) \right\} d^\times a.$$

Then the integral of (19.0.4) over $B(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ is $I_1 + I_2 + I_3 - I_4$, where

$$I_2 = \int_{|a|\leq 1} \sum_{y\in F^\times} \widehat{Y}(y/a) d^\times a,$$
$$I_3 = \widehat{Y}(0) \cdot \int_{|a|\leq 1} (1 - \widehat{\tau}_T(|a|^{-1})) d^\times a,$$
$$I_4 = Y(0) \int_{|a|\leq 1} |a| d^\times a.$$

Since $1 - \widehat{\tau}_T(|a|^{-1}) = \begin{cases} 0 & \text{if } |a| < T^{-1}, \\ 1 & \text{if } |a| \geq T^{-1}, \end{cases}$

$$I_3 = \widehat{Y}(0) \int_{T^{-1}}^1 d^\times a = \ln T \cdot |F^\times\backslash\mathbb{A}^1| \widehat{Y}(0).$$

Moreover, for $\mathrm{Re}(s) > 1$,

$$J(s) = \int_{\mathbb{A}^\times} Y(a)|a|^s d^\times a,] = \int_{F^\times\backslash\mathbb{A}^\times} \sum_{y\in F^\times} Y(ay)|a|^s d^\times a = J_1(s) + J^1(s),$$

where

$$J_1(s) = \int_{|a|\geq 1} \sum_{y\neq 0} Y(ay)|a|^s d^\times a$$

$$J^1(s) = \int_{|a|\leq 1} \sum_{y\neq 0} Y(ay)|a|^s d^\times a = \int_{|a|\leq 1} \left\{ \sum_{y\in F} Y(ay)|a|^s - Y(0)|a|^s \right\} d^\times a.$$

Using the Poisson summation formula again, we get

$$\begin{aligned} J^1(s) &= \int_{|a|\leq 1} \left\{ |a|^{s-1} \left(\sum_{y\in F^\times} \widehat{Y}(y/a) + \widehat{Y}(0) \right) - Y(0)|a|^s \right\} d^\times a \\ &= J_2(s) + J_3(s) - J_4(s), \end{aligned}$$

where

$$J_2(s) = \int_{|a|\leq 1} \left(\sum_{y\in F^\times} \widehat{Y}(y/a) \right) |a|^{s-1} d^\times a,$$

$$J_3(s) = \widehat{Y}(0) \int_{|a|\leq 1} |a|^{s-1} d^\times a = \widehat{Y}(0) \int_{|a|\leq 1} |a|^{s-2} da = \widehat{Y}(0) \frac{|F^\times \backslash \mathbb{A}^1|}{s-1},$$

$$J_4(s) = Y(0) \int_{|a|\leq 1} |a|^s d^\times a.$$

Thus $I_1 = J_1(1)$, $I_2 = J_2(1)$, and $I_4 = J_4(1)$, from which it follows that that the integral of (15.0.2d) over $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ is the sum of $I_3 = \ln T \cdot |F^\times \backslash \mathbb{A}^1| \cdot \widehat{Y}(0)$ and the value at $s = 1$ of

$$J(s) - J_3(s) = \int_{\mathbb{A}^\times} Y(a)|a|^s d^\times a - \widehat{Y}(0) \frac{|F^\times \backslash \mathbb{A}^1|}{s-1}.$$

Continuing, we have $J(s) = L(s, 1_F)\theta_f(s)$, where

$$\theta_f(s) = \frac{1}{L(s, 1_F)} \int_{\mathbb{A}^\times} \int_K f(k^{-1}u(a)k)|a|^s \, dk \, d^\times a.$$

For $\mathrm{Re}(s) > 0$, the product $\theta_f(s) = \prod_v \theta_{f_v}(s)$, where $\theta_{f_v}(s)$ is the local analogue, is convergent. Indeed, if $f_v = f_v^0$, then $\theta_{f_v^0}(s) = 1$ if $\mathrm{Re}(s) > 0$. Consequently, the function $\theta_f(s)$ is holomorphic at $s = 1$, and its Taylor series expansion at $s = 1$ is $\theta_f(s) = \theta_f(1) + (s-1)\theta_f'(1) + \cdots$. Thus the value at $s = 1$ of $J(s) - J_3(s) = L(s, 1_F)\theta_f(s) - \widehat{Y}(0)|F^\times \backslash \mathbb{A}^1|/(s-1)$ is $\lambda_0\theta_f(1) + \lambda_{-1}\theta_f'(1)$.

We get $\theta_f(1) = \prod_v \theta_{f_v}(1)$, where

$$\theta_{f_v}(1) = \frac{1}{L(1, 1_v)} \int_{F_v^\times} f_v^{K_v}(u(a_v))|a_v|_v \, d^\times a_v = \mathrm{OI}'(u(1), f_v d\dot{g}_v).$$

When $f_v = f_v^0$, $\theta_{f_v^0}(1) = 1$. The derivative of the finite product θ_f is $\theta_f'(s) = \sum_u \theta_{f_u}'(s) \prod_{v \neq u} \theta_{f_v}(s)$, where the sum ranges over the finite set of places where $f_u \neq f_u^0$, and

$$\theta_{f_u}'(s) = \frac{1}{L(s, 1_u)} \int_{F_u^\times} f_u^{K_u}(u(a_u))|a_u|_u^s \ln |a_u|_u \, d^\times a_u - \frac{L'(s, 1_u)}{L(s, 1_u)^2} \, \theta_{f_u}(s).$$

The first term here has the value $\mathrm{CWOI}'(d(1), f_u dg_u)$ at $s = 1$.

