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Hyperbolic Functions and Gaussian-Balancing-like Sequences

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Abstract

Motivated by the classical connection between Fibonacci numbers and hyperbolic functions, this paper develops hyperbolic representations for Gaussian balancing-like and Gaussian Lucas-balancing-like sequences. By expressing the real and imaginary components of these sequences in terms of hyperbolic sine and cosine functions, several addition, subtraction, and duplication identities are derived. A quadratic relation between the two Gaussian sequences and an explicit formula for their product are also obtained. These representations are further used to investigate the asymptotic relationship between Gx_nGy_n and $Gx_{2n}$and to determine the corresponding limiting ratio. The results demonstrate that hyperbolic functions provide a unified analytical framework for studying the algebraic and asymptotic properties of Gaussian balancing-like sequences.

Keywords: Gaussian balancing-like sequence; Hyperbolic function representation; Sequence identities.

1. Introduction

Second-order linear recurrence sequences and their generalizations constitute an active area of research in number theory and discrete mathematics. Their structural properties, closed-form representations, algebraic identities, asymptotic behavior, and extensions to complex domains have led to numerous developments in the theory of special sequences. Among these extensions, Gaussian analogues are of particular interest because they combine consecutive terms of a real recurrence sequence into complex-valued terms and thereby provide a richer algebraic structure.

Another useful perspective in the study of recurrence sequences arises from their connection with hyperbolic functions. In particular, whenever the characteristic roots of a second-order recurrence are reciprocal, the corresponding Binet-type expressions can naturally be represented in terms of hyperbolic sine and cosine functions. This relationship enables classical identities of hyperbolic functions to be transferred to recurrence sequences and provides an alternative analytical method for deriving structural identities.

The connection between recurrence sequences and hyperbolic functions has previously been investigated in the context of Fibonacci numbers. Stakhov and Tkachenko introduced hyperbolic Fibonacci trigonometry [3], and Trzaska subsequently studied related Fibonacci hyperbolic functions in greater detail [21]. These studies demonstrated that hyperbolic functions can provide an effective continuous representation of discrete recurrence structures and can be used to obtain identities that correspond to familiar formulas from classical hyperbolic analysis.

More recently, hyperbolic structures have been investigated for several generalized recurrence sequences and their higher-dimensional algebraic extensions. Özkan, Uysal, and Godase studied hyperbolic k-Jacobsthal and k-Jacobsthal-Lucas quaternions [8], while Özkan and Akkuş investigated hyperbolic k-Oresme and k-Oresme-Lucas quaternions [9]. Hyperbolic quaternionic structures associated with generalized Bronze Fibonacci sequences were considered in [10]. In addition, hyperbolic k-Mersenne and k-Mersenne-Lucas octonions were studied in [16]. These contributions indicate that hyperbolic methods provide a useful framework for extending recurrence sequences to more general algebraic settings.

Balancing-like sequences were introduced by Panda and Rout as a class of recurrence sequences exhibiting several properties analogous to those of balancing numbers [12]. Their structural properties were subsequently investigated in greater detail in [4], and k-balancing-like and k-Lucas-balancing-like sequences were later studied in [7]. Gaussian balancing numbers and Gaussian Lucas-balancing numbers were investigated by Taşçı [6], who established several fundamental properties of these complex-valued sequences. More recently, Gaussian balancing-like and Gaussian Lucas-balancing-like sequences were introduced as complex extensions of the corresponding balancing-like sequences [5]. Various algebraic extensions of balancing and Lucas-balancing numbers have also been investigated, including dual bicomplex balancing numbers [15] and balancing and Lucas-balancing hybrid numbers [17].

Let λ_1 and λ_2 denote the characteristic roots associated with the recurrence, where

$$\lambda_1\lambda_2 = 1.$$

The reciprocal relationship between these roots suggests introducing the parameter

$$z = \frac{1}{2} ln\left(\frac{\lambda_1}{\lambda_2}\right).$$

With this choice of z , the real and imaginary components of the Gaussian balancing-like and Gaussian Lucas-balancing-like sequences can be expressed naturally in terms of $\sinh nz$ and $\cosh nz$.

Consequently, the Gaussian sequences admit representations of the form

$$Gx_n = \frac{2}{\sqrt{A^2-4}}(\sinh nz - i\sinh(n-1)z)$$

and

$$Gy_n = \cosh nz - i\cosh(n-1)z.$$

These representations establish a direct correspondence between Gaussian recurrence sequences and classical hyperbolic functions.

The main purpose of this paper is to investigate this correspondence systematically and to use hyperbolic identities as an analytical tool for studying Gaussian balancing-like and Gaussian Lucas-balancing-like sequences. First, a fundamental quadratic relation between Gx_n and Gy_n is obtained, showing that the characteristic relationship between the associated real sequences extends naturally to the Gaussian setting.

