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Catalan transform of the incomplete Jacobsthal numbers and incomplete generalized Jacobsthal polynomials

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We introduce the Catalan transform of the Incomplete Jacobsthal numbers. We apply the Hankel transform to the Catalan transforms of these numbers. We calculate determinants of matrixes formed with $CJ_{k,n}^{l}$ by using Hankel transform. Then we define the incomplete $h(x)$-Jacobsthal polynomials. Then we examine the recurrence relation and some properties of these polynomials.

Keywords: Binet formula; generating function; $h(x)$-Jacobsthal polynomials; incomplete $h(x)$- Jacobsthal polynomials; Hankel transform.

AMS Subject Classification: 11B39, 11B83

1. Introduction

Number sequences continue to attract the attention of researchers for a long time [4–6, 8, 9, 11, 12, 19]. Number sequences, especially Fibonacci sequences, find application in many departments of mathematics as well as in other branches of science [3, 9, 13, 14, 16, 17, 20].

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While many generalizations of these number sequences are given, there are also many studies on the polynomials of these number sequences [2, 10, 13, 16].

One of these generalizations is the Jacobsthal numbers. Jacobsthal numbers were first defined by Horodam in 1996 as follows [6].

For $n \geq 2$, the Jacobsthal numbers J_n are defined by the following recurrence relation:

$$J_n = J_{n-1} + 2J_{n-2},$$

with $J_0 = 0$ and $J_1 = 1$.

In the same paper, he gave an explicit form of Jacobsthal numbers as follows:

$$J_n = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} 2^r,$$

where $\lfloor x \rfloor = \max\{m \epsilon \mathbb{Z} : m \leq x\}$.

Jacobsthal introduced Jacobsthal polynomials in 1919 as follows [7]. For $n \geq 2$, the Jacobsthal polynomials $J_n(x)$ are defined by

$$J_n(x) = xJ_{n-1}(x) + 2J_{n-2}(x),$$

with $J_0(x) = 0$ and $J_1(x) = 1$.

Horodam defined a new Jacobsthal polynomial with a different recurrence relation

$$J_n(x) = xJ_{n-1}(x) + 2xJ_{n-2}(x),$$

with $J_0(x) = 0$ and $J_1(x) = 1$ [6].

Catarino *et al.* gave a generalization of Jacobsthal polynomials in 2016 [2]. Philip defined incomplete Fibonacci and Lucas numbers in 1996 [5]. Djordjevic *et al.* introduced incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers in 2005 [12]. Ramírez [16] also created the incomplete $h(x)$-Fibonacci polynomials and incomplete $h(x)$-Lucas polynomials and gave some identities for them.

Uygun also defined a new generalization of Jacobsthal polynomials in 2018 [21].

In this work, we give the Catalan transform of the incomplete Jacobsthal numbers and find the generating functions of the Catalan transform of these numbers. Also, we apply the Hankel transform to the Catalan transforms of these numbers. We calculate determinants of matrixes formed with $CJ_{k,n}^{l}$by using Hankel transform. We introduce the definition of the incomplete $h(x)$-Jacobsthal polynomials by using the generalization of Jacobsthal polynomials defined in [21]. We analyze the recurrence relation and some properties of these polynomials.

Definition 1.1. Let $h(x)$ be a polynomial with a real coefficient. For $n \geq 2$, the $h(x)$-Jacobsthal polynomial $\{J_{h,n}(x)\}_{n \in N}$ is given by

$$J_{h,n}(x) = h(x)J_{h,n-1}(x) + 2J_{h,n-2}(x), \tag{1.1}$$

with $J_{h,0}(x) = 0$ and $J_{h,1}(x) = 1$ [21].

When $h(x) = x$, we get the Jacobsthal polynomials. If k is a fixed positive integer and $h(x) = k$, then we obtain k-Jacobsthal numbers. Especially, if we take $k = 1$, we find the Jacobsthal numbers.

Definition 1.2. An explicit form for $h(x)$-Jacobsthal polynomial $\{J_{h,n}(x)\}_{n\in N}$ is given by the following relation:

$$J_{h,n}(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} 2^i h^{n-2i-1}(x),$$

[21].