20 The geometric side of the corrected trace formula

Recall from Subsection 2.12 that

$$f_N^K(d(\alpha)) = \delta(d(\alpha))^{1/2} \int_{N(\mathbb{A})} \int_K f(k^{-1} d(\alpha) n k) dn \, dk.$$

Corollary 20.0.1. *The integral of the truncated kernel $\Lambda_2^T K_f$ satisfies*

$$\Lambda_2^T K_f(x, x) = \mathrm{vol}(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))f(e) \tag{20.0.1a}$$

$$+ \frac{1}{2} \sum_{\{T\}} \mathrm{vol}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}); dt) \sum_{\gamma \in Z(F)\backslash T(F)^{\mathrm{reg}}} \mathrm{OI}(\gamma, f dx) \tag{20.0.1b}$$

$$- \frac{\lambda_{-1}}{2} \sum_{\alpha \in F^\times} \sum_u \mathrm{CWOI}'(d(\alpha), f_u dg_u) \prod_{v \neq u} \mathrm{OI}'(d(\alpha), f_v dg_v) \tag{20.0.1c}$$

$$+ \left(\lambda_0 - \lambda_{-1} \sum_{\{u; f_u \neq f_u^0\}} \frac{L'(1, 1_v)}{L(1, 1_v)^2} \right) \mathrm{OI}'(d(1), f dg) \tag{20.0.1d}$$

$$+ (\ln T) \sum_{\alpha \in F^\times} f_N^K(d(\alpha)). \tag{20.0.2}$$

Remark. The term (20.0.2) depends on T, but $\mathrm{tr}\, r(f dg)$ does not. Thus we can expect (20.0.2) to cancel another term. This term, (21.0.1), is be discussed in Subsection 21.1. Note that (20.0.2) comes from (17.0.1)+(19.0.1)). The term (20.0.1a) comes from (16.0.1a) , (20.0.1b) from (16.0.2b), (20.0.1d) from (19.0.2), and (20.0.1c) from (17.0.2)+(19.0.3).

21 Contribution from the continuous spectrum

Using the kernel on the continuous spectrum computed in Chapter II, we prove the following proposition.

Proposition 21.0.1. *The contribution* $\int \Lambda_2^T K_{f,c}(x,x)dx$ *from the continuous spectrum is* (20.0.2) *plus*

(-v) $$+\frac{1}{4}\sum_{\mu=\widetilde{\mu}} \text{tr } M(\mu) i_B^G(\mu, fdg),$$

(-vi) $$+\frac{1}{4\pi}\sum_{\mu}\int_{-\infty}^{\infty} m(\mu(iy))^{-1}m'(\mu(iy))\cdot \text{tr}\, i_B^G(\mu(iy), fdg)dy,$$

(-vii) $$+\frac{1}{4\pi}\sum_{\mu}\int_{-\infty}^{\infty} \text{tr } [R(\mu(iy))^{-1}R'(\mu(iy))i_B^G(\mu(iy), fdg)]dy.$$

$$+o(T)$$

where $o(T)$ *is a term which tends to* 0 *as* $T\to\infty$. *In* (-v), $M(\mu)=-1$. *In* (-vi) *and* (-vii), *which equal the negatives of Theorem* 7.1(vi) *and* (vii), *respectively,* μ *ranges over the set specified in Subsection* 6.2.

Note that the contribution of Section 21.0.1 from the continuous spectrum to the trace formula appears with a negative coefficient; see (14.0.3). To obtain Theorem 7.1(vii) from (-vii) write $R=\otimes_v R_v$ and compute the logarithmic derivative $R'R^{-1}=\sum_v R_v' R_v^{-1}$.

Proof. Recall that the kernel $K_{f,c}(x,x')$ of the continuous spectrum is given by (13.3.2). Truncating with respect to the second variable, we get

$$\Lambda_2^T K_{f,c}(x,x') = \frac{1}{4\pi}\sum_{\mu}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, \varphi_\alpha^\mu)E(\varphi_\alpha^\mu, iy, x)\Lambda^T\bar{E}(\varphi_\beta^\mu, iy, x')dy.$$

Note that φ_α^μ, φ_β^μ range over an orthogonal basis for $\mathrm{H}(\mu(0))$. Integrating this over the diagonal $x'=x$ in $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ and interchanging the order of integration, we obtain

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\Lambda_2^T K_{f,c}(x,x)dx = \frac{1}{4\pi}\sum_{\mu}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, \varphi_\alpha^\mu)X(y)dy,$$

where

$$X(y)=\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}E(\varphi_\alpha^\mu, iy, x)\Lambda^T\bar{E}(\varphi_\beta^\mu, iy, x)dx.$$