Hyperbolic addition and subtraction formulas are then employed to derive identities involving$Gx_{m+n}, Gx_{m-n}, Gy_{m+n}, Gy_{m-n}$. In addition, explicit formulas for the even-indexed terms Gx_{2n} and Gy_{2n} are established. Furthermore, the product Gx_nGy_n is expressed explicitly in terms of hyperbolic functions. This representation is subsequently combined with the Binet-type forms of the Gaussian sequences to investigate their asymptotic behavior. In particular, an asymptotic relationship between Gx_nGy_n and Gx_{2n} is obtained, from which the limiting ratio

$$\lim_{n\to\infty}\frac{Gx_nGy_n}{Gx_{2n}} = \frac{1-i\lambda_2}{2}$$

is derived.

Thus, the hyperbolic approach developed in this paper provides a unified framework for deriving algebraic identities and asymptotic relations for Gaussian balancing-like and Gaussian Lucas-balancing-like sequences. Rather than viewing the hyperbolic representation merely as an alternative form of the corresponding Binet formulas, we employ it as a systematic method for revealing structural relationships within these Gaussian recurrence sequences.

2. Hyperbolic Functions and Gaussian Balancing-like Sequences

Formulas involving hyperbolic functions evaluated at $z = \frac{1}{2}ln\left(\frac{\lambda_1}{\lambda_2}\right)$ exhibit a close correspondence with the Gaussian balancing-like sequences Gx_n and the Gaussian Lucas-balancing-like sequences Gy_n.

Let $z = \frac{1}{2}ln\left(\frac{\lambda_1}{\lambda_2}\right)$ then,

$$\begin{aligned} Gx_n &= x_n - ix_{n-1} \\ &= \frac{2}{\sqrt{A^2-4}}(\sinh nz - i\sinh(n-1)z) \end{aligned}$$

and

$$\begin{aligned} Gy_n &= y_n - iy_{n-1} \\ &= \cosh nz - i\cosh(n-1)z. \end{aligned}$$

Hence

$$Gy_n - \frac{\sqrt{A^2-4}}{2}Gx_n = \lambda_2^n(1-i\lambda_1).$$

The following theorem shows that the fundamental relationship between balancing-like and Lucas-balancing-like sequences extends naturally to their Gaussian counterparts.

Theorem 2.1. For $n \in \mathbb{Z}^+$,

$$Gy_n^2 = \frac{A^2-4}{4}Gx_n^2 - Ai.$$

Proof. Since

$$Gy_n^2 = cosh^2 nz - cosh^2(n-1)z + 2i\cosh nz\cosh(n-1)z$$

and

$$\frac{A^2-4}{4}Gx_n^2 = sinh^2 nz - sinh^2(n-1)z + 2i\sinh nz\sinh(n-1)z$$

we get

$$\begin{aligned} Gy_n^2 - \frac{A^2-4}{4}Gx_n^2 &= 2i\left(y_n y_{n-1} - \frac{A^2-4}{4}x_n x_{n-1}\right) \\ &= 2i\left(\frac{A+\lambda_1^{2n-1}+\lambda_2^{2n-1}}{4} - \frac{\lambda_1^{2n-1}-\lambda_1-\lambda_2+\lambda_2^{2n-1}}{4}\right) \\ &= -Ai. \end{aligned}$$

Now, the part on hyperbolic addition identities for Gaussian-balancing-like sequences will be given.

Theorem 2.2. For $n, m \in \mathbb{Z}^+$ with $m > n$,

$$i)\ Gx_{m+n} + Gx_{m-n} = 2Gx_m y_n,$$
$$ii)\ Gx_{m+n} - Gx_{m-n} = 2Gy_m x_n.$$

Proof. i) Since $\sinh(m+n)z + \sinh(m-n)z = 2\sinh mz\cosh nz$,

$$Gx_{m+n} = \frac{2}{\sqrt{A^2-4}}[\sinh(m+n)\,z - i\sinh(m+n-1)\,z]$$

and

$$Gx_{m-n} = \frac{2}{\sqrt{A^2-4}}[\sinh(m-n)\,z - i\sinh(m-n-1)\,z].$$

From here we obtain,

$$Gx_{m+n} + Gx_{m-n} = \frac{2}{\sqrt{A^2-4}}[2\sinh m\,z\cosh n\,z - i2\sinh(m-1)\,z\cosh n\,z]$$
$$= \frac{4\cosh n\,z}{\sqrt{A^2-4}}[\sinh m\,z - i\sinh(m-1)\,z]$$
$$= 2\cosh n\,zGx_m$$
$$= 2Gx_m y_n.$$