For (1.1), the characteristic equation is given by

$$r^2 - h(x)r - 2 = 0$$

and its roots are

$$\alpha(x) = \frac{h(x) + \sqrt{h(x)^2 + 8}}{2}, \quad \beta(x) = \frac{h(x) - \sqrt{h(x)^2 + 8}}{2}.$$

We get the following equations related to the roots:

$$\alpha(x) + \beta(x) = h(x), \quad \alpha(x)\beta(x) = -2, \quad \alpha(x) - \beta(x) = \sqrt{h(x)^2 + 8}.$$

The Binet formula for these sequences is

$$J_{h,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}.$$

Definition 1.3. The generating function for $h(x)$-Jacobsthal polynomial $\{J_{h,n}(x)\}_{n\in N}$ is defined by

$$G(t) = \sum_{n=0}^{\infty} J_{h,n}(x)t^n = \frac{t}{1 - p(x)t - 2t^2}$$

[21].

Definition 1.4. The Catalan numbers with general term $C(n)$ are defined in [1] by

$$C(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Generating functions of the Catalan numbers $C(n)$ are given in [1] by

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

2. Incomplete Jacobsthal Numbers and Catalan Transform

Definition 2.1. The incomplete Jacobsthal numbers are defined by

$$J_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j} 2^j, \tag{2.1}$$

for $n = 1, 2, \ldots; 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

We give some values of the incomplete Jacobsthal numbers in Table 1.
It is seen easily that $J_n(\lfloor \frac{n-1}{2} \rfloor) = J_n$.

Theorem 2.1. *The terms of the above incomplete Jacobsthal numbers satisfy the following recurrence relation*:

$$J_{n+2}(k+1) = J_{n+1}(k+1) + 2J_n(k), \tag{2.2}$$

for $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

Proof. Let's write the left side of the equality according to Definition 2.1 for the proof.

$$\begin{aligned}
J_{n+1}(k) + 2J_n(k) &= \sum_{j=0}^{k+1} \binom{n-j}{j} 2^j + 2\sum_{j=0}^{k} \binom{n-1-j}{j} 2^j \\
&= \sum_{j=0}^{k+1} \binom{n-j}{j} 2^j + \sum_{j=0}^{k} \binom{n-1-j}{j} 2^{j+1} \\
&= \sum_{j=0}^{k+1} \binom{n-j}{j} 2^j + \sum_{j=0}^{k+1} \binom{n-j}{j-1} 2^j \\
&= \sum_{j=0}^{k+1} \left[\binom{n-j}{j} + \binom{n-j}{j-1} \right] 2^j \\
&= \sum_{j=0}^{k+1} \binom{n-j+1}{j} 2^j = J_{n+2}(k+1).
\end{aligned}$$

Thus, the proof is complete. □

Table 1. Some values of $J_n(k)$.

n/k	0	1	2	3
1	1			
2	1			
3	1	3		
4	1	5		
5	1	7	11	
6	1	9	21	
7	1	11	35	43

From (2.2), we get the following relation:

$$J_{n+2}(k) = J_{n+1}(k) + 2J_n(k) - \binom{n-1-k}{k} 2^{k*1}. \tag{2.3}$$

Lemma 2.1 (see [14, p. 592]). *For $n > 1$, let $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ be a complex sequence with*

$$s_n = as_{n-1} + bs_{n-2} + r_n,$$

where a, b are complex numbers. Then the generating function $U(t)$ for $\{s_n\}$ is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2},$$

where $G(t)$ is the generating function for $\{r_n\}$.

Theorem 2.2. *The generating functions for the incomplete Jacobsthal numbers $J_n(k)$ is given by*

$$R_k(t) = \sum_{i=0}^{\infty} J_k(i)(x)t^i = t^{2k+1} \frac{(J_{2k+1} + 2tJ_{2k})(1-t)^{k+1} - t^2 2^{k+1}}{(1-t)^{k+1}(1-t-2t^2)}.$$

Proof. From (2.1) and (2.3), for $0 \leq n < 2k+1$, we get

$$J_{2k+2}(k) = J_{2k+2} \quad \text{and} \quad J_{2k+1}(k) = J_{2k+1}$$

and

$$J_n(k) = J_{n-1}(k) + 2J_{n-2}(k) - \binom{n-3-k}{k} 2^{k*1}.$$

If we show $s_0 = J_{2k+1}(k)$ and $s_1 = J_{2k+2}(k)$then we get $s_n = J_{n+2k+1}(k)$.

Similarly, if $r_0 = r_1 = 0$ then we get

$$r_n = \binom{n+k-1}{n-2} 2^{k+1}.$$

The generating function for $\{r_n\}$ is

$$G(t) = \frac{t^2}{(1-t)^{k+1}} 2^{k+1} = t^2 \left(\frac{2}{1-t}\right)^{k+1}$$

see [18, p. 355].