Use (12.1.8) with $\varphi_{1\mu}(s) = \varphi_\alpha^\mu(s)$, $\varphi_{2\mu}(s) = \varphi_\beta^\mu(s)$ to get

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \Lambda_2^T K_{f,c}(x,x)dx$$

$$= \frac{\ln T}{2\pi} \sum_\mu \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, \varphi_\alpha^\mu)(\varphi_\alpha^\mu, \varphi_\beta^\mu)dy, \tag{21.0.1}$$

$$+ \frac{1}{4\pi} \sum_\mu \sum_{\alpha,\beta} \int_{-\infty}^{\infty} \{(M(\mu(-iy))^{-1}M'(\mu(iy))\varphi_\alpha^\mu, \varphi_\beta^\mu) \cdot (i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, \varphi_\alpha^\mu)\}dy \tag{21.0.2}$$

$$+ \frac{1}{4\pi} \sum_\mu \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, \varphi_\alpha^\mu) \cdot \left\{(\varphi_\alpha^\mu, M(\mu(iy))\varphi_\beta^\mu)\frac{T^{2iy}}{2iy} - (M(\mu(iy))\varphi_\alpha^\mu, \varphi_\beta^\mu)\frac{T^{-2iy}}{2iy}\right\}dy. \tag{21.0.3}$$

To complete the proof of the proposition we need to compute (21.0.1)-(21.0.3).

21.1 The term (21.0.1)**.** After changing the order of $\sum$ and $\int$, we can write the term (21.0.1) as $(\ln T/2\pi)\sum_\mu \int_{-\infty}^{\infty} \operatorname{tr}\ i_B^G(\mu(iy), fdg)dy$. But $i_B^G(\mu(iy))$ is an induced representation whose trace can be computed (the standard computation is left to the reader) to be

$$\sum_{\alpha\in F^\times} \int_{F^\times\backslash\mathbb{A}^1} \int_{F_\infty^+} f_N^K(d(a\alpha t))\mu_1(a)|t|^{iy+\frac{1}{2}}\, d^\times t\, d^\times a.$$

Hence our term is

$$\ln T \sum_{\alpha\in F^\times} \sum_{\mu_1} \int_{F^\times\backslash\mathbb{A}^1} \int_{-\infty}^{\infty} \int_{F_\infty^+} f_N^K(d(a\alpha t))\mu_1(a)|t|^{2\pi iy} d^\times t\, dy\, d^\times a.$$

Since μ_1 ranges over the group of characters of $F^\times\backslash\mathbb{A}^1$, in the Fourier inversion formula ($\widehat{\widehat{F}}(a) = F(a^{-1})$ at $a = 1$), applied to the function $F = f_N^K$ and the group $F^\times\backslash\mathbb{A}^1$, the sum over μ_1 and integral over $F^\times\backslash\mathbb{A}^1$ cancel, and we set $a = 1$. Similarly, in the Fourier inversion formula, applied to the same function on the group $F_\infty^+ = \mathbb{R}_{>0}^\times$ and its character group $i\mathbb{R} \ni iy : t \mapsto t^{2\pi iy}$, the integrals $\int_{-\infty}^{\infty}$ and $\int_{F_\infty^+}$ cancel, and we set $t = 1$. We obtain $(\ln T)\sum_{\alpha\in F^\times} f_N^K(d(\alpha))$, which is (21.0.1).

21.2 The term (21.0.2)**.** This term can be written as

$$\frac{1}{4\pi} \sum_\mu \int_{-\infty}^{\infty} \operatorname{tr}\ M(\mu(iy))^{-1}M'(\mu(iy))i_B^G(\mu(iy), fdg)dy.$$

Recall that the intertwining operator $M(\mu)$ is $m(\mu)R(\mu)$, and $R(\mu)$ is a product $\otimes_v R(\mu_v)$ of local operators $R(\mu_v)$, each acting as 1 on the unit vector in the space of the induced representation $i_B^G(\mu_v)$ when μ_v is unramified. Formally, taking the logarithm of $M(\mu(s)) = m(\mu(s)) \otimes_v R(\mu_v)$, noting that on each vector in the induced representation the product is finite, and differentiating with respect to the parameter s, we obtain

$$M(\mu(s))^{-1}M'(\mu(s)) = m(\mu(s))^{-1}m'(\mu(s)) + \sum_v R(\mu_v(s))^{-1}R'(\mu_v(s)),$$

where, on each vector, the sum over v is finite. In particular, if μ_v is unramified and f_v is spherical, the operator $R(\mu_v(s))i_B^G(\mu_v(s), f_v)$ factorizes through the projection on the K_v-fixed vector in $i_B^G(\mu_v(s))$, the operator $R(\mu_v(s))$ is a constant on this vector, and its derivative is 0. Thus we obtain, for the term in question,

(-vi) $$\frac{1}{4\pi}\sum_\mu \int_{-\infty}^{\infty} m(\mu(iy))^{-1}m'(\mu(iy))\ \text{tr}\ i_B^G(\mu(iy), fdg)dy$$

(-vii) $$+\frac{1}{4\pi}\sum_\mu \int_{-\infty}^{\infty} \text{tr}\ [R(\mu(iy))^{-1}R'(\mu(iy))i_B^G(\mu(iy), fdg)]dy.$$

Here, $R(\mu(iy))^{-1}R'(\mu(iy))$ denotes $\sum_v R(\mu_v(iy))^{-1}R'(\mu_v(iy))$, where the sum over v is finite and ranges over the places at which μ_v is not unramified.