ii) Since $\sinh(m+n)\,z - \sinh(m-n)\,z = 2\cosh m\,z\sinh n\,z$,

$$Gx_{m+n} = \frac{2}{\sqrt{A^2-4}}[\sinh(m+n)\,z - i\sinh(m+n-1)\,z]$$

and

$$Gx_{m-n} = \frac{2}{\sqrt{A^2-4}}[\sinh(m-n)\,z - i\sinh(m-n-1)\,z].$$

From here we obtain,

$$Gx_{m+n} - Gx_{m-n} = \frac{2}{\sqrt{A^2-4}}[2\cosh m\,z\sinh n\,z] - \frac{2i}{\sqrt{A^2-4}}[2\cosh(m-1)\,z\sinh n\,z]$$
$$= \frac{4}{\sqrt{A^2-4}}\sinh n\,z[\cosh m\,z - i\cosh(m-1)\,z]$$
$$= 2x_n Gy_m.$$

Theorem 2.3. For $n, m \in Z^+$ with $m > n$,

i) $Gy_{m+n} + Gy_{m-n} = 2y_n Gy_m$

ii) $Gy_{m+n} - Gy_{m-n} = \frac{A^2-4}{2}x_n Gx_m$.

Proof. i) Since $\cosh(m+n)\,z + \cosh(m-n)\,z = 2\cosh m\,z\cosh n\,z$,

$$Gy_{m+n} = \cosh(m+n)\,z - i\cosh(m+n-1)\,z$$

and

$$Gy_{m-n} = \cosh(m-n)\,z - i\cosh(m-n-1)\,z.$$

From here we obtain,

$$Gy_{m+n} + Gy_{m-n} = 2\cosh m\,z\cosh n\,z - 2i\cosh(m-1)\,z\cosh n\,z$$
$$= 2\cosh n\,z[\cosh m\,z - i\cosh(m-1)\,z]$$

$$= 2y_n G y_m$$

ii) Proof is obtained in a similar way.

The following theorem provides explicit formulas for computing the even-indexed Gaussian balancing-like numbers and Gaussian-Lucas-balancing-like numbers.

Theorem 2.4. For $n \in Z^+$,

i) $Gx_{2n} = x_n G y_n + y_n G x_n$

ii) $Gy_{2n} = y_n G y_n + \frac{A^2-4}{4} x_n G x_n$.

Proof. i) Since $Gx_{2n} = x_{2n} - ix_{2n-1}$, $x_{2n} = 2x_n y_n$ and

$$x_{2n-1} = \frac{2\sinh(2n-1)\,z}{\sqrt{A^2-4}}$$

$$= \frac{2}{\sqrt{A^2-4}}[\sinh(nz + (n-1)z)]$$

$$= \frac{2}{\sqrt{A^2-4}}[\sinh(nz)\cosh(n-1)\,z + \cosh(nz)\sinh(n-1)\,z]\backslash$$

$$= \frac{2}{\sqrt{A^2-4}}\sinh n\,z\cosh(n-1)\,z + \frac{2}{\sqrt{A^2-4}}\cosh n\,z\sinh(n-1)\,z$$

$$x_n y_{n-1} + x_{n-1} y_n.$$

we get

$$Gx_{2n} = 2x_n y_n - i(x_n y_{n-1} + x_{n-1} y_n)$$

$$= x_n(y_n - iy_{n-1}) + y_n(x_n - ix_{n-1})$$

$$= x_n G y_n + y_n G x_n.$$

ii) Since $Gy_{2n} = y_{2n} - iy_{2n-1}$, $y_{2n} = y_n^2 + \frac{A^2-4}{4}x_n^2$ and

$$y_{2n-1} = \cosh(2n-1)\,z$$

$$= \cosh n\,z\cosh(n-1)\,z + \sinh n\,z\sinh(n-1)\,z$$

$$= y_n y_{n-1} + \frac{A^2-4}{4} x_n x_{n-1}$$

we get,

$$Gy_{2n} = y_n^2 + \frac{A^2-4}{4}x_n^2 - i\left[y_n y_{n-1} + \frac{A^2-4}{4}x_n x_{n-1}\right]$$

$$= y_n^2 - iy_n y_{n-1} + \frac{A^2-4}{4} x_n^2 - i\frac{A^2-4}{4} x_n x_{n-1}$$

$$= y_n G y_n + \frac{A^2-4}{4} x_n G x_n.$$

Theorem 2.5. For $n \in Z^+$,

$$Gx_n Gy_n = \frac{1}{\sqrt{A^2-4}}[\sinh(2nz) - 2i\sinh(2n-1)\,z - \sinh 2(n-1)\,z].$$