From Lemma 2.1, we obtain the generating function $R_k(t)$ for $\{s_n\}$. □

Definition 2.2. Following [14], the Catalan transform of the incomplete Jacobsthal numbers are defined as follows:

$$CJ_n(k) = \sum_{I=0}^{n} \left(\frac{i}{2n-r}\right) \binom{2n-i}{n-i} J_i(k)$$

for $n < 1$ with $CJ_n(k) = 0$.

We can give terms some of the Catalan transform of the incomplete Jacobsthal numbers as follows:

$$\begin{bmatrix} CJ_1(k) \\ CJ_2(k) \\ CJ_3(k) \\ CJ_4(k) \\ CJ_5(k) \\ CJ_6(k) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 12 \\ 42 \\ 132 \\ \vdots \end{bmatrix}.$$

The following equation can be written as a product of two matrices:

$$\begin{bmatrix} CJ_1(k) \\ CJ_2(k) \\ CJ_3(k) \\ CJ_4(k) \\ CJ_5(k) \\ CJ_6(k) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ & & \vdots & & & \end{bmatrix} \begin{bmatrix} J_1(k) \\ J_2(k) \\ J_3(k) \\ J_4(k) \\ J_5(k) \\ J_6(k) \\ \vdots \end{bmatrix}.$$

The entries of the matrix C verify the recurrence relation $C_{i,j} = \sum_{r=j-1}^{i-1} C_{i-1,r}$. For $i > 1$, note that the first and second columns are the Catalan numbers.

The matrix $C_{n,n-i}$ is the Catalan triangle. For $1 \leq i \leq n$, the entries of the matrix $C_{n,n-i}$ verify

$$C_{n,n-i} = \frac{(2n-i)!(i+1)}{(n-i)!(n+1)!}.$$

Definition 2.3. The generating function of Catalan numbers is defined in paper [1] as follows:

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Definition 2.4. Let $c(x)$ and $A(x)$, respectively, be the generating function of the sequence of the Catalan numbers $\{C_n\}$ and any sequence $\{a_n\}$. It was proved in [1] that $A(c(x))$ is the generating function of the Catalan transform of this last sequence.

Consequently, from the composition of functions, we obtain the generating function of the incomplete Jacobsthal numbers.

Definition 2.5. The generating function of the Catalan form of incomplete Jacobsthal numbers is as follows:

$$CJ_n(k)(x) = J_n(k)(x * c(x)) = R_k(c(x)),$$

where $R_k(t)$ is the generating function of incomplete Jacobsthal numbers,

$$R_k(t) = t^{2k+1}\frac{(J_{2k+1} + 2tJ_{2k})(1-t)^{k+1} - t^2 2^{k+1}}{(1-t)^{k+1}(1-t-2t^2)}.$$

The generating function of the Catalan form of incomplete Jacobsthal numbers according to Definition 2.4 is as follows:

$$R_k(c(x)) = R_k\left(\frac{1-\sqrt{1-4x}}{2x}\right)$$

$$= \left(\frac{1-\sqrt{1-4x}}{2x}\right)^{2k+1} \frac{\begin{array}{c}(J_{2k+1} + 2(\frac{1-\sqrt{1-4x}}{2x})J_{2k})(1-\frac{1-\sqrt{1-4x}}{2x})^{k+1} \\ -(\frac{1-\sqrt{1-4x}}{2x})^2 2^{k+1}\end{array}}{(1-\frac{1-\sqrt{1-4x}}{2x})^{k+1}(1-t-2t^2)}.$$

Definition 2.6. Let $A = \{a_0, a_1, a_2, \ldots\}$ be a sequence of real numbers. The sequence of determinat $H_n = \mathrm{Det}[a_{i+j-2}]$ is the Hankel transform of the sequence A, i.e. [15]

$$H_n = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{vmatrix}.$$

The upper-left $n \times n$ sub-determinant of H_n is called the Hankel determinant of order n of the sequence A. The Hankel transform of the Catalan transform of the incomplete Jacobsthal numbers for $k = 0$,

$$H_2 = \begin{vmatrix} J_1(k) & J_2(k) \\ J_2(k) & J_3(k) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0,$$

$$H_3 = \begin{vmatrix} J_1(k) & J_2(k) & J_3(k) \\ J_2(k) & J_3(k) & J_4(k) \\ J_3(k) & J_4(k) & J_5(k) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

If we continue in this way, in the case of $k = 0$, the determinants of the other matrices will also be 0.