21.3 The term (21.0.3)**.** As for this term, it can be written as

$$\frac{1}{4\pi}\sum_\mu\sum_\beta \int_{-\infty}^{\infty} \Big[(i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, M(\mu(iy))\varphi_\beta^\mu)\frac{T^{2iy}}{2iy}\cdot$$
$$-(i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, M(\mu(-iy))^{-1}\varphi_\beta^\mu)\frac{T^{-2iy}}{2iy}\Big]dy$$

or

$$\frac{1}{4\pi}\sum_\mu\sum_\beta \int_{-\infty}^{\infty} (M(\mu(-iy))^{-1}i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, \varphi_\beta^\mu)\frac{T^{2iy}-T^{-2iy}}{2iy}dy$$

(21.3.1) $$+\frac{1}{4\pi}\sum_\mu\sum_\beta \int_{-\infty}^{\infty} \Big[(i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, M(\mu(iy))\varphi_\beta^\mu)$$
$$-(i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, M(\mu(-iy))\varphi_\beta^\mu)\Big]\frac{T^{-2iy}}{2iy}dy.$$

The last term here is the Fourier transform $\widehat{F}(x) = \int_{-\infty}^{\infty} F(y)e^{-2\pi ixy}dy$ of the integrable function

$$F(y) = \sum_{\mu}\sum_{\beta}[(i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, M(\mu(iy))\varphi_\beta^\mu) - (i_B^G(\mu(iy), fdg)\varphi_\beta^\mu, M(\mu(-iy))\varphi_\beta^\mu)]/8\pi iy$$

at $\ln T/\pi$, and so tends to 0 as $T \to \infty$. The following lemma shows that the function $F(y)$ is integrable.

Lemma 21.4. *The improper integral*

$$\int_{-\infty}^{\infty}\sum_{\mu}\sum_{\beta}(i(\mu(iy), f)\psi_\beta)(g)\overline{\psi}_\beta(g)d|y|$$

converges.

Proof. For $z \in \mathbb{C}$, the induced representation $i(z) = i_{N(\mathbb{A})A(F)A_\infty^+}^{G(\mathbb{A})}(\delta^z)$, which acts by right translation on the Hilbert space $\mathrm{H}(z)$ of functions $\psi : G(\mathbb{A}) \to \mathbb{C}$ with $\int_K |\psi(k)|^2dk < \infty$ and $\psi(nag) = \delta(a)^{z+1/2}\psi(g)$, $n \in N(\mathbb{A})$, $a \in A(F)A_\infty^+$, decomposes as the direct sum over μ of $i(\mu(z))$. Hence it suffices to show the convergence of

$$\int_{-\infty}^{\infty}\sum_{\beta}(i(iy, f)\psi_\beta)(g)\overline{\psi}_\beta(g)d|y|.$$

To this end, we may replace i by i', where $i'(z)$ acts on the space

$$\mathrm{H} = \left\{\psi : N(\mathbb{A})A(F)A_\infty^+\backslash G(\mathbb{A}) \to \mathbb{C};\ \int_K |\psi(k)|^2dk < \infty\right\}$$

by $(i'(g, z)\psi)(h) = \delta(hg)^{z+\frac{1}{2}}\delta(h)^{-z-\frac{1}{2}}\psi(hg)$. Indeed, $i(z)$ and $i'(z)$ are equivalent via the intertwining operator $I(z) : i(z) \to i'(z)$ given by $I(z) : \psi(g) \mapsto \psi'(g) = \delta(g)^{-z-\frac{1}{2}}\psi(g)$, as $\psi'(ang) = \delta(ang)^{-z-\frac{1}{2}}\psi(ang) = \delta(g)^{-z-\frac{1}{2}}\psi(g)$, since $\delta(ang) = \delta(a)\delta(g)$.

Let us compute the convolution operator

$$\begin{aligned}(i'(z, f)\psi)(h) &= \int_{Z_\infty^+\backslash G(\mathbb{A})} f(g)(i'(g, z)\psi)(h)dg\\ &= \int_{Z_\infty^+\backslash G(\mathbb{A})} f(g)\psi(hg)\delta(hg)^{z+\frac{1}{2}}\delta(h)^{-z-\frac{1}{2}}dg\\ &= \delta(h)^{-z-\frac{1}{2}}\int_{Z_\infty^+\backslash G(\mathbb{A})} f(h^{-1}g)\psi(g)\delta(g)^{z+\frac{1}{2}}dg.\end{aligned}$$

Decomposing $g = n\alpha tk$, we get that this is $\int_K \mathcal{S}(z, f; h, k)\psi(k)dk$, where

$$\mathcal{S}(z, f; h, k) = \delta(h)^{-z-\frac{1}{2}} \sum_{\alpha \in A(F)} \int_{t \in F_\infty^+} \int_{N(\mathbb{A})} f(h^{-1}n\alpha d(t)g)\delta(d(t))^{z-\frac{1}{2}} dt\, dn,$$

with $\delta(d(t)) = t > 0$.