Proof. $\mathrm{Gx_n Gy_n} = (\mathrm{x_n} - \mathrm{ix_{n-1}})(\mathrm{y_n} - \mathrm{iy_{n-1}})$

$$= (x_n y_n - x_{n-1} y_{n-1}) - i(x_n y_{n-1} + x_{n-1} y_n).$$

Since $x_n = \frac{2}{\sqrt{A^2-4}}\sinh n\,z$ and $y_n = \cosh n\,z$, then real part of the equation:

$$x_n y_n - x_{n-1} y_{n-1} = \frac{2}{\sqrt{A^2-4}}[\sinh n\,z\ \cosh n\,z - \sinh(n-1)\,z\cosh(n-1)\,z]$$

$$= \frac{1}{\sqrt{A^2-4}}\sinh(2nz) - \sinh 2(n-1)\,z$$

and imaginary part of the equation:

$$x_n y_{n-1} + x_{n-1} y_n = \frac{2}{\sqrt{A^2-4}}[\sinh n\,z\cosh(n-1)\,z + \sinh(n-1)\,z\cosh n\,z]$$

$$= \frac{2}{\sqrt{A^2-4}}\sinh(2n-1)\,z.$$

We obtain,

$$Gx_n Gy_n = \frac{1}{\sqrt{A^2-4}}[\sinh 2\,nz - \sinh 2(n-1)\,z] - i\frac{2}{\sqrt{A^2-4}}\sinh(2n-1)\,z$$

$$= \frac{1}{\sqrt{A^2-4}}[\sinh 2\,nz - i2\sinh(2n-1)\,z - \sinh 2(n-1)\,z].$$

Theorem 2.6. Let $A > 2$,

$$Gx_n Gy_n = \frac{\lambda_1 - i}{2\lambda_1} Gx_{2n} + o(\lambda_2^{2n})$$

or equivalently

$$Gx_n Gy_n = \frac{1 - i\lambda_2}{2} Gx_{2n} + o(\lambda_2^{2n}).$$

Proof.

$$\begin{aligned} Gx_n &= x_n - ix_{n-1} \\ &= \frac{\lambda_1^{n-1}(\lambda_1 - i) - \lambda_2^{n-1}(\lambda_2 - i)}{\lambda_1 - \lambda_2}. \end{aligned}$$

Similarly,

$$\begin{aligned} Gy_n &= y_n - iy_{n-1} \\ &= \frac{1}{2}[\lambda_1^{n-1}(\lambda_1 - i) + \lambda_2^{n-1}(\lambda_2 - i)]. \end{aligned}$$

$$Gx_nGy_n = \frac{1}{2(\lambda_1 - \lambda_2)}[\lambda_1^{2n-2}(\lambda_1 - i)^2 - \lambda_2^{2n-2}(\lambda_2 - i)^2].$$

On the other hand,

$$Gx_{2n} = \frac{\lambda_1^{2n-1}(\lambda_1 - i) - \lambda_2^{2n-1}(\lambda_2 - i)}{\lambda_1 - \lambda_2}.$$

Since the dominant term of Gx_{2n} is

$$\frac{\lambda_1^{2n-1}(\lambda_1 - i)}{\lambda_1 - \lambda_2}.$$

The coefficient is :

$$C = \frac{\frac{1}{2(\lambda_1 - \lambda_2)}\lambda_1^{2n-2}(\lambda_1 - i)^2}{\frac{1}{\lambda_1 - \lambda_2}\lambda_1^{2n-1}(\lambda_1 - i)} = \frac{\lambda_1 - i}{2\lambda_1}.$$

Therefore

$$Gx_nGy_n = \frac{\lambda_1 - i}{2\lambda_1}Gx_{2n} + o(\lambda_2^{2n}).$$

As a consequence of the preceding theorem, the limit of the ratio of consecutive terms of the Gaussian balancing-like and Gaussian Lucas-balancing-like sequences is established in the following result.

Corollary 2.7.

$$\lim_{n\to\infty} \frac{\mathrm{Gx_nGy_n}}{\mathrm{Gx_{2n}}} = \frac{1 - \mathrm{i}\lambda_2}{2}.$$

3. Conclusion

In this paper, hyperbolic function representations of Gaussian balancing-like and Gaussian Lucas-balancing-like sequences were investigated. By expressing the real and imaginary components of these Gaussian sequences in terms of hyperbolic sine and cosine functions, several algebraic identities, including addition, subtraction, and duplication formulas, were obtained. A quadratic relation between the Gaussian balancing-like and Gaussian Lucas-balancing-like sequences was also established. Furthermore, an explicit representation for the product Gx_nGy_n was derived and used to obtain an asymptotic relationship between Gx_nGy_n and Gx_{2n}. Consequently, the limiting ratio between these quantities was determined. These results show that hyperbolic functions provide a useful analytical framework for investigating both the algebraic and asymptotic structure of Gaussian recurrence sequences.

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