Now let's examine the determinants for the case $k = 1$.

$$H_2 = \begin{vmatrix} J_3(k) & J_4(k) \\ J_4(k) & J_5(k) \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -4.$$

$$H_3 = \begin{vmatrix} J_3(k) & J_4(k) & J_5(k) \\ J_4(k) & J_5(k) & J_6(k) \\ J_5(k) & J_6(k) & J_7(k) \end{vmatrix} = \begin{vmatrix} 3 & 5 & 7 \\ 5 & 7 & 9 \\ 7 & 9 & 11 \end{vmatrix} = 0.$$

We can continue in this form $4 \times 4, 5 \times 5, \ldots, n \times n$ determinats will all come out zero. For $k = 2$,

$$H_2 = \begin{vmatrix} J_5(k) & J_6(k) \\ J_6(k) & J_7(k) \end{vmatrix} = \begin{vmatrix} 11 & 21 \\ 21 & 35 \end{vmatrix} = -56.$$

$$H_3 = \begin{vmatrix} J_5(k) & J_6(k) & J_7(k) \\ J_6(k) & J_7(k) & J_8(k) \\ J_7(k) & J_8(k) & J_9(k) \end{vmatrix} = \begin{vmatrix} 11 & 21 & 35 \\ 21 & 35 & 53 \\ 35 & 53 & 75 \end{vmatrix} = -64.$$

$$H_4 = \begin{vmatrix} J_5(k) & J_6(k) & J_7(k) & J_8(k) \\ J_6(k) & J_7(k) & J_8(k) & J_9(k) \\ J_7(k) & J_8(k) & J_9(k) & J_{10}(k) \\ J_8(k) & J_9(k) & J_{10}(k) & J_{11}(k) \end{vmatrix} = \begin{vmatrix} 11 & 21 & 35 & 53 \\ 21 & 35 & 53 & 75 \\ 35 & 53 & 75 & 101 \\ 53 & 75 & 101 & 131 \end{vmatrix} = 0.$$

We can continue in this form $5 \times 5, \ldots, n \times n$ determinats will all come out zero.

3. Incomplete $h(x)$-Jacobsthal Polynomials

Definition 3.1. The incomplete $h(x)$-Jacobsthal polynomials are given by

$$J_{h,n}^{l}(x) = \sum_{i=0}^{l} \binom{n-1-i}{i} 2^i h^{n-2i-1}(x), \tag{3.1}$$

where $0 \leq l \leq \lfloor \frac{n-1}{2} \rfloor$.

We give some values of incomplete $h(x)$-Jacobsthal polynomials in Table 2.

It is seen easily that $J_{1,n}^{\lfloor \frac{n-1}{2} \rfloor}(x) = J_n$.

We can see some n values that satisfy the above equation in Table 3.

From (3.1), we can easily write the following equations:

- $J_{h,n}^{0}(x) = h^{n-1}(x)$, $n \geq 1$.
- $J_{h,n}^{1}(x) = h^{n-1}(x) + (n-2)2h^{n-3}(x)$, $n \geq 3$.
- $J_{h,n}^{2}(x) = h^{n-1}(x) + (n-2)2h^{n-3}(x) + \frac{(n-3)(n-4)}{2} 2^2 h^{n-5}(x)$, $n \geq 5$.

Table 2. Some values of $J^l_{h,n}(x)$.

n/l	0	1	2	3
1	1			
2	h			
3	h^2	h^2+2		
4	h^3	h^3+4h		
5	h^4	h^4+6h^2	h^4+6h^2+4	
6	h^5	h^5+8h^3	h^5+8h^3+12h	
7	h^6	h^6+10h^4	$h^6+10h^4+24h^2$	$h^6+10h^4+24h^2+8$

Table 3. Some values for $J^{\lfloor\frac{n-1}{2}\rfloor}_{1,n}(x) = J_n$.

n	$J^{\lfloor\frac{n-1}{2}\rfloor}_{1,n}(x)$	J_n
1	$J^0_{1,1}(x) = 1$	$J_1 = 1$
2	$J^0_{1,2}(x) = 1$	$J_2 = 1$
3	$J^1_{1,3}(x) = 3$	$J_3 = 3$
4	$J^1_{1,4}(x) = 5$	$J_4 = 5$
5	$J^2_{1,5}(x) = 11$	$J_5 = 11$

- $J^{\lfloor\frac{n-1}{2}\rfloor}_{h,n}(x) = J_{h,n}(x)$, $n \geq 1$.
- $\sum_{n=1}^{k} J^0_{h,n}(x) = 1 + h(x) + h^2(x) + \cdots + h^{k-1}(x) = \frac{1-h^n(x)}{1-h(x)}$.