Since f is compactly supported on $Z_\infty^+\backslash G(\mathbb{A})$ and $\mathcal{S}(z, f; h, g)$ is a continuous function in (h, g) on the compact space $(N(\mathbb{A})A(F)A_\infty^+\backslash G(\mathbb{A}))^2$, the operator $i'(z, f)$ is Hilbert-Schmidt on H with continuous kernel $\mathcal{S}(z, f; h, g)$.

For fixed h, g, f, the function $y \mapsto \mathcal{S}(iy, f; h, g)$ is Bruhat-Schwartz on $\mathbb{R}$, namely, it is Fourier transform of a compactly supported function. Hence the integral $\int_{-\infty}^{\infty} \mathcal{S}(iy, f; h, g)d|y|$ is finite. The sum $\sum_\beta (i'(iy, f)\psi_\beta)(h)\overline{\psi}_\beta(g)$ defines a Hilbert-Schmidt kernel for $i'(iy, f)$ also, which equals $\mathcal{S}(iy, f; h, g)$ since both kernels are continuous in h and g. □

To evaluate the first term in (21.3.1) we need the following lemma.

Lemma 21.5. *If F is continuous on $\mathbb{R}$, and F, $\widehat{F}$ are integrable, then*

$$\lim_{x \to +\infty} \int \frac{e^{2\pi ixy} - e^{-2\pi ixy}}{y} F(y)dy = 2\pi i F(0).$$

Proof. Set $G(x) = \int_{-\infty}^{\infty}((e^{2\pi ixy} - e^{-2\pi ixy})/y)F(y)dy$. Then $G(0) = 0$, and

$$G'(x) = 2\pi i \int (e^{2\pi ixy} + e^{-2\pi ixy})F(y)dy = 2\pi i(\widehat{F}(x) + \widehat{F}(-x)).$$

Therefore,

$$G(x) = 2\pi i \int_0^x (\widehat{F}(t) + \widehat{F}(-t))dt = 2\pi i \int_{-x}^x \widehat{F}(t)dt \quad \text{for } x > 0.$$

So as $x \to \infty$, $G(x) \to 2\pi i F(0)$. □

21.6 Completion of the proof of Proposition 21.0.1. Applying Lemma 21.5 to (21.3.1), we conclude that the limit of (21.3.1) as $T \to \infty$ is

$$\text{(-v)} \qquad \frac{1}{4} \sum_\mu \sum_\beta (M(\mu(0))i_B^G(\mu(0), fdg)\varphi_\beta^\mu, \varphi_\beta^\mu) = \frac{1}{4} \sum_\mu \operatorname{tr} M(\mu)i_B^G(\mu, fdg).$$

By (5.3.1), since μ is unitary, $i_B^G(\mu)$ is irreducible. Recall that the mapping $M(\mu) : i_B^G(\mu) \to i_B^G(\widetilde{\mu})$ is a $G(\mathbb{A})$-equivariant isomorphism. Hence, unless $\mu = \widetilde{\mu}$, $\operatorname{tr} M(\mu)i_B^G(\mu, fdg) = 0$. Hence $\sum \operatorname{tr} M(\mu)i_B^G(\mu, f)$ ranges only over $\mu = (\mu_1, \mu_1)$, $\mu_1^2 = \omega$.

Note that $M(\mu)$ in (v) is a scalar, since $i_B^G(\mu)$ is irreducible. It equals $m(\mu)$ (see (5.2.1) for $m(\mu_v)$) and this equals -1, by the the functional equation for L-functions (see Subsection 5.2 for the local case) and L'Hôpital's rule. The proof of Proposition 21.0.1 is now complete. □

Chapter IV
Invariant Trace Formula

In this chapter, we present the invariant trace formula for GL(2).

22 Invariant trace formula for GL(2)

22.1 Equivalence classes. The trace formula has been rearranged by Arthur to take the form

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} J_{\chi}(f).$$

Here $\mathcal{O}$ is the set of equivalence classes of elements in $G(F)$, where $\gamma \sim \gamma'$ if they have same eigenvalues. The set $\mathfrak{X}$ consists of equivalence classes of irreducible representations of $G(\mathbb{A})$, where $\pi \sim \pi'$ if both are constituents of $i_B^G(\mu)$ or $i_B^G(\widetilde{\mu})$ (with the same μ), or are equivalent cuspidal representations. We write $\mathcal{O}_e(G)$ for the set of $\mathfrak{o} \in \mathcal{O}$ with eigenvalues not in F (elliptic regular classes). Then $\mathcal{O}_h(G)$ is the set of $\mathfrak{o} \in \mathcal{O}$ with distinct eigenvalues in $F^\times$, and $\mathcal{O}_u(G)$ is the set of the $\mathfrak{o} \in \mathcal{O}$ with equal eigenvalues. In our case of $Z(F)\backslash G(F)$, $\mathcal{O}_u(F)$ consists of the identity and the unipotent regular elements. Thus the terms (ii) are parametrized by $\mathfrak{o} \in \mathcal{O}_e(G)$, the terms (iii)′ by $\mathfrak{o} \in \mathcal{O}_h(G)$, and the terms (iv)′ and (i) by $\mathcal{O}_u(G)$. We write $\mathfrak{X}_0(G)$ for the cuspidal representations in $L^2_\omega(G)$, $\mathfrak{X}_1(G)$, for the one-dimensional representations and $\mathfrak{X}_c(G)$ for the disjoint union of the $\mathfrak{X}_\mu(G) = i_B^G(\mu)$ over the characters $\mu = (\mu_1, \mu_2) : A(\mathbb{A})/A(F)A^+_\infty \to \mathbb{C}^1$ with $\mu_1\mu_2 = \omega$, up to the equivalence $\mu \sim \widetilde{\mu}$.

22.2 Study of invariance. All terms in the trace formula except for (iii) and (vii) are invariant. A distribution D on G (i.e., a linear function $D : C^\infty_{c,\omega}(G) \to \mathbb{C}$) is called **invariant** if its value at f^y is independent of $y \in G(\mathbb{A})$, where $f^y(x) = f(y^{-1}xy)$. Our aim is to incorporate (vii) into (iii) to form a sum over $\mathfrak{o} \in \mathcal{O} - \mathcal{O}_e(G)$ of invariant distributions. For this, we first need to study the failure of the terms in (iii) and (vii) to be invariant. This has been done

by Arthur [Ar 81] for a general reductive group. We work out here our example of $G = \mathrm{GL}(2)$.

We introduce some notation.

(1) Let $H(g) = \ln \delta(g)$, and $t = \ln T > 0$.

(2) For $a \in A(\mathbb{A})$ and $f \in C^\infty_{c,\omega}(G)$, put

$$f_{B,y}(a) = \delta(a)^{1/2} \int_K \int_{N(\mathbb{A})} f(k^{-1}ank)H(ky)dn\,dk.$$

Here $y \in G(\mathbb{A})$; and, if $ky = d(\alpha, \beta)n'k'$, then $H(ky) = \ln \delta(ky) = \ln|\alpha/\beta|$.

(3) Denote by $J^T_{\mathfrak{o}}(f)$ the integral over $x \in Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ of

$$K^f_{G,\mathfrak{o}}(x,x) - \sum_{\eta \in B(F)\backslash G(F)} K^f_{B,\mathfrak{o}}(\eta x, \eta x)\widehat{\tau}_B(H(\eta x) - t),$$

where $\hat{\tau}_B$ is the characteristic function of $\mathbb{R}_{>0}$. This is the contribution to the trace formula from the elements of type $\mathfrak{o}$. Here $K^f_{G,\mathfrak{o}}(x,y)$ is the part $\sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma y)$ of the kernel $K^f_G(x,y) = \sum_{\gamma\in Z(F)\backslash G(F)} f(x^{-1}\gamma y)$ of terms in $\mathfrak{o}$. In general,

$$K^f_{P,\mathfrak{o}}(x,y) = \sum_{\gamma\in M_P(F)\cap\mathfrak{o}} \int_{N_P(\mathbb{A})} f(x^{-1}\gamma \mathbf{n} y)d\mathbf{n}$$

if P is a standard F-parabolic subgroup of G with standard Levi subgroup M_P and unipotent radical N_P. In our case of $P = B$, M_B is A and N_B is N.

Proposition 22.2.1. *For all $f \in C^\infty_{c,\omega}(G)$ and $y \in G(\mathbb{A})$ and $\mathfrak{o} \in \mathcal{O}$,*

$$J^T_{\mathfrak{o}}(f^y) - J^T_{\mathfrak{o}}(f) = \sum_{\gamma\in A\cap\mathcal{O}} f_{B,y}(\gamma).$$