Theorem 3.1. *For $n \geq 3$, we have the following equality*:

$$J^{\lfloor\frac{n-3}{2}\rfloor}_{h,n}(x) = \begin{cases} J_{h,n}(x) - n\left\lfloor\dfrac{n-1}{2}\right\rfloor h, & \text{if } n \text{ is an even number,} \\ J_{h,n}(x) - 2^{\lfloor\frac{n-1}{2}\rfloor}, & \text{if } n \text{ is an odd number.} \end{cases}$$

Proof. First let's prove the theorem by mathematical induction on n, if n is an even integer.

For $n = 4$, the equation is true since$J^0{}_{h,4}(x) = h^3 = h^3 + 4h - 4h$.

Now, suppose it is true for all even numbers for $n \leq 2k$. Then we have

$$J^{\lfloor\frac{2k-3}{2}\rfloor}_{h,2k}(x) = J_{h,2k}(x) - 2k\left\lfloor\frac{2k-1}{2}\right\rfloor h. \tag{3.2}$$

For $n = 2k+2$, let's find the following equation:

$$J^{\lfloor\frac{2k-1}{2}\rfloor}_{h,2k+2}(x) = J_{h,2k+2}(x) - (2k+2)\left\lfloor\frac{2k+1}{2}\right\rfloor h.$$

For $k \geq 2$, since for $k+1 \leq 2k$, if we get $k+1$ instead of k then we find the desired result.

Similar proof is made for odd numbers. □

Theorem 3.2. *The terms of the above incomplete $h(x)$-Jacobsthal polynomials satisfy the following recurrence relation*:

$$J^{l+1}_{h,n+2}(x) = h(x)J^{l+1}_{h,n+1}(x) + 2J^{l}_{h,n}(x), 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{3.3}$$

Proof. Let's write the left side of the equality according to (3.1) for the proof.

$$\begin{aligned}
&h(x)\sum_{i=0}^{l+1}\binom{n-i}{i}2^i h^{n-2i}(x) + 2\sum_{i=0}^{l}\binom{n-1-i}{i}2^i h^{n-2i-1}(x)\\
&= \sum_{i=0}^{l+1}\binom{n-i}{i}2^i h^{n-2i+1}(x) + \sum_{i=0}^{l}\binom{n-1-i}{i}2^{i+1} h^{n-2i-1}(x)\\
&= \sum_{i=0}^{l+1}\binom{n-i}{i}2^i h^{n-2i+1}(x) + \sum_{i=0}^{l+1}\binom{n-i}{i-1}2^i h^{n-2i+1}(x) - \binom{n}{-1}2^0 h^{n+1}(x)\\
&= \sum_{i=0}^{l+1}\binom{n-i}{i-1}2^i h^{n-2i+1}(x) + \binom{n-i}{i}2^i h^{n-2i+1}(x)\\
&= \sum_{i=0}^{l+1}\left(\binom{n-i}{i-1} + \binom{n-i}{i}\right)2^i h^{n-2i+1}(x)\\
&= \sum_{i=0}^{l+1}\binom{n-1-i}{i}2^i h^{n-2i+1}(x) = J^{l+1}_{h,n+2}.
\end{aligned}$$

Thus, the proof is complete. □

From (3.3), we get the following recurrence relation:

$$J^{l}_{h,n+2}(x) = h(x)J^{l}_{h,n+1}(x) + 2J^{l}_{h,n}(x) - \binom{n-1-l}{l}2^{l+1}h^{n-2l-1}(x). \tag{3.4}$$

Theorem 3.3. *We have the following equality*:

$$J^{l+s+1}_{h,n+2s+2}(x) = h(x)J^{l+s+1}_{h,n+2s+1}(x) + 2J^{l+s}_{h,n+2s}, \quad 0 \leq l \leq \frac{n-s-1}{2}.$$

Proof.