Proof. Since

$$J^T_{\mathfrak{o}}(f^y) = \int \Big[K^f_{G,\mathfrak{o}}(x,x) - \sum_{\eta\in B(F)\backslash G(F)} K^f_{B,\mathfrak{o}}(\eta x, \eta x)\widehat{\tau}_B(H(\eta xy) - t)\Big]dx,$$

we need to consider

$$\begin{aligned}\widehat{\tau}_B(H(\eta xy) - t) - \widehat{\tau}_B(H(\eta x)) &= \widehat{\tau}_B(H(\eta xk^{-1}ky) - t) - \widehat{\tau}_B(H(\eta x)) \\ &= \widehat{\tau}_B(H(\eta x) - t + H(ky)) - \widehat{\tau}_B(H(\eta x)),\end{aligned}$$

where $k = k_B(\eta x) \in K$ is such that $\eta xk^{-1} \in B(\mathbb{A})$. Put $H = H(\eta x) - t$, $X = -H(ky)$. As a function of H, $\Gamma'_B(H,X) = \widehat{\tau}_B(H) - \widehat{\tau}_B(H - X)$ is the characteristic function of $\{H \in \mathbb{R}; 0 < H < X\}$, since $\widehat{\tau}_B$ is the characteristic function of $\{H > 0\}$. Hence $\int_{\mathbb{R}} \Gamma'_B(H,X)dH = X$. We conclude that

$$J^T_{\mathfrak{o}}(f^y - f) = -\int_{Z(\mathbb{A})B(F)\backslash G(\mathbb{A})} K^f_{B,\mathfrak{o}}(x,x)\Gamma'_B(H(x) - t, -H(k_B(x)y))dx.$$

Writing $x = nak$ and noting that

$$K^f_{B,\mathfrak{o}}(nak, nak) = \sum_{\gamma \in A(F)\cap \mathfrak{o}} \int_{N(\mathbb{A})} f(k^{-1}a^{-1}n^{-1}\gamma \mathbf{n} nak) d\mathbf{n} = \delta(a) K^f_{B,\mathfrak{o}}(k,k)$$

since $dx = \delta(a)^{-1} dn\, da\, dk$, we see that

$$\begin{aligned} J^T_{\mathfrak{o}}(f^y - f) &= -\int_K \Big[\int_{A(\mathbb{A})/Z(\mathbb{A})A(F)} \Gamma'_B(H(a) - t, -H(ky)) da \Big] K^f_{B,\mathfrak{o}}(k,k) dk \\ &= \int_K K^f_{B,\mathfrak{o}}(k,k) H(ky) dk \\ &= \int_K \sum_{\gamma \in A(F)\cap \mathcal{O}} \int_{N(\mathbb{A})} f(k^{-1}\gamma \mathbf{n} k) d\mathbf{n} H(ky) dk = \sum_{\gamma \in A(F)\cap \mathcal{O}} f_{B,y}(\gamma). \end{aligned}$$

□

22.3 Continuous spectrum. The kernel on the continuous spectrum is $K^f_{c,\omega}(x,y) = \sum_\mu K^f_\mu(x,y)$, sum over the pairs of characters $\mu = (\mu_1, \mu_2)$ of $F^+_\infty F^\times \backslash \mathbb{A}^\times$ with $\mu_1\mu_2 = \omega$. Here,

$$K^f_\mu(x,g) = \frac{1}{4\pi} \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (i^G_B(\mu(iy), fdg)\varphi^\mu_\beta, \varphi^\mu_\alpha) E(\varphi^\mu_\alpha, iy, x) \overline{E}(\varphi^\mu_\beta, iy, g) dy.$$

The equivalence class χ of μ consists of $\{\mu, \widetilde{\mu}\}$. Put

$$I_\chi(fdg) = \frac{1}{4\pi} \sum_{\mu \in \chi} \int_{-\infty}^{\infty} m(\mu(iy))^{-1} m'(\mu(iy)) \operatorname{tr}\, i^G_B(\mu(iy), fdg) dy$$

from (-vi) of Subsection 21.2.

Let $K^f_{B,\mu}(x,y)$ be the kernel of the restriction of the operator $r_B(fdg)$ to the invariant subspace $L^2_\chi = L^2_\omega(N(\mathbb{A})A(F)Z(\mathbb{A})\backslash G(\mathbb{A}))_\chi$ of $L^2_\omega(N(\mathbb{A})A(F)Z(\mathbb{A})\backslash G(\mathbb{A}))$ of automorphic forms type μ or $\widetilde{\mu}$. This kernel can be obtained by projecting $K^f_{B,\omega}(x,y) = \sum_{\gamma \in A(F)/Z(F)} \int_{N(\mathbb{A})} f(x^{-1}\gamma n y) dn$, regarded as a function of x or y, onto L^2_χ. In other words,

$$K^f_{B,\mu}(x,y) = \int_{Z(\mathbb{A})A\backslash A(\mathbb{A})} \int_{N(\mathbb{A})} f(x^{-1}\mathbf{a}\mathbf{n}y) \delta(\mathbf{a})^{1/2} \mu(\mathbf{a})\, d\mathbf{a}\, d\mathbf{n}.$$

Note that $\mu(z) = \omega(z)$ on $z \in Z(\mathbb{A})$, and $f(zg) = \omega^{-1}(z) f(g)$.

Proposition 22.3.1. *For $f \in C^\infty_{c,\omega}(G)$ and $y \in G(\mathbb{A})$ and $\chi \in \mathfrak{X}$,*

$$J^T_\chi(f^y) - J^T_\chi(f) = \sum_{\mu \in \chi} \widehat{f}_{B,y}(\mu).$$

Proof. Repeating the computation of $J_{\mathfrak{o}}^T(f^y - f)$ for J_χ^T (defined on replacing $\mathcal{O}$ by χ in $J_{\mathfrak{o}}^T$), we obtain that $J_\chi^T(f^y - f)$ is

$$J_\chi^T(f^y - f) = -\int_{Z(\mathbb{A})B(F)\backslash G(\mathbb{A})} K_{B,\chi}^f(x,x)\Gamma'_B(H(x) - T, -H(k_B(x)y))dx.$$

Using the Iwasawa decomposition $x = nak$, $a \in Z(\mathbb{A})A(F)\backslash A(\mathbb{A})$, $k \in K$, $n \in N(F)\backslash N(\mathbb{A})$, with $dx = \delta(a)^{-1}dn\,da\,dk$, we compute the integral over n, a, k, $\mathbf{a}$, $\mathbf{n}$ of

$$-f(k^{-1}a^{-1}n^{-1}\mathbf{a}\,\mathbf{n}nak)\delta(\mathbf{a})^{1/2}\mu(\mathbf{a})\cdot\Gamma'_B(H(a) - T, -H(ky))\delta(a)^{-1}.$$

The integral over $n \in N(F)\backslash N(\mathbb{A})$ can be absorbed into that of $\mathbf{n}$ and be eliminated. Conjugating $\mathbf{n}$ by a replaces $\mathbf{n} = u(\mathbf{x})$ by $u(\mathbf{x}/\alpha)$ if $a = d(\alpha)$. By the change of variables $\mathbf{x} \mapsto \alpha\mathbf{x}$, we gain a factor $\delta(a)$ which cancels the $\delta(a)^{-1}$. So $a = d(\alpha)$ is no longer present in the argument of the function f. Now $H(d(\alpha)) = \ln|\alpha|$. The change of variables $a \mapsto H = \ln|\alpha|$, transforms $d^\times a$ into dH. We also have $\int_{\mathbb{R}}\Gamma'_B(H, -H(ky))dH = -H(ky)$. Thus

$$\int_{A(F)Z(\mathbb{A})\backslash A(\mathbb{A})}\delta(\mathbf{a})^{1/2}\int_K\int_{N(\mathbb{A})} f(k^{-1}\mathbf{a}nk)dn\,H(ky)dk\;\mu(\mathbf{a})da$$
$$= \int_{A(F)Z(\mathbb{A})\backslash A(\mathbb{A})} f_{B,y}(\mathbf{a})\mu(\mathbf{a})d\mathbf{a} = \widehat{f}_{B,y}(\mu). \qquad \square$$

22.4 Invariant trace formula. Define

$$\phi_A(\mu, fdg) = \mathrm{tr}(R(\mu,0)^{-1}R'(\mu,0)i_B^G(\mu, fdg)).$$

Then the function $\phi_A(fdg) : \mu \mapsto \phi_A(\mu, fdg)$ is a Schwartz function on the space of characters μ on $A(\mathbb{A})/A(F)$ (with $\mu = \omega$ on $Z(\mathbb{A})$) and is the Fourier transform of a Schwartz function on $A(\mathbb{A})/A(F)$. Denote the restriction of this function to $A^1/A(F) \simeq (\mathbb{A}^1/F^\times)^2$ by $\phi_A^\vee(fdg)$. Then the term (vii) in the trace formula is

$$\frac{1}{4\pi}\sum_\mu\int_{-\infty}^{\infty}\phi_A(\mu(iy), fdg)dy,$$

where $\mu(iy) = (\mu_1\nu^{iy}, \mu_2\nu^{-iy})$. This is the sum over $\mu \in \chi$ of the values of the Fourier transform of $\phi_A^\vee(fdg)$ at μ. Since $\phi_A^\vee(fdg)$ is a Schwartz function, the Poisson summation formula on $A^1/A(F)$ can be applied, which yields $\sum_{\alpha,\beta\in F^\times}\phi_A^\vee((\alpha,\beta), fdg)$.

In fact, using the product formula and the fact that $\phi_A(\mu, fdg)$ is a finite sum over v of products where all but one factors are the Fourier transform of smooth and compactly supported local functions, it can be seen that $\phi_A^\vee(fdg)$ is compactly supported on $(\mathbb{A}^1/F^\times)^2$

If $\mathfrak{o}$ is the equivalence class (union of conjugacy classes) with eigenvalues $\alpha, \beta \in F^\times$, put

$$I_\mathfrak{o}(fdg) = J_\mathfrak{o}(fdg) - \sum_{(\alpha,\beta)\in\mathcal{O}} \phi_A^\vee((\alpha,\beta), fdg).$$

It follows from our computation of $J_\mathfrak{o}^T(f^y - f)$ and $J_\chi^T(f^y - f)$ that $I_\mathfrak{o}(fdg)$ is an invariant distribution. We obtain a trace formula

$$\text{tr } r_0(fdg) = \sum_{\mathfrak{o}\in\mathcal{O}} I_\mathfrak{o}(fdg) - \sum_{\chi\in\mathfrak{X}-\mathfrak{X}_0(G)} I_\chi(fdg),$$

in which all distributions are invariant.

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