$$\begin{aligned}
&h(x)J^{l+s+1}_{h,n+2s+1} + 2J^{l+s}_{h,n+2s}\\
&= h(x)\sum_{i=0}^{l+s+1}\binom{n+2s+1-1-i}{i}2^i h^{n+2s+1-2i-1}(x)\\
&\quad + 2\sum_{i=0}^{l+s}\binom{n+2s-1-i}{i}2^i h^{n+2s-2i-1}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{l+s+1} \binom{n+2s-i}{i} 2^i h^{n+2s-2i+1}(x) \\
&\quad + \sum_{i=0}^{l+s} \binom{n+2s-1-i}{i} 2^{i+1} h^{n+2s-2i-1}(x) \\
&= \sum_{i=0}^{l+s+1} \binom{n+2s-i}{i} 2^i h^{n+2s-2i+1}(x) + \sum_{i=1}^{l+s+1} \binom{n+2s-i}{i-1} 2^i h^{n+2s-2i+1}(x) \\
&= \sum_{i=0}^{l+s+1} \binom{n+2s-i}{i} 2^i h^{n+2s-2i+1}(x) + \sum_{i=0}^{l+s+1} \binom{n+2s-i}{i-1} \\
&\quad \times 2^i h^{n+2s-2i+1}(x) - \binom{n+2s}{-1} h^{n+2s+1}(x) \\
&= \sum_{i=0}^{l+s+1} \left(\binom{n+2s-i}{i} + \binom{n+2s-i}{i-1} \right) 2^i h^{n+2s-2i+1}(x) - 0 \\
&= \sum_{i=0}^{l+s+1} \left(\binom{n+2s-i}{i} + \binom{n+2s-i}{i-1} \right) 2^i h^{n+2s-2i+1}(x) \\
&= \sum_{i=0}^{l+s+1} \binom{n+2s-i+1}{i} 2^i h^{n+2s-2i+1}(x) = J_{h,n+2s+2}^{l+s+1}(x).
\end{aligned}
$$

Thus, the proof is complete. □

Theorem 3.4. *We have*

$$\sum_{i=0}^{s} \binom{s}{i} 2^{i+1} J_{h,n}^{l}(x) = 3^s (J_{h,n+2}^{l+1}(x) - h(x) J_{h,n+1}^{l+1}).$$

Proof. We proceed by introduction on s.

Let's first see that it is true for $s = 0$.

For $s = 0$, we get

$$\sum_{i=0}^{0} \binom{0}{i} 2^{i+1} J_{h,n}^{l} = 2 J_{h,n}^{l} = J_{h,n+2}^{l+1}(x) - h(x) J_{h,n+1}^{l+1}.$$

Now suppose that the result is true for $s = k$. So, we have the following equation:

$$\sum_{i=0}^{k} \binom{k}{i} 2^{i+1} J_{h,n}^{l}(x) = 3^k (J_{h,n+2}^{l+1}(x) - h(x) J_{h,n+1}^{l+1}).$$

Now, we prove it for $s = k + 1$

$$\sum_{i=0}^{k+1} \binom{k+1}{i} 2^{i+1} J_{h,n}^{l}(x) = 3^{k+1} (J_{h,n+2}^{l+1}(x) - h(x) J_{h,n+1}^{l+1})$$

$$\sum_{i=0}^{k+1}\binom{k+1}{i}2^{i+1}J_{h,n}^{l}(x)=\sum_{i=0}^{k+1}\left(\binom{k}{i}+\binom{k}{i-1}\right)2^{i+1}J_{h,n}^{l}(x)$$

$$=\sum_{i=0}^{k+1}\binom{k}{i}2^{i+1}J_{h,n}^{l}(x)+\sum_{i=0}^{k+1}\binom{k}{i-1}2^{i+1}J_{h,n}^{l}(x)$$

$$=3^{k}(J_{h,n+2}^{l+1}(x)-h(x)J_{h,n+1}^{l+1})+\binom{k}{k+1}2^{k+2}J_{h,n}^{l}(x)$$

$$+\sum_{i=-1}^{k}\binom{k}{i}2^{i+2}J_{h,n}^{l}(x)$$

$$=3^{k}(J_{h,n+2}^{l+1}(x)-h(x)J_{h,n+1}^{l+1})+0+\sum_{i=0}^{k}\binom{k}{i}2^{i+2}J_{h,n}^{l}(x)$$

$$+\binom{k}{-1}2^{1}J_{h,n}^{l}(x)$$

$$=3^{k}(J_{h,n+2}^{l+1}(x)-h(x)J_{h,n+1}^{l+1})+2\sum_{i=0}^{k}\binom{k}{i}2^{i+1}J_{h,n}^{l}(x)+0$$

$$=3^{k}(J_{h,n+2}^{l+1}(x)-h(x)J_{h,n+1}^{l+1})+23^{k}(J_{h,n+2}^{l+1}(x)$$

$$-h(x)J_{h,n+1}^{l+1})$$

$$=3^{k+1}(J_{h,n+2}^{l+1}(x)-h(x)J_{h,n+1}^{l+1}).$$

Thus, the proof is complete. □

Theorem 3.5. *We get*

$$J_{h,n}^{l+1}(x)=J_{h,n}^{l}(x)+2^{l+1}\binom{n-l-2}{l+1}h^{n-2l-3}(x).$$

Proof. From (3.1), we obtain

$$J_{h,n}^{l+1}(x)=\sum_{i=0}^{l+1}\binom{n-1-i}{i}2^{i}h^{n-2i-1}(x)$$

$$=\sum_{i=0}^{l}\binom{n-1-i}{i}2^{i}h^{n-2i-1}(x)+\binom{n-2-l}{l+1}2^{l+1}h^{n-2l-3}(x)$$

$$=J_{h,n}^{l}(x)+\binom{n-2-l}{l+1}2^{l+1}h^{n-2l-3}(x).$$

Thus, the proof is complete. □

Theorem 3.6. *We get the following equality*:

$$J'_{h,n}(x) = \frac{h'(x)h(x)(n - J_{h,n}(x))}{h^2(x)+8} = \frac{h'(x)L_{h,1}(n - J_{h,n}(x))}{h^2(x)+8}.$$

Proof. We recall the Binet formula for $J_{h,n}(x)$,

$$J_{h,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$

where $\alpha(x) = \frac{h(x)+\sqrt{h(x)^2+8}}{2}, \beta(x) = \frac{h(x)-\sqrt{h(x)^2+8}}{2}$.

Let's first find the derivatives of $\alpha(x)$ and $\beta(x)$.

$$\alpha'(x) = \frac{h'(x)\alpha(x)}{\sqrt{h(x)^2+8}}, \quad \beta'(x) = -\frac{h'(x)\beta(x)}{\sqrt{h(x)^2+8}}.$$

Let's write this $\alpha'(x)$ and $\beta'(x)$ derivatives in the derivative of $J'_{h,n}(x)$.

$$\begin{aligned}
J'_{h,n}(x) &= \frac{\begin{array}{c}(n\alpha^{n-1}(x)\alpha'(x) - n\beta^{n-1}(x)\beta'(x))(\alpha(x) - \beta(x)) - (\alpha^n(x) \\ - \beta^n(x))(\alpha'(x) - \beta'(x))\end{array}}{(\alpha(x) - \beta(x))^2} \\
&= \frac{\begin{array}{c}(n\alpha^{n-1}(x)\frac{h'(x)\alpha(x)}{\sqrt{h(x)^2+8}} + n\beta^{n-1}(x)\frac{h'(x)\beta(x)}{\sqrt{h(x)^2+8}})(\alpha(x) - \beta(x)) \\ - (\alpha^n(x) - \beta^n(x))(\frac{h'(x)\alpha(x)}{\sqrt{h(x)^2+8}} + \frac{h'(x)\beta(x)}{\sqrt{h(x)^2+8}})\end{array}}{(\alpha(x) - \beta(x))^2} \\
&= \frac{[\frac{nh'(x)\alpha(x)+nh'(x)\beta(x)}{\sqrt{h(x)^2+8}}]\sqrt{h(x)^2+8} - \frac{(\alpha^n(x)-\beta^n(x))h'(x)(\alpha(x)+\beta(x))}{\sqrt{h(x)^2+8}}}{h(x)^2+8} \\
&= \frac{nh'(x)(\alpha(x)+\beta(x)) - h'(x)(\alpha(x)+\beta(x))(\frac{\alpha^n(x)-\beta^n(x)}{\sqrt{h(x)^2+8}})}{h(x)^2+8} \\
&= \frac{nh'(x)h(x) - h'(x)h(x)J_{h,n}(x)}{h(x)^2+8} = \frac{h'(x)h(x)(n - J_{h,n}(x))}{h(x)^2+8}.
\end{aligned}$$

Thus, the first equality is obtained. For the other part,

$$L_{h,n} = \alpha^n(x) + \beta^n(x).$$

If we write $n = 1$ in the last equity,

$$L_{h,1} = \alpha(x) + \beta(x),$$

is obtained.

If we write $h(x) = \alpha(x) + \beta(x) = L_{h,1}$ in the derivative equation, we obtain

$$J'_{h,n}(x) = \frac{nh'(x)(\alpha(x)+\beta(x)) - h'(x)(\alpha(x)+\beta(x))(\frac{\alpha^n(x)-\beta^n(x)}{\sqrt{h(x)^2+8}})}{h(x)^2+8}$$

$$= \frac{h'(x)L_{h,1}(n - J_{h,n}(x))}{h(x)^2+8}.$$

Thus, the proof is complete. □

Theorem 3.7. *The following equality is provided*:

$$\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor} J^l_{h,n}(x) = \left(\left\lfloor\frac{n-1}{2}\right\rfloor + 1\right) J_{h,n}(x) - \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} i\binom{n-1-i}{i} 2^i h^{n-2i-1}(x).$$

Proof. Let's expand the left side of the equality.

$$\begin{aligned}
\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor} J^l_{h,n}(x) &= \binom{n-1-0}{0} 2^0 h^{n-1}(x) \\
&\quad + \left[\binom{n-1-0}{0} 2^0 h^{n-1}(x) + \binom{n-1-1}{1} 2^1 h^{n-3}(x)\right] \\
&\quad + \cdots + \left[\binom{n-1-0}{0} 2^0 h^{n-1}(x) + \binom{n-1-1}{1} 2^1 h^{n-3}(x)\right. \\
&\quad \left. + \cdots + \binom{n-1-\left\lfloor\frac{n-1}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor} 2^{\lfloor\frac{n-1}{2}\rfloor} h^{n-1-2\lfloor\frac{n-1}{2}\rfloor}(x)\right] \\
&= \left(\left\lfloor\frac{n-1}{2}\right\rfloor + 1\right)\binom{n-1-0}{0} 2^0 h^{n-1}(x) \\
&\quad + \left\lfloor\frac{n-1}{2}\right\rfloor \binom{n-1-1}{1} 2^1 h^{n-3}(x) \\
&\quad + \cdots + \binom{n-1-\left\lfloor\frac{n-1}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor} 2^{\lfloor\frac{n-1}{2}\rfloor} h^{n-1-2\lfloor\frac{n-1}{2}\rfloor}(x) \\
&= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-i}{i} 2^i h^{n-2i-1}(x) \left(\left\lfloor\frac{n-1}{2}\right\rfloor + 1 - i\right)
\end{aligned}$$

$$= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} 2^i h^{n-2i-1}(x) \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right)$$

$$- \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} i \binom{n-1-i}{i} 2^i h^{n-2i-1}(x)$$

$$= \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right) J_{h,n}(x) - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} i \binom{n-1-i}{i} 2^i h^{n-2i-1}(x). \qquad \square$$

Theorem 3.8. *The derivative of* $J^l_{h,n}(x)$ *is as follows*:

$$(J^l_{h,n}(x))' = \sum_{i=0}^{l} \binom{n-1-i}{i} 2^i (n-2i-1) h^{n-2i-2}(x) h'(x).$$

Proof. The proof is easily seen if the right side of Eq. (3.1) is taken as the derivative with respect to x. $\square$

Theorem 3.9. *The generating functions for the incomplete* $h(x)$*-Jacobsthal polynomials* $J^l_{h,n}(x)$ *is given by*

$$R_{h,l}(x) = \sum_{i=0}^{\infty} J^l_{h,n}(x) t^i$$

$$= \frac{t^{2l+1}[J_{h,2l+1}(x) + (J_{h,2l+2}(x) - h(x)J_{h,2l+1}(x)t - t^2(\frac{2}{1-h(x)t})^{l+1})]}{(1-h(x)t-t^2)}.$$

Proof. From (3.1) and (3.4), for $0 \le n < 2l+1$, we get

$$J^l_{h,2l+1}(x) = J_{h,2l+1}(x) \quad \text{and} \quad J^l_{h,2l+2}(x) = J_{h,2l+2}(x)$$

and

$$J^l_{h,n}(x) = h(x)J^l_{h,n-1}(x) + 2J^l_{h,n-2}(x) - \binom{n-3-l}{l} 2^{l+1} h^{n-2l-3}(x).$$

If we let $s_0 = J^l_{h,2l+1}(x)$ and $s_1 = J^l_{h,2l+2}(x)$ then we get $s_n = J^l_{h,n+2l+1}(x)$.

Similarly, if $r_0 = r_1 = 0$ then we get

$$r_n = \binom{n+l-1}{n-2} 2^{l+1} h^{n-2}(x).$$

The generating function for $\{r_n\}$ is

$$G(t) = t^2 \left(\frac{2}{1-h(x)t} \right)^{l+1}$$

see [18, p. 355].

From Lemma 2.1, we have the generating function $R_{h,l}(x)$ for $\{s_n\}$. $\square$

4. Conclusions

We give the Catalan transform of the incomplete Jacobsthal numbers. Also, we apply the Hankel transform to the Catalan transforms of these numbers. We give the definition of the incomplete $h(x)$-Jacobsthal polynomials by using the generalization of Jacobsthal polynomials defined in [21]. We analyze the recurrence relation and some properties of these polynomials. Then we calculate determinants of matrixes formed with $CJ_{k,n}^{l}$ by using Hankel transform.

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